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On ρ-labeling the union of three cycles

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Abstract

Let G be a graph of size n with vertex set $V(G)$ and edge set $E(G)$. A *ρ-labeling* of G is a one-to-one function $f : V(G) \to \{0, 1, \ldots, 2n\}$ such that $\{|f(u) - f(v)| : \{u, v\} \in E(G)\} = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in \{1, 2, \ldots, n\}$ either $x_i = i$ or $x_i = 2n + 1 - i$. Such a labeling of G yields a cyclic G-decomposition of K_{2n+1}. It is conjectured that every 2-regular graph G has a ρ-labeling. We show that this conjecture holds when G has at most three components.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively.

Let $V(K_k) = \mathbb{Z}_k$ and let G be a subgraph of K_k. By *clicking* G, we mean applying the isomorphism $i \to i+1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of pairwise disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. If H is K_k, a G-decomposition Γ of H is *cyclic* if clicking is a permutation of Γ. If G is a graph and r is a positive integer, rG denotes the vertex disjoint union of r copies of G.

For any graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [19], Rosa introduced a hierarchy of labelings. We add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. Let $\bar{E}(G) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

($\ell 1$) $f(V(G)) \subseteq [0, 2n]$,

($\ell 2$) $f(V(G)) \subseteq [0, n]$,

($\ell 3$) $\bar{E}(G) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,

($\ell 4$) $\bar{E}(G) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ (with every edge in G having one endvertex in A and the other in B) such that

($\ell 5$) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

($\ell 6$) there exists an integer λ (called the *boundary value* of f) such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

($\ell 1$), ($\ell 3$) is called a *ρ-labeling*;

($\ell 1$), ($\ell 4$) is called a *σ-labeling*;

($\ell 2$), ($\ell 4$) is called a *β-labeling.*

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. If G is bipartite and a ρ, σ or β-labeling of G also satisfies ($\ell 5$), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition ($\ell 6$) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [19].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [19] and [11], respectively.

Theorem 1. *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 2. *Let G be a graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

A non-bipartite graph G is *almost-bipartite* if G contains an edge e whose removal renders the remaining graph bipartite (for example, odd cycles are almost-bipartite). In [5], Blinco et al. introduced a variation of a ρ-labeling of an almost-bipartite graph G of size n that yields cyclic G-decompositions of K_{2nx+1}. They called this labeling a γ-labeling. Rather than restate the (lengthy) definition of a γ-labeling here, we direct the interested reader to [5]. We do note however that a γ-labeling is necessarily a ρ-labeling.

Let G be a graph with n edges and Eulerian components and let h be a σ-labeling of G. It is well-known (see [3]) that we must have $n \equiv 0$ or $3 \pmod 4$. Moreover, if such a G is bipartite, then $n \equiv 0 \pmod 4$. We shall refer to this restriction as the *parity condition.* There are no such restrictions on $|E(G)|$ if h is a ρ-labeling.

Theorem 3. (Parity Condition) *If a graph G with Eulerian components and n edges has a σ-labeling, then $n \equiv 0$ or $3 \pmod 4$. If such a G is bipartite, then $n \equiv 0 \pmod 4$.*

In [19], Rosa presented α- and β-labelings of C_{4m} and of C_{4m+3}, respectively. It is also known that both C_{4m+1} and C_{4m+2} admit ρ-labelings. It was shown in [11] that there exists a ρ^+-labeling of C_{4m+2}, for all positive integers m. It can be easily checked that this labeling is actually a ρ^{++}-labeling.

In this manuscript, we will focus on labelings of 2-regular graphs (i.e., the vertex-disjoint union of cycles). If a 2-regular graph G is bipartite, then it is known that G admits a σ^+-labeling if the parity condition is satisfied (see [11]) and a ρ^{++}-labeling otherwise (see [4]). A 2-regular graph G need not admit a β-labeling even if the parity condition is satisfied. For example, it is shown in [16] that rC_3 does not admit a β-labeling for all $r > 1$ and rC_5 never admits a β-labeling. Moreover, it is known that $C_3 \cup C_3 \cup C_5$ is the smallest 2-regular graph that satisfies the parity condition, yet fails to have a β-labeling (see [2]). It is thus reasonable to focus on labelings that are less restrictive than β-labelings when studying 2-regular graphs.

Here, we shall show that every 2-regular graph G consisting of at most three components has a ρ-labeling. In a companion article [3], it is shown that if the parity condition is satisfied, then such a G necessarily admits a σ-labeling. These results are already known if G has at most two components (see [10]). Our results provide further evidence in support of a conjecture of El-Zanati and Vanden Eynden that every 2-regular graph admits a σ-labeling if the parity condition is satisfied and a ρ-labeling otherwise.

Let r, s and t be positive integers ≥ 3 and let $G = C_r \cup C_s \cup C_t$. If we consider the congruences of r, s and t modulo 4, then G belongs to one of 20 types of graphs (see Table 1). In each of the ten cases where the parity condition is not satisfied, we will show that G has a ρ-labeling.

2 Summary of Some of the Known Results

As stated in the previous section, the following is known for cycles (see [18], [19] and [11]).

Theorem 4. *Let $m \geq 3$ be an integer. Then, C_m admits an α-labeling if $m \equiv 0 \pmod 4$, a ρ-labeling if $m \equiv 1 \pmod 4$, a ρ^{++}-labeling if $m \equiv 2 \pmod 4$, and a β-labeling if $m \equiv 3 \pmod 4$.*

For 2-regular graphs with two components, we have the following important result from Abrham and Kotzig [2].

