Leaves, Excesses and Neighbourhoods in Triple Systems

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Abstract

A (partial) triple system of *order* v and *index* λ, denoted $TS(v,\lambda)$ ($PTS(v,\lambda)$, resp.) is a set V of v elements and a collection $\mathcal{B}$ of 3-element subsets of V called *triples*; each 2-subset of V appears in exactly (at most, respectively) λ triples of $\mathcal{B}$. A *covering by triples* $CT(v,\lambda)$ is a collection $\mathcal{B}$ of triples covering each pair on a v-set at least λ times. The *leave* of a partial triple system is the multiset of 2-subsets of V that contains $\{x,y\}$ s times whenever $\{x,y\}$ appears in precisely $\lambda - s$ triples. Usually, the pairs of the leave are taken as edges of a multigraph on v vertices. The *excess* of a covering similarly contains a pair s times if it appears $\lambda + s$ times in triples of the covering. The *neighbourhood* of an element x in a triple system is the multiset of pairs appearing in triples with the element x; these pairs are often taken as edges of a multigraph on $v-1$ vertices.

In this survey paper, known necessary and sufficient conditions for a multigraph to be the leave of a $PTS(v,\lambda)$, the excess of a $CT(v,\lambda)$, or to be a neighbourhood in a $TS(v,\lambda)$ are described.

1 Introduction

There has been a large amount of research activity in recent times on graph-theoretic problems in combinatorial design theory, and on the use of graph-theoretic tools in the construction and analysis of designs. In this survey, we examine three problems that concern the structure of triple systems, packings by triples (partial triple systems), and coverings by triples. In each of the problems, we exploit both graph-theoretic and design-theoretic constructions.

The basic theme of the three problems studied is to represent some structural property of the triple system as a graph or multigraph. Then questions about the structure can often be rephrased as questions about the corresponding multigraph.

We assume familiarity with standard graph-theoretic terminology (see Bondy and Murty (1976), for example), and with basic results in combinatorial design theory (see Street and Street (1987)). We employ a convenient notation to describe multigraphs, as follows. If G_1 and G_2 are multigraphs, then $G_1 \cup G_2$ is their disjoint union. For integer k, kG is a multigraph obtained from G by replacing each edge of G by k copies of the edge, and $k\{G\}$ is k vertex-disjoint copies of G (i.e., the disjoint union $G \cup G \cup \ldots \cup G$).

2 Leaves

A multigraph M on v vertices is a λ-*leave* if there exists a partial triple system of order v and index λ whose leave is M. For example, the 4-vertex multigraph $K_{1,3}$ with edge set $\{\{1,2\},\{1,3\},\{1,4\}\}$ is a 1-leave, but is not a 2-leave. In this section, we examine the question: Which multigraphs are λ-leaves?

The characterization of λ-leaves is far from complete, even in the case $\lambda = 1$. We first establish some basic necessary conditions, and then establish that recognizing λ-leaves is NP-hard. We then present the known sufficient conditions for a multigraph to be a λ-leave.

2.1 Necessary Conditions

Let $M = (V, E)$ be a multigraph with $|V| = v$ vertices and $|E| = m$ edges. When can M be a λ-leave?

The most basic necessary condition is that M have no edge of multiplicity exceeding λ, and we assume this throughout. Now if M is to be a λ-leave, there must be a partial triple system $(V, \mathcal{B})$ of index λ whose leave is M. Consider the following process. Starting with $G = \lambda K_v$ (the complete graph on v vertices with every edge having multiplicity λ), form a sequence of graphs by removing, for each triple of $\mathcal{B}$, the corresponding triangle of G. At the end of this process, G and M are identical. We derive an elementary necessary condition:

Lemma 2.1 *A λ-leave on v vertices and m edges has all vertex degrees congruent to $\lambda(v-1)$ (mod 2), and has $m \equiv \lambda\binom{v}{2}$ (mod 3).*

Proof: λK_v has all vertex degrees equal to $\lambda(v-1)$ and has $\lambda\binom{v}{2}$ edges. The removal of triangles leaves the vertex degrees the same modulo 2, and leaves the number of edges the same modulo 3. □

This basic lemma provides parity and congruence conditions. Not surprisingly, they are not sufficient. Consider the multigraph

$$C = 2\{K_3\} = \{\{1,2\},\{1,3\},\{2,3\},\{4,5\},\{4,6\},\{5,6\}\}$$

on the seven vertices $\{1,2,3,4,5,6,7\}$. It has six vertices of degree two, and one of degree zero; it has 6 ($\equiv \binom{7}{2} \pmod 3$) edges. But is it a 1-leave?

