AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **37** (2007), Pages 265–269

Complete-factors and (g, f)-covered graphs

Sizhong Zhou*

School of Mathematics and Physics
Jiangsu University of Science and Technology
Zhenjiang, Jiangsu 212003
P.R. China
zsz_cumt@163.com

Xiuqian Xuan

School of Science
China University of Mining and Technology
Xuzhou, Jiangsu 221008
P.R. China

Abstract

A factor F of a graph is called a complete-factor if each component of F is complete. Let G be a graph, F be a complete-factor of G with $\omega(F) \geq 2$ and g, f be two integer-valued functions defined on $V(G)$. If $G - V(C)$ is a (g, f)-covered graph for each component C of F, then G is a (g, f)-covered graph.

1 Introduction

In this paper, we consider finite undirected graphs which may have loops and multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges, respectively. For $x \in V(G)$, we denote the degree of x in G by $d_G(x)$. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S,T)$ the number of edges joining S and T. We denote by $\omega(G)$ the number of components of a graph G. For a subset S of $V(G)$, we denote by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. Let g and f be integer-valued functions defined on $V(G)$. A (g, f)-factor of G is defined as a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for each $x \in V(G)$. And if $f(x) = g(x)$ for all $x \in V(G)$, then F is called an f-factor; if $f(x) = g(x) = r$ for all $x \in V(G)$, then F is called an r-factor. A graph G is called a (g, f)-covered graph if any edge e of G belongs to a (g, f)-factor of G. A graph G is called a (g, f)-k-covered

* Corresponding author.

graph if any k edges of G belong to a (g, f)-factor of G. Let $\mathcal{F}$ be a set of graphs. If each component H of F is isomorphic to some member of $\mathcal{F}$,then F is called an $\mathcal{F}$-factor. A $\{K_n|n \geq 1\}$-factor is called a complete-factor. The other terminologies and notations may be found in [1].

In [2], Katerinis proved the following result.

Theorem 1 *Let G be a graph of order at least two, and let r be a positive integer. If $G - x$ has a $2r$-factor for each $x \in V(G)$, then G itself has a $2r$-factor.*

In [3], Egawa et al. proved a similar result to Theorem 1.

Theorem 2 *Let G be a graph of order at least three, and let r be a positive integer. If $G - \{x, y\}$ has an r-factor for any pair of adjacent vertices x and y, then G itself has an r-factor.*

In [4], Wang et al. proved the following theorem and generalized Theorem 2.

Theorem 3 *Let G be a graph, and let F be a 1-factor of G. Let g and f be integer-valued functions defined on $V(G)$ such that $0 \leq g(x) < f(x) \leq d_G(x)$ for all $x \in V(G)$. If $f(x) = f(y)$ and $G - \{x, y\}$ has a (g, f)-factor for all $xy \in E(F)$, then G itself has a (g, f)-factor.*

Moreover, Saito [5] proved the following result.

Theorem 4 *Let G be a graph of order at least four, F be a 1-factor of G, and r be a positive integer. If $G - V(e)$ has an r-factor for each $e \in E(F)$, then G itself has an r-factor.*

Enomoto and Tokuda [6] proved the following result and generalized Theorem 1 and Theorem 4.

Theorem 5 *Let G be a graph, F be a complete-factor of G with $\omega(F) \geq 2$, and f be an integer-valued function defined on $V(G)$ with $\sum_{x \in V(G)} f(x)$ even. If $G - V(C)$ has an f-factor for each component C of F, then G itself has an f-factor.*

Li and Ma [7] proved the following theorem and extended Theorem 5 to (g, f)-factors.

Theorem 6 *Let G be a graph, and F be a complete-factor of G with $\omega(F) \geq 2$. Let g and f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ and $g(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$, and $f(V(G))$ even. If $G - V(C)$ has a (g, f)-factor for each component C of F, then G itself has a (g, f)-factor.*

In this paper, we prove the following result, which is an extension of Theorems 3 and 6. We extend Theorems 3 and 6 to (g, f)-covered graphs.

Theorem 7 *Let G be a graph, and F be a complete-factor of G with $\omega(F) \geq 2$. Let g and f be integer-valued functions defined on $V(G)$ such that $0 \leq g(x) < f(x)$ for all $x \in V(G)$. If $G - V(C)$ is a (g, f)-covered graph for each component C of F, then G itself is a (g, f)-covered graph.*

2 Proof of Theorem 7

Let G, F, g and f be as in Theorem 7, and we assume that $G - V(C)$ is a (g, f)-covered graph for each component C of F. In order to prove Theorem 7, we depend on the following lemma.