Theorem 5. *Let $m \geq 3$ and $n \geq 3$ be integers. Then the graph $C_m \cup C_n$ has a β-labeling if and only if $m + n \equiv 0$ or $3 \pmod 4$. Moreover, $C_m \cup C_n$ has an α-labeling if and only if both m and n are even and $m + n \equiv 0 \pmod 4$.*

If the parity condition is not satisfied, then we have the following from [4] and [10].

Theorem 6. *Let $m \geq 3$ and $n \geq 3$ be integers such that $m + n \equiv 1$ or $2 \pmod 4$. Then $C_m \cup C_n$ has a ρ^{++}-labeling if both m and n are even and a ρ-labeling otherwise.*

mod 4			Labeling of $C_r \cup C_s \cup C_t$	Reference
r	s	t		
0	0	0	σ^+ if $r = s = t = 4$ α otherwise	[11] [12]
0	0	1	γ (thus ρ)	[5]
0	0	2	ρ^{++}	[4]
0	0	3	σ	[14]
0	1	1	ρ	This paper
0	1	2	σ	[14]
0	1	3	σ	[14]
0	2	2	α	[12]
0	2	3	γ (thus ρ)	[7]
0	3	3	ρ	This paper
1	1	1	σ	[3]
1	1	2	σ	[3]
1	1	3	ρ	This paper
1	2	2	γ (Thus ρ)	[5]
1	2	3	ρ	This paper
1	3	3	σ	[3]
2	2	2	ρ^{++}	[4]
2	2	3	σ	[14]
2	3	3	σ	[3]
3	3	3	ρ	This paper

Table 1. Labelings of $C_r \cup C_s \cup C_t$, $r, s, t \geq 3$

For 2-regular graphs with more than two components, the following is known. In [15], Kotzig shows that if $r > 1$, then rC_3 does not admit a β-labeling. Similarly, he shows that rC_5 does not admit a β-labeling for any r. In [16], Kotzig shows that $3C_{4k+1}$ admits a β-labeling for all $k \geq 2$. From results in [8], it can be shown that rC_3 admits a ρ-labeling for all $r \geq 1$. The ρ-labeling in [8] can be modified to produce a σ-labeling of rC_3 when the parity condition is satisfied. In [12], Eshghi shows that $C_{2m} \cup C_{2n} \cup C_{2k}$ has an α-labeling for all m, n, and $k \geq 2$ with $m + n + k \equiv 0 \pmod 2$ except when $m = n = k = 2$. In [1], Abrham and Kotzig show that rC_4 has an α-labeling for all positive integers $r \neq 3$. In [9], it is shown that $3C_m$ and $4C_m$ admit σ-labelings if the parity condition is satisfied and ρ-labelings otherwise. An additional result follows by combining results from [11] and from [4].

Theorem 7. *Let G be a 2-regular bipartite graph of order n. Then G has a σ^+-labeling if $n \equiv 0 \pmod 4$ and a ρ^{++}-labeling if $n \equiv 2 \pmod 4$.*

A result of Hevia and Ruiz [14] proves very useful.

Theorem 8. *The disjoint union of a graph with a β-labeling, together with a collection of graphs with α-labelings, has a σ-labeling.*

When applied to 2-regular graphs and combined with the results of Abrham and Kotzig [2], Theorem 8 yields the following.

Corollary 9. *Let* $x \geq 0$ *and* $y \geq 1$ *be integers and let* $G_1 \in \{C_{4x+3}, C_{4x+3} \cup C_{4y+1}, C_{4x+1} \cup C_{4y+2}\}$*. If* G_2 *is a 2-regular bipartite graph of order* $0 \pmod 4$*, then* $G_1 \cup G_2$ *admits a* σ*-labeling.*

In [5], it is shown that if G admits an α-labeling and $j > 1$, then $G \cup C_{2j+1}$ admits a γ-labeling. Thus for example, both $C_{4x} \cup C_{4y} \cup C_{4z+1}$ and $C_{4x+1} \cup C_{4y+2} \cup C_{4z+2}$ admit γ-labelings. These results are generalized in [7], where it is shown that every 2-regular almost-bipartite graph $G \neq C_3 \cup (kC_4)$, $k \in \{0, 1\}$, has a γ-labeling.

3 Main results

Let r, s and t be positive integers ≥ 3 and let $G = C_r \cup C_s \cup C_t$. We shall show that G admits a ρ-labeling. If $r + s + t \equiv 0$ or $3 \pmod 4$, then G admits a σ-labeling (see [3]). Thus it suffices to show that G admits a ρ-labeling when $r + s + t \equiv 1$ or $2 \pmod 4$. Table 1 summarizes the results for labeling $C_r \cup C_s \cup C_t$.

Before proceeding, some additional definitions and notational conventions are necessary. We denote the path with consecutive vertices $a_1, a_2, \ldots, a_k$ by $(a_1, a_2, \ldots, a_k)$. By $(a_1, a_2, \ldots, a_k) + (b_1, b_2, \ldots, b_j)$, where $a_k = b_1$, we mean the path $(a_1, \ldots, a_k, b_2, \ldots, b_j)$.

To simplify our consideration of various labelings, we will sometimes consider graphs whose vertices are named by distinct nonnegative integers, which are also their labels.

Let a, b, and k be integers with $0 \leq a \leq b$ and $k > 0$. Set $d = b - a$. We define the path

$$P(a, k, b) = (a, a + k + 2d - 1, a + 1, a + k + 2d - 2, a + 2, \ldots, b - 1, b + k, b).$$

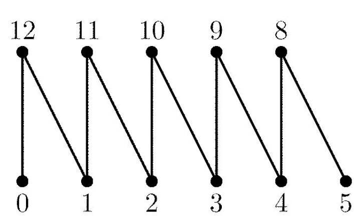

Figure 1. The path $P(0, 3, 5)$.