Consider the partition of the vertex set $V_1 = \{1,2,3\}$ and $V_2 = \{4,5,6,7\}$. Call an edge a *cross edge* if it contains one vertex from each class; otherwise call it an *inside edge*. Then C has no cross edges and six inside edges. Now suppose that C is a 1-leave of the partial triple system $(V,\mathcal{B})$. Each triple of $\mathcal{B}$ can contain pairs corresponding to three inside edges, or to one inside edge and two cross edges. In total, triples of $\mathcal{B}$ must account for twelve cross edges, and for the remaining three inside edges (those not in C). However, since the number of cross edges to deal with is more than twice the number of inside edges, there can be no such partial triple system $\mathcal{B}$.

Graham (see Nash-Williams (1970)) gives an elegant construction for an infinite family of graphs that meet the parity and congruence conditions, but are not leaves. We give it in slightly more general form here. Choose a positive index λ, and an order $v = 4t \equiv 0 \pmod 4$ for which $\lambda(t-1) \equiv 0 \pmod 2$ and $\lambda t \equiv 0 \pmod 3$. Let G be the multigraph on vertex set Z_{4t}, whose edges are those pairs of vertices $\{i,j\}$ with $i - j \equiv 2 \pmod 4$, λ times each. G meets the necessary conditions of Lemma 2.1. Now consider $\overline{G}$; it contains $4\lambda t^2$ *odd* edges (edges $\{i,j\}$ with $i-j \equiv 1 \pmod 2$), and $4\lambda\binom{t}{2}$ *even* edges (edges with $i-j \equiv 0 \pmod 2$). For G to be a λ-leave, $\overline{G} = \lambda K_{4t} \setminus G$ must have an edge-partition into triangles. But every triangle in $\overline{G}$ contains at least one even edge, and hence we require that $4\lambda t^2 \leq 2 \cdot 4\lambda\binom{t}{2}$ which holds only if $t < 1$.

In general, we obtain the necessary density condition:

Lemma 2.2 *Let M be a v-vertex multigraph with m edges. If the vertices of M can be partitioned into sets of size s and $v-s$ so that M has c cross edges, then M is a λ-leave only if*

$$2\left(\lambda\binom{s}{2} + \lambda\binom{v-s}{2} - m + c\right) \geq \lambda s(v-s) - c\ .$$

Proof: λK_v has $\lambda\binom{s}{2} + \lambda\binom{v-s}{2}$ inside edges and $\lambda s(v-s)$ cross edges; M accounts for c of the latter and $m-c$ of the former. The number of remaining inside edges must be at least half the number of remaining cross edges. □

Together with the parity and congruence conditions, this density condition is still not strong enough even to eliminate as candidates for λ-leaves all multigraphs whose complements (with respect to λK_v) have edges appearing in no triangles. An easy computation shows that the density condition cannot eliminate any graph having fewer than $\frac{\lambda}{4}\binom{v}{2} - \frac{3v}{8}$ edges; to see this, observe that if when $s = v - s = \frac{v}{2}$ and $c = 0$ the inequality of the lemma holds, then it holds for all $1 \leq s \leq v-1$ and all $0 \leq c \leq m$. So we may assume that $s = \frac{v}{2}$ and $c = 0$, and extract the inequality on m. Thus, for regular graphs, the density condition cannot be violated if the degree is less than $\lambda(\frac{v}{4} - 1)$. Graham's construction gives a regular graph of degree $\lambda(\frac{v}{4})$ that does violate the density condition (using the partition into odd and even vertices), and hence is not a 1-leave.

Stinson and Wallis (1987) describe infinite families of graphs meeting the parity, congruence and density conditions that cannot be realized as leaves for a simple reason: the complement, which must be partitioned into triangles, contains either an edge appearing in no triangle, or a pair of disjoint sets of vertices having no common neighbours, and an odd number of cross edges between the sets.