Lemma 2.1 [8] *Let G be a graph, and g and f be integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x)$ for all $x \in V(G)$. Then G is a (g, f)-covered graph if and only if*

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \ge \varepsilon(S, T)$$

for all $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S \quad \text{and} \quad d_{G-S}(x) < g(x)\}$, where $\varepsilon(S, T)$ is defined as follows:

(1) $\varepsilon(S, T) = 2$, if S is not independent.

(2) $\varepsilon(S, T) = 1$, if S is independent and $e_G(S, V(G) \setminus (S \cup T)) \ge 1$.

(3) $\varepsilon(S, T) = 0$, if neither (1) nor (2) holds.

By Lemma 2.1, to prove the theorem we only need to show that

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \ge \varepsilon(S, T)$$

for all $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S \quad \textit{and} \quad d_{G-S}(x) < g(x)\}$.

Let $U = V(G) - (S \cup T)$. For each component C of F, let $S_C = V(C) \cap S$, $T_C = V(C) \cap T$ and $U_C = V(C) \cap U$. Note that

$$\begin{aligned}
&d_{(G-V(C))-(S-V(C))}(T-V(C)) = d_{G-V(C)}(T-V(C)) - e_{G-V(C)}(T-V(C), S - V(C)) \\
&= d_G(T - V(C)) - e_G(V(C), T - V(C)) - e_G(T - V(C), S - V(C)) \\
&= d_{G-S}(T - V(C)) + e_G(T - V(C), S) - e_G(T - V(C), V(C)) \\
&\quad -e_G(T - V(C), S - V(C)) \\
&= d_{G-S}(T - V(C)) + e_G(T - V(C), S) - e_G(T - V(C), T_C \cup S_C \cup U_C) \\
&\quad -e_G(T - V(C), S - V(C)) \\
&= d_{G-S}(T - V(C)) + e_G(T - V(C), S) - e_G(T - V(C), T_C) - e_G(T - V(C), S_C) \\
&\quad -e_G(T - V(C), U_C) - e_G(T - V(C), S) + e_G(T - V(C), S_C) \\
&= d_{G-S}(T - V(C)) - e_G(T - V(C), T_C) - e_G(T - V(C), U_C)
\end{aligned}$$

for each component C of F. Since $G - V(C)$ is a (g, f)-covered graph for each component C of F,

$$\begin{aligned}
&\varepsilon(S - V(C), T - V(C)) \le \delta_{G-V(C)}(S - V(C), T - V(C)) \\
&= f(S - V(C)) + d_{(G-V(C))-(S-V(C))}(T - V(C)) - g(T - V(C)) \\
&= \delta_G(S, T) - f(S_C) - d_{G-S}(T_C) + g(T_C) - e_G(T - V(C), T_C) - e_G(T - V(C), U_C) \\
&\le \delta_G(S, T) - f(S_C) - d_{G-S}(T_C) + g(T_C).
\end{aligned}$$

Therefore,

$$\delta_G(S,T) \geq f(S_C) + d_{G-S}(T_C) - g(T_C) + \varepsilon(S - V(C), T - V(C)).$$

Here, let $C_1, C_2, \cdots, C_{\omega(F)}$ be the components of F. We divide this proof into three cases.

Case 1 If $xy \in E(G)$ for $x, y \in S$, then $\varepsilon(S,T) = 2$.

Subcase 1.1 Say $x, y \in S_{C_i}$ for some i.

Then $\varepsilon(S - V(C_i), T - V(C_i)) \geq 0$ and $\varepsilon(S - V(C_j), T - V(C_j)) = 2$ $(j \neq i)$. Hence

$$\begin{aligned}\sum_{j=1}^{\omega(F)} \delta_G(S,T) &\geq \sum_{j=1}^{\omega(F)} (f(S_{C_j}) + d_{G-S}(T_{C_j}) - g(T_{C_j}) + \varepsilon(S - V(C_j), T - V(C_j))) \\ &\geq \delta_G(S,T) + 2(\omega(F) - 1),\end{aligned}$$

that is, $(\omega(F) - 1)\delta_G(S,T) \geq 2(\omega(F) - 1)$. Hence $\delta_G(S,T) \geq 2 = \varepsilon(S,T)$.

Subcase 1.2 Say $x \in S_{C_i}, y \in S_{C_j}, j \neq i$.

Then $e_{G-V(C_i)}(S - V(C_i), V(G) - ((S - V(C_i)) \cup (T - V(C_i))) \geq 1$ and $e_{G-V(C_j)}(S - V(C_j), V(G) - ((S - V(C_j)) \cup (T - V(C_j))) \geq 1$.

Therefore, $\varepsilon(S - V(C_i), T - V(C_i)) \geq 1$ and $\varepsilon(S - V(C_j), T - V(C_j)) \geq 1$.