We note that the labeling of $P(a, k, b)$ is a translation of a k-graceful labeling of the path P_{2d+1} (as introduced in 1982 by Slater [20] and by Maheo and Thuillier [17]). It is easily checked that $P(a,k,b)$ is simple and

$$V(P(a,k,b)) = [a,b] \cup [b+k, b+k+d-1].$$

Furthermore, the edge labels of $P(a,k,b)$ are distinct and

$$\bar{E}(P(a,k,b)) = [k, k+2d-1].$$

These formulas will be used extensively in the proofs that follow.

As can be seen from Table 1, $G = C_r \cup C_s \cup C_t$ satisfies the parity condition in 10 of the 20 possible cases. We shall present the new results in four theorems, followed by our main theorem.

Theorem 10. *Let x, y, z be positive integers with $x \geq y$ and let $G = C_{4x+1} \cup C_{4y+1} \cup C_{4z}$. Then G has a ρ-labeling.*

Proof. The three cycles $G_1 = C_{4x+1}$, $G_2 = C_{4y+1}$, and $G_3 = C_{4z}$ are defined as follows:

$$\begin{aligned}
G_1 &= P(4x+4y+4z+1, 2x+4y+4z+4, 5x+4y+4z) \\
&+ P(5x+4y+4z, 4y+4z+3, 6x+4y+4z) \\
&+ (6x+4y+4z, 6x+4y+4z+1, 8x+8y+8z+4, 4x+4y+4z+1), \\
G_2 &= P(0, 2y+4z+2, y) + P(y, 4z+3, 2y-1) + (2y-1, 2y+1, 4y+4z+2, 0), \\
G_3 &= P(6x+4y+4z+2, 2z+4, 6x+4y+5z+1) + P(6x+4y+5z+1, 3, 6x+4y+6z+1) \\
&+ (6x+4y+6z+1, 6x+4y+8z+4, 6x+4y+4z+2).
\end{aligned}$$

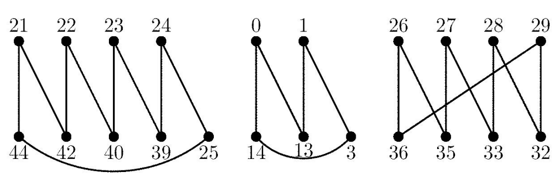

Figure 2. A ρ-labeling of $C_9 \cup C_5 \cup C_8$

Now we compute

$$\begin{aligned}
V(G_1) &= [4x+4y+4z+1, 6x+4y+4z] \cup [7x+8y+8z+4, 8x+8y+8z+2] \\
&\cup [6x+8y+8z+3, 7x+8y+8z+2] \cup \{6x+4y+4z+1, 8x+8y+8z+4\}, \\
V(G_2) &= [0, 2y-1] \cup [3y+4z+2, 4y+4z+1] \cup [2y+4z+2, 3y+4z] \\
&\cup \{2y+1, 4y+4z+2\}, \\
V(G_3) &= [6x+4y+4z+2, 6x+4y+6z+1] \cup [6x+4y+7z+5, 6x+4y+8z+3] \\
&\cup [6x+4y+6z+4, 6x+4y+7z+3] \cup \{6x+4y+8z+4\}.
\end{aligned}$$

We can order these as follows.

Cycle	Vertex Labels	Cycle	Vertex Labels
G_2	$[0, 2y-1]$	G_3	$[6x+4y+4z+2, 6x+4y+6z+1]$
G_2	$2y+1$	G_3	$[6x+4y+6z+4, 6x+4y+7z+3]$
G_2	$[2y+4z+2, 3y+4z]$	G_3	$[6x+4y+7z+5, 6x+4y+8z+3]$
G_2	$[3y+4z+2, 4y+4z+1]$	G_3	$6x+4y+8z+4$
G_2	$4y+4z+2$	G_1	$[6x+8y+8z+3, 7x+8y+8z+2]$
G_1	$[4x+4y+4z+1, 6x+4y+4z]$	G_1	$[7x+8y+8z+4, 8x+8y+8z+2]$
G_1	$6x+4y+4z+1$	G_1	$8x+8y+8z+4$

The vertices of the three cycles are distinct and contained in $[0, 8x+8y+8z+4]$. Note that if $x=1$, the set $[7x+8y+8z+4, 8x+8y+8z+2]$ will be empty. If in addition $y=1$, the set $[2y+4z+2, 3y+4z]$ is empty. Finally, if in addition $z=1$, then the set $[6x+4y+7z+5, 6x+4y+8z+3]$ is also be empty. This however does not change the proof.

Likewise we compute

$$\begin{aligned}
\overline{E}(G_1) &= [2x+4y+4z+4, 4x+4y+4z+1] \cup [4y+4z+3, 2x+4y+4z+2] \\
&\cup \{1, 2x+4y+4z+3, 4x+4y+4z+3\}, \\
\overline{E}(G_2) &= [2y+4z+2, 4y+4z+1] \cup [4z+3, 2y+4z] \cup \{2, 2y+4z+1, 4y+4z+2\}, \\
\overline{E}(G_3) &= [2z+4, 4z+1] \cup [3, 2z+2] \cup \{2z+3, 4z+2\}.
\end{aligned}$$