Colbourn and Mathon (1987) developed a stronger necessary condition that extends the density condition. A pair of subsets X, Y of a multigraph $M = (V, E)$ form a *fence* for M if X and Y are disjoint and no vertex $z \in V \setminus (X \cup Y)$ is adjacent to vertices both in X and in Y. The *fence-degree* of a vertex $z \in X \cup Y$, $fd(z)$, is the degree of the vertex z in the submultigraph of M induced on $X \cup Y$. The *X-defect* $def_X(X,Y)$ of the fence is the minimum number of edges in a subgraph H on X whose vertex degrees deg_H satisfy $deg_H(x) \equiv fd(x) \pmod 2$ for all $x \in X$. The Y-defect $def_Y(X,Y)$ is defined similarly, and the defect $def(X,Y)$ is the sum of the X-defect and the Y-defect.

Finally, for fence (X,Y), let $\epsilon(X,Y)$ equal the number of edges between vertices of X and vertices of Y, and $\iota(Z)$ be the number of edges inside a subset Z of vertices of M. Then we have:

Lemma 2.3 *Let (X,Y) be a fence of $\overline{M} = \lambda K_v - M$. M is a λ-leave only if $\epsilon(X,Y)$ is even, and*

$$\epsilon(X,Y) \geq 2(\iota(X) + \iota(Y) - def(X,Y)) .$$

Proof: Consider those edges having one endpoint in X and the other in Y. Since (X,Y) is a fence, any triangle containing such an edge has its third endpoint in X or in Y. Thus $\epsilon(X,Y)$ must be even. Moreover, for every two edges between X and Y, either an edge inside X or one inside Y is accounted for. Removing all such triangles leaves the fence-degrees unchanged modulo 2. Hence the number of edges inside X and Y that remain is at least $def(X,Y)$. □

This lemma includes the previous one: take $X \cup Y = V$, which always forms a fence. It further eliminates the trivial case for $\lambda = 1$ when $\overline{G}$ has an edge appearing in no triangle; for if $e = \{x,y\}$ is such an edge, the fence $(\{x\},\{y\})$ establishes the impossibility of G being a λ-leave.

It does not appear to be feasible to check these necessary conditions efficiently, nor are these conditions sufficient for a graph to be a λ-leave.

2.2 Recognizing λ-leaves

In this section, we examine the computational complexity of determining whether a given multigraph is a λ-leave. Colbourn (1983) has shown that the problem is

NP-complete, even when $\lambda = 1$.

We require some preliminaries in order to sketch the proof. Let $G = (V, E)$ be a simple undirected r-regular graph on n vertices. A *latin background* for G, $B(G; m, s)$, is an $s \times s$ array $L = (\ell_{ij})$ which is a symmetric square with symbols from $\{1, 2, \ldots, m\}$ (where $m \geq s$). Every diagonal position ℓ_{ii} contains the symbol m. In the first n rows, each cell of L is either empty or contains a symbol from $\{r + 1, \ldots, m\}$. In the latter $s - n$ rows, every cell is occupied by a symbol from $\{1, \ldots, m\}$. L is a partial latin square: every symbol appears at most once in each row and at most once in each column. Finally, the $n \times n$ matrix $A(L) = (a_{ij})$ defined by $a_{ij} = 1$ if ℓ_{ij} is an empty cell, and 0 otherwise, is an adjacency matrix of the specified graph G.

Given an arbitrary r-regular n-vertex graph $G = (V, E)$, our first task is to construct a latin background for G. We assume without loss of generality that $V = \{1, 2, \ldots, n\}$. We can form a background $B = (b_{ij})$ of size $m \geq 2n$ by setting $b_{ii} = m$ for all $1 \leq i \leq m$. For $1 \leq i < j \leq m$, if $\{i, j\} \in E$ we leave cells b_{ij} and b_{ji} empty; otherwise, we set $b_{ij} = b_{ji} = ((i + j) \bmod n) + n$. B is then a latin background $B(G; m, n)$.

Our next task is to extend the latin background L so produced to form a $B(G; m, m)$. To do this, Colbourn (1983) proves the following lemma using a "bordering" method of Cruse (1974). Let $P_L(i)$ be the number of occurrences of symbol i in L. Let $N_L(i) = P_L(i) + n$ for $1 \leq i \leq r$, and $N_L(i) = P_L(i)$ for $r + 1 \leq i \leq m$.