Moreover, $S - V(C_k)$ $(k \neq i, j)$ is not independent. Hence $\varepsilon(S - V(C_k), T - V(C_k)) = 2$ $(k \neq i, j)$. Therefore,

$$\begin{aligned}\sum_{j=1}^{\omega(F)} \delta_G(S,T) &\geq \sum_{j=1}^{\omega(F)} (f(S_{C_j}) + d_{G-S}(T_{C_j}) - g(T_{C_j}) + \varepsilon(S - V(C_j), T - V(C_j))) \\ &\geq \delta_G(S,T) + 2(\omega(F) - 2) + 1 + 1 \\ &= \delta_G(S,T) + 2(\omega(F) - 1),\end{aligned}$$

that is, $(\omega(F) - 1)\delta_G(S,T) \geq 2(\omega(F) - 1)$. Hence $\delta_G(S,T) \geq 2 = \varepsilon(S,T)$.

Case 2 S is independent, and $xy \in E(G)$ for $x \in S, y \in V(G) \setminus (S \cup T)$. Then $\varepsilon(S,T) = 1$.

We assume $x \in S_{C_i}$. Therefore, $\varepsilon(S - V(C_i), T - V(C_i)) \geq 0$ and $\varepsilon(S - V(C_j), T - V(C_j)) = 1$ $(j \neq i)$. Hence

$$\begin{aligned}\sum_{j=1}^{\omega(F)} \delta_G(S,T) &\geq \sum_{j=1}^{\omega(F)} (f(S_{C_j}) + d_{G-S}(T_{C_j}) - g(T_{C_j}) + \varepsilon(S - V(C_j), T - V(C_j))) \\ &\geq \delta_G(S,T) + (\omega(F) - 1),\end{aligned}$$

that is, $(\omega(F) - 1)\delta_G(S,T) \geq \omega(F) - 1$. Hence $\delta_G(S,T) \geq 1 = \varepsilon(S,T)$.

Case 3 Neither Case 1 nor Case 2 holds. Then $\varepsilon(S,T) = 0$, and $\varepsilon(S - V(C), T - V(C)) \geq 0$ for each component C of F. Hence

$$\begin{aligned}\sum_{j=1}^{\omega(F)} \delta_G(S,T) &\geq \sum_{j=1}^{\omega(F)} (f(S_{C_j}) + d_{G-S}(T_{C_j}) - g(T_{C_j}) + \varepsilon(S - V(C_j), T - V(C_j))) \\ &\geq \delta_G(S,T),\end{aligned}$$

that is, $(\omega(F)-1)\delta_G(S,T)\geq 0$. Hence $\delta_G(S,T)\geq 0=\varepsilon(S,T)$.

By Lemma 2.1, the theorem is proved.

Corollary 1 *Let G be a graph, and let F be a 1-factor of G. Let g and f be integer-valued functions defined on $V(G)$ such that $0\leq g(x)<f(x)$ for all $x\in V(G)$. If $G-\{x,y\}$ is a (g,f)-covered graph for each $xy\in E(F)$, then G itself is a (g,f)-covered graph.*

Finally, we pose a conjecture as the conclusion of this paper.

Conjecture 1 *Let G be a graph, and let F be a complete-factor of G with $\omega(F)\geq 2$. Let g and f be integer-valued functions defined on $V(G)$ such that $0\leq g(x)<f(x)$ for all $x\in V(G)$. If $G-V(C)$ is a (g,f)-k-covered graph for each component C of F, then G itself is a (g,f)-k-covered graph.*

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, New York, 1976.

[2] P. Katerinis, Some results on the existence of $2n$-factors in terms of vertex-deleted subgraphs, *Ars Combin.* 16B (1993), 271–277.

[3] Y. Egawa, H. Enomoto and A. Saito, Factors and induced subgraphs, *Discrete Math.* 68 (1998), 179–189.

[4] Ch.P. Wang, Ch.M. Ji, One-factors, f-factors and (g,f)-factors of graphs, *Mathematica Acta Scientia*, 19(1) (1999), 85–88.

[5] A. Saito, One-factors and $k-$factors, *Discrete Math.* 91 (1991), 323–326.

[6] H. Enomoto and T. Tokuda, Complete-factors and f-factors, Discrete Math. 220 (2000), 239–242.

[7] J.X. Li and Y.H. Ma, Complete-factors and (g,f)-factors, *Discrete Math.* 265 (2003), 385–391.

[8] G.Zh. Liu, $(g<f)$-factors of graphs, *Mathematica Acta Scientia* 14(3) (1994), 285–290.

(Received 19 Feb 2006; revised 22 Oct 2006)