We can order these as follows

Cycle	Edge Labels	Cycle	Edge Labels
G_1	1	G_2	$2y+4z+1$
G_2	2	G_2	$[2y+4z+2, 4y+4z+1]$
G_3	$[3, 2z+2]$	G_2	$4y+4z+2$
G_3	$2z+3$	G_1	$[4y+4z+3, 2x+4y+4z+2]$
G_3	$[2z+4, 4z+1]$	G_1	$2x+4y+4z+3$
G_3	$4z+2$	G_1	$[2x+4y+4z+4, 4x+4y+4z+1]$
G_2	$[4z+3, 2y+4z]$	G_1	$4x+4y+4z+3$

Thus $\overline{E}(G) = [1, 4x+4y+4z+1] \cup \{4x+4y+4z+3\}$. Since $2(4x+4y+4z+2)+1-(4x+4y+4z+3) = 4x+4y+4z+2$, we have a ρ-labeling. If $x=1$ the set $[2x+4y+4z+4, 4x+4y+4z+1]$ is empty. If in addition $y=1$, then the set $[4z+3, 2y+4z]$ is empty. Finally, if in addition $z=1$, then $[2z+4, 4z+1]$ is also empty. This however does not change the proof. □

Theorem 11. *Let x, y, z be nonnegative integers with $x \geq y$ and $z \geq 1$ and let $G = C_{4x+3} \cup C_{4y+3} \cup C_{4z}$. Then G has a ρ-labeling.*

Proof. The three cycles $G_1 = C_{4x+3}$, $G_2 = C_{4y+3}$, and $G_3 = C_{4z}$ are defined as follows:

$$\begin{aligned}
G_1 &= P(4x+4y+4z+5, 2x+4y+4z+6, 5x+4y+4z+5) \\
&+ P(5x+4y+4z+5, 4y+4z+5, 6x+4y+4z+5) \\
&+ (6x+4y+4z+5, 6x+4y+4z+7, 8x+8y+8z+12, 4x+4y+4z+5), \\
G_2 &= P(0, 2y+4z+4, y) + P(y, 4z+3, 2y) + (2y, 2y+1, 4y+4z+4, 0), \\
G_3 &= P(6x+4y+4z+8, 2z+4, 6x+4y+5z+7) + P(6x+4y+5z+7, 3, 6x+4y+6z+7) \\
&+ (6x+4y+6z+7, 6x+4y+8z+10, 6x+4y+4z+8).
\end{aligned}$$

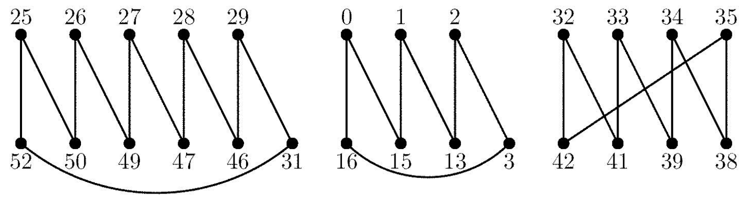

Figure 3. A ρ-labeling of $C_{11} \cup C_7 \cup C_8$

Now we compute

$$\begin{aligned}
V(G_1) &= [4x+4y+4z+5, 6x+4y+4z+5] \cup [7x+8y+8z+11, 8x+8y+8z+10] \\
&\cup [6x+8y+8z+10, 7x+8y+8z+9] \cup \{6x+4y+4z+7, 8x+8y+8z+12\}, \\
V(G_2) &= [0, 2y] \cup [3y+4z+4, 4y+4z+3] \cup [2y+4z+3, 3y+4z+2] \cup \{2y+1, 4y+4z+4\}, \\
V(G_3) &= [6x+4y+4z+8, 6x+4y+6z+7] \cup [6x+4y+7z+11, 6x+4y+8z+9] \\
&\cup [6x+4y+6z+10, 6x+4y+7z+9] \cup \{6x+4y+8z+10\}.
\end{aligned}$$

We can order these as follows.

Cycle	Vertex Labels	Cycle	Vertex Labels
G_2	$[0, 2y]$	G_3	$[6x+4y+4z+8, 6x+4y+6z+7]$
G_2	$2y+1$	G_3	$[6x+4y+6z+10, 6x+4y+7z+9]$
G_2	$[2y+4z+3, 3y+4z+2]$	G_3	$[6x+4y+7z+11, 6x+4y+8z+9]$
G_2	$[3y+4z+4, 4y+4z+3]$	G_3	$6x+4y+8z+10$
G_2	$4y+4z+4$	G_1	$[6x+8y+8z+10, 7x+8y+8z+9]$
G_1	$[4x+4y+4z+5, 6x+4y+4z+5]$	G_1	$[7x+8y+8z+11, 8x+8y+8z+10]$
G_1	$6x+4y+4z+7$	G_1	$8x+8y+8z+12$

The vertices of the three cycles are distinct and contained in $[0, 8x+8y+8z+12]$. Note that if $x = 0$, the sets $[6x+8y+8z+10, 7x+8y+8z+9]$ and $[7x+8y+8z+11, 8x+8y+8z+10]$ are empty. If in addition $y = 0$, the sets $[2y+4z+3, 3y+4z+2]$ and $[3y+4z+4, 4y+4z+3]$ are empty. Finally, if in addition $z = 1$, then the set $[6x+4y+7z+11, 6x+4y+8z+9]$ is also be empty. This however does not change the proof.

Likewise we compute

$$\begin{aligned}
\overline{E}(G_1) &= [2x+4y+4z+6, 4x+4y+4z+5] \cup [4y+4z+5, 2x+4y+4z+4] \\
&\cup \{2, 2x+4y+4z+5, 4x+4y+4z+7\}, \\
\overline{E}(G_2) &= [2y+4z+4, 4y+4z+3] \cup [4z+3, 2y+4z+2] \cup \{1, 2y+4z+3, 4y+4z+4\}, \\
\overline{E}(G_3) &= [2z+4, 4z+1] \cup [3, 2z+2] \cup \{2z+3, 4z+2\}.
\end{aligned}$$

We can order these as follows.