Lemma 2.4 *Let L be a $B(G; m, s)$ with m even, for G an r-regular n-vertex graph with n even. If for each $1 \leq i \leq m$, $N_L(i) \geq 2s - m$, then there is a $B(G; m, s + 1)$, L', for which $N_{L'}(i) \geq 2(s + 1) - m$ for all symbols $1 \leq i \leq m$.* $\square$

Now applying the lemma repeatedly to the initial latin background $B(G; m, n)$ with $m \geq 2n$ yields a latin background $B(G; m, m)$. Moreover, an explicit construction of the background can be accomplished by using standard bipartite matching algorithms to produce the required "bordering" in the lemma.

For such a latin background $B(G; m, m)$ $B = (b_{ij})$, we define an *idempotent latin background* $IB(G; m - 1, m - 1)$ to be an $m - 1 \times m - 1$ square $L = (\ell_{ij})$ obtained by setting $\ell_{ij} = \ell_{ji} = b_{ij}$ for $1 \leq i < j \leq m - 1$, and setting diagonal cells $\ell_{ii} = b_{im}$.

Now we are in a position to prove the main complexity result.

Theorem 2.5 *Deciding whether a graph is a 1-leave is NP-complete.*

Proof: Membership in NP follows from the observation that given a partial triple system, one can easily verify whether or not its leave is a specified graph.

Let G be a cubic graph on vertex set $\{1, \ldots, n\}$. To show that recognizing 1-leaves is NP-hard, we first construct an $IB(G; 2n - 1, 2n - 1)$ *and* an $IB(G; 2n + 1, 2n + 1)$ using the efficient techniques based on bordering. Let $v \in \{2n - 1, 2n + 1\}$ be the order of a Steiner triple system. We construct a partial triple system on $V = \{x_1, \ldots x_v\} \cup \{y_1, \ldots, y_v\} \cup \{z\}$. On the x_is, we place the triples of a Steiner triple system of order v. Next we include blocks $\{x_i, y_i, z\}$ for $1 \leq i \leq v$. Finally, whenever the (i, j) cell of the idempotent background $IB(G; v, v)$ is nonempty, containing the element k, we add a triple $\{y_i, y_j, x_k\}$. Idempotence ensures that $k \notin \{i, j\}$.

The leave R of the partial triple system so constructed contains all edges of the form $\{x_i, y_j\}$ with $1 \le i \le 3$ and $1 \le j \le n$, and also all edges of the form $\{y_i, y_j\}$ for $\{i, j\}$ an edge of G. Let $\overline{R}$ denote the complement of R with respect to the complete graph on V. $\overline{R}$ is a 1-leave if and only if R has an edge-partition into triangles. Now R has an edge-partition into triangles if and only if G is 3-edge-colourable. Holyer (1981) has shown that determining whether a cubic graph is 3-edge-colourable is NP-complete. Hence determining whether R has an edge-partition into triangles is NP-hard, and thus determining whether $\overline{R}$ is a 1-leave is NP-hard. □

It is important to remark that the proof establishes that for very dense graphs, it is apparently difficult to determine whether or not the graph is a 1-leave. The graphs employed have $4n - 1$ or $4n + 3$ vertices and have minimum degree $3n - 2$ or $3n + 2$. If the edge density is restricted to be "small", the complexity of recognizing 1-leaves is then open.

2.3 Maximal partial triple systems

In most investigations of partial triple systems, we are concerned with those partial triple systems to which no further triple can be added without forcing at least one pair to appear more than λ times. We call such partial triple systems *maximal*, and denote such a maximal partial triple system of order v and index λ by $MPT(v, \lambda)$. A partial triple system is maximal if and only if its leave is triangle-free; hence leaves of maximal systems have a structural constraint in addition to the parity, congruence and fence constraints given before. It remains open whether or not recognizing leaves of MPTs is NP-complete.

In this section, we examine the possible number of edges in a maximal partial triple system. A *maximum* $MPT(v, \lambda)$, or $MMPT(v, \lambda)$, is a maximal partial triple system of order v and index λ with the most triples of any $MPT(v, \lambda)$; a *minimum* $MPT(v, \lambda)$, or $mMPT(v, \lambda)$, is analogously one with the fewest triples.