Cycle	Edge Labels	Cycle	Edge Labels
G_2	1	G_2	$2y+4z+3$
G_1	2	G_2	$[2y+4z+4, 4y+4z+3]$
G_3	$[3, 2z+2]$	G_2	$4y+4z+4$
G_3	$2z+3$	G_1	$[4y+4z+5, 2x+4y+4z+4]$
G_3	$[2z+4, 4z+1]$	G_1	$2x+4y+4z+5$
G_3	$4z+2$	G_1	$[2x+4y+4z+6, 4x+4y+4z+5]$
G_2	$[4z+3, 2y+4z+2]$	G_1	$4x+4y+4z+7$

Thus $\overline{E}(G) = [1, 4x+4y+4z+5] \cup \{4x+4y+4z+7\}$. Since $2(4x+4y+4z+6)+1-(4x+4y+4z+7) = 4x+4y+4z+6$ we have a ρ-labeling. If $x=0$ the sets $[4y+4z+5, 2x+4y+4z+4]$ and $[2x+4y+4z+6, 4x+4y+4z+5]$ are empty. If in addition $y=0$, then the sets $[4z+3, 2y+4z+2]$ and $[2y+4z+4, 4y+4z+3]$ are empty. Finally, if in addition $z=1$, then $[2z+4, 4z+1]$ is also empty. This however does not change the proof. □

Theorem 12. *Let x, y, z be nonnegative integers with $x \geq y \geq 1$ and let $G = C_{4x+1} \cup C_{4y+1} \cup C_{4z+3}$. Then G has a ρ-labeling.*

Proof. The three cycles $G_1 = C_{4x+1}$, $G_2 = C_{4y+1}$, and $G_3 = C_{4z+3}$ are defined as follows:

$$\begin{aligned}
G_1 &= P(4x+4y+4z+4, 2x+4y+4z+5, 5x+4y+4z+4) \\
&+ P(5x+4y+4z+4, 4y+4z+6, 6x+4y+4z+3) \\
&+ (6x+4y+4z+3, 6x+4y+4z+6, 8x+8y+8z+10, 4x+4y+4z+4), \\
G_2 &= P(0, 2y+4z+5, y) + P(y, 4z+6, 2y-1) + (2y-1, 2y+1, 4y+4z+5, 0), \\
G_3 &= P(6x+4y+4z+7, 2z+5, 6x+4y+5z+7) + P(6x+4y+5z+7, 4, 6x+4y+6z+7) \\
&+ (6x+4y+6z+7, 6x+4y+6z+8, 6x+4y+8z+12, 6x+4y+4z+7).
\end{aligned}$$

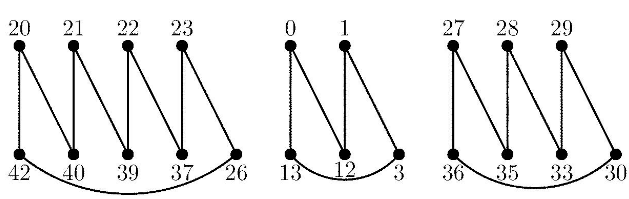

Figure 4. A ρ-labeling of $C_9 \cup C_5 \cup C_7$

Now we compute

$$\begin{aligned}
V(G_1) &= [4x+4y+4z+4, 6x+4y+4z+3] \cup [7x+8y+8z+9, 8x+8y+8z+8] \\
&\cup [6x+8y+8z+9, 7x+8y+8z+7] \cup \{6x+4y+4z+6, 8x+8y+8z+10\}, \\
V(G_2) &= [0, 2y-1] \cup [3y+4z+5, 4y+4z+4] \cup [2y+4z+5, 3y+4z+3] \\
&\cup \{2y+1, 4y+4z+5\}, \\
V(G_3) &= [6x+4y+4z+7, 6x+4y+6z+7] \cup [6x+4y+7z+12, 6x+4y+8z+11] \\
&\cup [6x+4y+6z+11, 6x+4y+7z+10] \cup \{6x+4y+6z+8, 6x+4y+8z+12\}.
\end{aligned}$$

We can order these as follows.

Cycle	Vertex Labels	Cycle	Vertex Labels
G_2	$[0, 2y-1]$	G_3	$6x+4y+6z+8$
G_2	$2y+1$	G_3	$[6x+4y+6z+11, 6x+4y+7z+10]$
G_2	$[2y+4z+5, 3y+4z+3]$	G_3	$[6x+4y+7z+12, 6x+4y+8z+11]$
G_2	$[3y+4z+5, 4y+4z+4]$	G_3	$[6x+4y+8z+12$
G_2	$4y+4z+5$	G_1	$[6x+8y+8z+9, 7x+8y+8z+7]$
G_1	$[4x+4y+4z+4, 6x+4y+4z+3]$	G_1	$[7x+8y+8z+9, 8x+8y+8z+8]$
G_1	$6x+4y+4z+6$	G_1	$8x+8y+8z+10$
G_3	$[6x+4y+4z+7, 6x+4y+6z+7]$		

The vertices of the three cycles are distinct and contained in $[0, 8x+8y+8z+10]$. Note that if $x=1$, the set $[6x+8y+8z+9, 7x+8y+8z+7]$ is empty. If in addition $y=1$, the set $[2y+4z+5, 3y+4z+3]$ is empty. Finally, if in addition $z=0$, then the sets $[6x+4y+6z+11, 6x+4y+7z+10]$ and $[6x+4y+7z+12, 6x+4y+8z+11]$ are also empty. This however does not change the proof.

Likewise we compute

$$\begin{aligned}
\overline{E}(G_1) &= [2x+4y+4z+5, 4x+4y+4z+4] \cup [4y+4z+6, 2x+4y+4z+3] \\
&\cup \{3, 2x+4y+4z+4, 4x+4y+4z+6\}, \\
\overline{E}(G_2) &= [2y+4z+5, 4y+4z+4] \cup [4z+6, 2y+4z+3] \cup \{2, 2y+4z+4, 4y+4z+5\}, \\
\overline{E}(G_3) &= [2z+5, 4z+4] \cup [4, 2z+3] \cup \{1, 2z+4, 4z+5\}.
\end{aligned}$$

We can order these as follows.

Cycle	Edge Labels	Cycle	Edge Labels
G_3	1	G_2	$2y+4z+4$
G_2	2	G_2	$[2y+4z+5, 4y+4z+4]$
G_1	3	G_2	$4y+4z+5$
G_3	$[4, 2z+3]$	G_1	$[4y+4z+6, 2x+4y+4z+3]$
G_3	$2z+4$	G_1	$2x+4y+4z+4$
G_3	$[2z+5, 4z+4]$	G_1	$[2x+4y+4z+5, 4x+4y+4z+4]$
G_3	$4z+5$	G_1	$4x+4y+4z+6$
G_2	$[4z+6, 2y+4z+3]$		

Thus $\overline{E}(G) = [1, 4x+4y+4z+4] \cup \{4x+4y+4z+6\}$. Since $2(4x+4y+4z+5)+1-(4x+4y+4z+6) = 4x+4y+4z+5$ we have a ρ-labeling. If $x=1$ the set $[4y+4z+6, 2x+4y+4z+3]$ is empty. If in addition $y=1$, then the set $[4z+6, 2y+4z+3]$ is empty. Finally, if in addition $z=0$, then the sets $[4, 2z+3]$ and $[2z+5, 4z+4]$ are also empty. This however does not change the proof. □

Theorem 13. *Let x, y, z be nonnegative integers with $x, z \geq 1$ and let $G = C_{4x+2} \cup C_{4y+3} \cup C_{4z+1}$. Then G has a ρ-labeling.*

Proof. The three cycles $G_1 = C_{4x+2}$, $G_2 = C_{4y+3}$, and $G_3 = C_{4z+1}$ are defined as follows:

$$\begin{aligned}
G_1 &= P(0, 2x+4y+4z+8, x-1) + P(x-1, 4y+4z+5, 2x) + (2x, 4x+4y+4z+7, 0),\\
G_2 &= P(4x+4y+4z+8, 2y+4z+4, 4x+5y+4z+8)\\
&+ P(4x+5y+4z+8, 4z+3, 4x+6y+4z+8)\\
&+ (4x+6y+4z+8, 4x+6y+4z+9, 4x+8y+8z+12, 4x+4y+4z+8),\\
G_3 &= P(2x+1, 2z+2, 2x+z+1) + P(2x+z+1, 3, 2x+2z)\\
&+ (2x+2z, 2x+2z+2, 2x+4z+3, 2x+1).
\end{aligned}$$

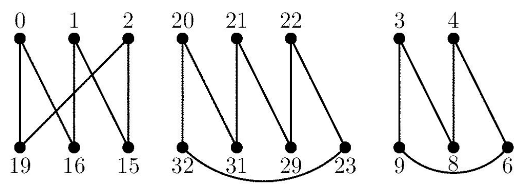

Figure 5. A ρ-labeling of $C_6 \cup C_7 \cup C_5$

Now we compute

$$\begin{aligned}
V(G_1) &= [0, 2x] \cup [3x+4y+4z+7, 4x+4y+4z+5] \cup [2x+4y+4z+5, 3x+4y+4z+5]\\
&\cup \{4x+4y+4z+7\},\\
V(G_2) &= [4x+4y+4z+8, 4x+6y+4z+8] \cup [4x+7y+8z+12, 4x+8y+8z+11]\\
&\cup [4x+6y+8z+11, 4x+7y+8z+10] \cup \{4x+6y+4z+9, 4x+8y+8z+12\},\\
V(G_3) &= [2x+1, 2x+2z] \cup [2x+3z+3, 2x+4z+2]\\
&\cup [2x+2z+3, 2x+3z+1] \cup \{2x+2z+2, 2x+4z+3\}.
\end{aligned}$$

We can order these as follows.

Cycle	Vertex Labels	Cycle	Vertex Labels
G_1	$[0, 2x]$	G_1	$[3x+4y+4z+7, 4x+4y+4z+5]$
G_3	$[2x+1, 2x+2z]$	G_1	$4x+4y+4z+7$
G_3	$2x+2z+2$	G_2	$[4x+4y+4z+8, 4x+6y+4z+8]$
G_3	$[2x+2z+3, 2x+3z+1]$	G_2	$4x+6y+4z+9$
G_3	$[2x+3z+3, 2x+4z+2]$	G_2	$[4x+6y+8z+11, 4x+7y+8z+10]$
G_3	$2x+4z+3$	G_2	$[4x+7y+8z+12, 4x+8y+8z+11]$
G_1	$[2x+4y+4z+5, 3x+4y+4z+5]$	G_2	$4x+8y+8z+12$

The vertices of the three cycles are distinct and contained in $[0, 8x+8y+8z+12]$. Note that if $x = 1$, the set $[3x+4y+4z+7, 4x+4y+4z+5]$ is empty. If in addition $y = 0$, then the sets $[4x+6y+8z+11, 4x+7y+8z+10]$ and $[4x+7y+8z+12, 4x+8y+8z+11]$ are empty. Finally, if $z = 1$, the set $[2x+2z+3, 2x+3z+1]$ is also empty. This however does not change the proof.

Likewise we compute

$$\begin{aligned}\overline{E}(G_1) &= [2x+4y+4z+8, 4x+4y+4z+5] \cup [4y+4z+5, 2x+4y+4z+6] \\ &\cup \{2x+4y+4z+7, 4x+4y+4z+7\}, \\ \overline{E}(G_2) &= [2y+4z+4, 4y+4z+3] \cup [4z+3, 2y+4z+2] \cup \{1, 2y+4z+3, 4y+4z+4\}, \\ \overline{E}(G_3) &= [2z+2, 4z+1] \cup [3, 2z] \cup \{2, 2z+1, 4z+2\}.\end{aligned}$$

We can order these as follows.

Cycle	Edge Labels	Cycle	Edge Labels
G_2	1	G_2	$2y+4z+3$
G_3	2	G_2	$[2y+4z+4, 4y+4z+3]$
G_3	$[3, 2z]$	G_2	$4y+4z+4$
G_3	$2z+1$	G_1	$[4y+4z+5, 2x+4y+4z+6]$
G_3	$[2z+2, 4z+1]$	G_1	$2x+4y+4z+7$
G_3	$4z+2$	G_1	$[2x+4y+4z+8, 4x+4y+4z+5]$
G_2	$[4z+3, 2y+4z+2]$	G_1	$4x+4y+4z+7$

Hence $\overline{E}(G) = [1, 4x+4y+4z+5] \cup \{4x+4y+4z+7\}$. Since $2(4x+4y+4z+6)+1-(4x+4y+4z+7) = 4x+4y+4z+6$ we have a ρ-labeling.

As with the vertex labels, note that if $x = 1$, then the set $[2x+4y+4z+8, 4x+4y+4z+5]$ is empty. If in addition $y = 0$, then the sets $[4z+3, 2y+4z+2]$ and $[2y+4z+4, 4y+4z+3]$ are empty. Finally, if $z = 1$, the set $[3, 2z]$ is empty. Neither condition would however change the proof. □

Theorem 14. *Let x, y, z be nonnegative integers with $x \le y \le z$ and let $G = C_{4x+3} \cup C_{4y+3} \cup C_{4z+3}$. Then G has a ρ-labeling.*

Proof. We will distinguish two cases according to whether $z = 0$ or $z \ge 1$.

Case 1: $z = 0$

If $z = 0$, then $x = y = 0$ as well, thus all we need is a ρ-labeling of the graph $C_3 \cup C_3 \cup C_3$, which is given in Figure 6.

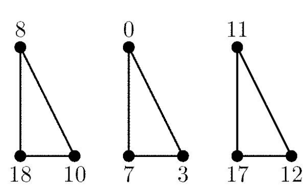

Figure 6. A ρ-labeling of $C_3 \cup C_3 \cup C_3$

Case 2: $z \ge 1$

The three cycles $G_1 = C_{4x+3}$, $G_2 = C_{4y+3}$, and $G_3 = C_{4z+3}$ are defined as follows:

$$\begin{aligned}
G_1 &= P(4x+4y+4z+8, 2x+4y+4z+9, 5x+4y+4z+8) \\
&+ P(5x+4y+4z+8, 4y+4z+8, 6x+4y+4z+8) \\
&+ (6x+4y+4z+8, 6x+4y+4z+10, 8x+8y+8z+18, 4x+4y+4z+8), \\
G_2 &= P(0, 2y+4z+7, y) + P(y, 4z+6, 2y) + (2y, 2y+1, 4y+4z+7, 0), \\
G_3 &= P(6x+4y+4z+11, 2z+3, 6x+4y+5z+12) \\
&+ P(6x+4y+5z+12, 4, 6x+4y+6z+11) \\
&+ (6x+4y+6z+11, 6x+4y+6z+14, 6x+4y+8z+16, 6x+4y+4z+11).
\end{aligned}$$

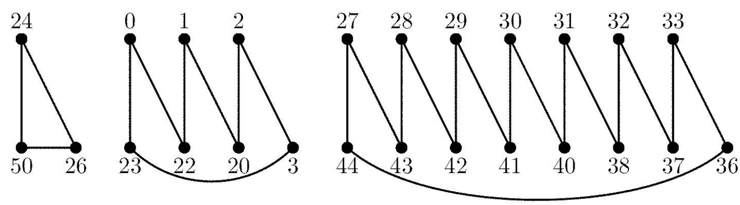

Figure 7. A ρ-labeling of $C_3 \cup C_7 \cup C_{15}$

Now we compute

$$\begin{aligned}
V(G_1) &= [4x+4y+4z+8, 6x+4y+4z+8] \cup [7x+8y+8z+17, 8x+8y+8z+16] \\
&\cup [6x+8y+8z+16, 7x+8y+8z+15] \cup \{6x+4y+4z+10, 8x+8y+8z+18\}, \\
V(G_2) &= [0, 2y] \cup [3y+4z+7, 4y+4z+6] \cup [2y+4z+6, 3y+4z+5] \\
&\cup \{2y+1, 4y+4z+7\}, \\
V(G_3) &= [6x+4y+4z+11, 6x+4y+6z+11] \cup [6x+4y+7z+15, 6x+4y+8z+15] \\
&\cup [6x+4y+6z+15, 6x+4y+7z+13] \cup \{6x+4y+6z+14, 6x+4y+8z+16\}.
\end{aligned}$$

We can order these as follows.

Cycle	Vertex Labels	Cycle	Vertex Labels
G_2	$[0, 2y]$	G_3	$6x+4y+6z+14$
G_2	$2y+1$	G_3	$[6x+4y+6z+15, 6x+4y+7z+13]$
G_2	$[2y+4z+6, 3y+4z+5]$	G_3	$[6x+4y+7z+15, 6x+4y+8z+15]$
G_2	$[3y+4z+7, 4y+4z+6]$	G_3	$6x+4y+8z+16$
G_2	$4y+4z+7$	G_1	$[6x+8y+8z+16, 7x+8y+8z+15]$
G_1	$[4x+4y+4z+8, 6x+4y+4z+8]$	G_1	$[7x+8y+8z+17, 8x+8y+8z+16]$
G_1	$6x+4y+4z+10$	G_1	$8x+8y+8z+18$
G_3	$[6x+4y+4z+11, 6x+4y+6z+11]$		

The vertices of the three cycles are distinct and contained in $[0, 8x+8y+8z+18]$. Note that if $x = 0$, the sets $[6x+8y+8z+16, 7x+8y+8z+15]$ and $[7x+8y+8z+17, 8x+8y+8z+16]$ are empty. If in addition $y = 0$, the sets $[2y+4z+6, 3y+4z+5]$ and $[3y+4z+7, 4y+4z+6]$ are empty. Finally, if in addition $z = 1$, then the set

$[6x+4y+6z+15, 6x+4y+7z+13]$ is also empty. This however does not change the proof.

Likewise we compute

$$\begin{aligned}
\overline{E}(G_1) &= [2x+4y+4z+9, 4x+4y+4z+8] \cup [4y+4z+8, 2x+4y+4z+7] \\
&\cup \{2, 2x+4y+4z+8, 4x+4y+4z+10\}, \\
\overline{E}(G_2) &= [2y+4z+7, 4y+4z+6][4z+6, 2y+4z+5] \cup \{1, 2y+4z+6, 4y+4z+7\}, \\
\overline{E}(G_3) &= [2z+3, 4z+4] \cup [4, 2z+1] \cup \{3, 2z+2, 4z+5\}.
\end{aligned}$$

We can order these as follows.

Cycle	Edge Labels	Cycle	Edge Labels
G_2	1	G_2	$2y+4z+6$
G_1	2	G_2	$[2y+4z+7, 4y+4z+6]$
G_3	3	G_2	$4y+4z+7$
G_3	$[4, 2z+1]$	G_1	$[4y+4z+8, 2x+4y+4z+7]$
G_3	$2z+2$	G_1	$2x+4y+4z+8$
G_3	$[2z+3, 4z+4]$	G_1	$[2x+4y+4z+9, 4x+4y+4z+8]$
G_3	$4z+5$	G_1	$4x+4y+4z+10$
G_2	$[4z+6, 2y+4z+5]$		

Thus $\overline{E}(G) = [1, 4x+4y+4z+8] \cup \{4x+4y+4z+10\}$. Since $2(4x+4y+4z+9)+1-(4x+4y+4z+10) = 4x+4y+4z+9$ we have a ρ-labeling. If $x=0$ the sets $[4y+4z+8, 2x+4y+4z+7]$ and $[2x+4y+4z+9, 4x+4y+4z+8]$ are empty. If in addition $y=0$, then the sets $[4z+6, 2y+4z+5]$ and $[2y+4z+7, 4y+4z+6]$ are empty. Finally, if in addition $z=1$, then the set $[4, 2z+1]$ is also empty. This however does not change the proof. □

We conclude this section with our main result and a corollary.

Theorem 15. *Let G be a 2-regular graph with at most three components. Then G admits a ρ-labeling.*

Proof. Let G have size n. If G has one or two components, then by Theorems 4, 5 and 6, G has a ρ-labeling (or a more restricted labeling). Now let $G = C_r \cup C_s \cup C_t$ (see Table 1). If the parity condition is satisfied, then G has a σ-labeling [3]. Now suppose $n \equiv 1$ or $2 \pmod 4$. If G is bipartite, then it has a ρ^{++}-labeling [4]. If G is almost-bipartite (i.e., if exactly one of r, s or t is odd), then it has a γ-labeling [5, 7]. The remaining five cases are covered by the previous five theorems. □

Using Theorems 1 and 15, we get the following.

Corollary 16. *Let G be a 2-regular graph of size n and at most three components. Then there exists a cyclic G-decomposition of K_{2n+1}.*

4 Concluding Remarks

The study of graph decompositions is a popular branch of modern combinatorial design theory (see [6] for an overview). In particular, the study of G-decompositions of K_{2n+1} (and of K_{2nx+1}) when G is a graph with n edges (and x is a positive integer) has attracted considerable attention. The study of graph labelings is also quite popular (see Gallian [13] for a dynamic survey). Theorems 1 and 2 provide powerful links between the two areas. Much of the attention on labelings has been on graceful labelings (i.e., β-labelings). Unfortunately, the parity condition "disqualifies" large classes of graphs from admitting graceful labelings.

In conclusion, we note that our results here, along with results from [3], [8] and [14] among others, provide further evidence in support of the following conjecture of El-Zanati and Vanden Eynden.

Conjecture 17. *Every* 2*-regular* G *graph of size* n *has a* ρ*-labeling. Moreover, if* $n \equiv 0$ *or* $3 \pmod 4$ *then* G *has a* σ*-labeling.*

As a final comment, we note that this work was done while the first, third, fourth and fifth authors were enrolled in an undergraduate research program at Illinois State University.

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