2.3.1 Maximum partial triple systems

How many triples can a maximal partial triple system of order v and index λ have? Schönheim (1966,1969) and Spencer (1968) solved this problem for $\lambda = 1$, and Stanton and Rogers (1982) settled it in general.

A maximum partial triple system has the leave with the fewest edges; hence it is natural to examine what the possible leaves are in order to determine $MMPT(v, \lambda)$s. First observe that if $\lambda \ge gcd(v - 2, 6)$, we can do no better than to include a triple system of order v and index $gcd(v - 2, 6)$ and an $MMPT(v, \lambda - gcd(v - 2, 6))$. At the other extreme, if $\lambda = 0$, the $MMPT$ has no triples at all. Thus we only need to treat the cases with $0 < \lambda < gcd(v - 2, 6)$.

Consider the necessary conditions for a leave L. If $\lambda(v - 1) \equiv 0 \pmod 2$, L must have all vertex degrees even; otherwise L has all vertex degrees odd. Similarly, the number of edges of L must be congruent to $\lambda\binom{v}{2} \pmod 3$. Specifically, we have the following nontrivial cases:

λ	$v \pmod 6$ 0	2	4	5
1	odd degrees 0 (mod 3) edges	odd degrees 1 (mod 3) edges	odd degrees 0 (mod 3) edges	even degrees 1 (mod 3) edges
2		even degrees 2 (mod 3) edges		even degrees 2 (mod 3) edges
3		odd degrees 0 (mod 3) edges		
4		even degrees 1 (mod 3) edges		
5		odd degrees 2 (mod 3) edges		

When $v \equiv 0, 2 \pmod 6$ and $\lambda = 1$, the smallest possible leave has all vertices of degree 1. An $MMPT(v, 1)$ with such a leave is realized by taking a Steiner triple system of order $v+1$, eliminating all triples containing a fixed element x, and removing the element x.

When $v \equiv 5 \pmod 6$ and $\lambda = 1$, form a pairwise balanced design with one block of size five, and all other blocks of size three (see Street and Street (1987) for existence of these). Replace the block $\{u, w, x, y, z\}$ of size five by two triples, $\{u, w, z\}$ and $\{u, x, y\}$. The resulting partial triple system has a leave which is a 4-cycle (z, x, w, y). Since even degrees are required, a leave of one edge is impossible and hence four is the minimum.

For $v \equiv 4 \pmod 6$ and $\lambda = 1$, we proceed similarly and then eliminate the element x and all triples containing it. The resulting partial triple system has a leave consisting of $\frac{v-4}{2}$ disjoint edges and a 3-star $\{\{y, u\}, \{y, v\}, \{y, w\}\}$. Hence it has all degrees odd and has 0 (mod 3) edges. Since odd degrees are required, at least $\frac{v}{2}$ edges are needed in the leave, and since $\frac{v}{2} \equiv 2 \pmod 3$, $\frac{v-4}{2} + 3$ is indeed the minimum possible number of edges in the leave.

When $\lambda = 2$ and $v \equiv 5 \pmod 6$, form $MMPT(v, 1)$'s with leaves

$$\{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}\} \text{ and } \{\{a, b\}, \{b, d\}, \{c, d\}, \{a, c\}\}.$$

Take the union of these and add the triples $\{\{a, b, c\}, \{a, b, d\}\}$. The result has leave $\{\{c, d\}, \{c, d\}\}$ and is an $MMPT(v, 2)$.

When $\lambda = 2$ and $v \equiv 2 \pmod 6$, the smallest admissible leave is a pair of parallel edges $\{\{a, b\}, \{a, b\}\}$. To obtain an $MPT(v, 2)$ with such a leave, begin with a triple system of order $v + 1$ containing the triple $\{a, b, c\}$. Delete the triple $\{a, b, c\}$ and replace every triple of the form $\{c, x, y\}$ by the triple $\{b, x, y\}$. Then delete element c, and duplicate every triple not containing b. The result is an $MPT(v, 2)$ that leaves the pair $\{a, b\}$ uncovered, but covers each other pair exactly twice.

For the remaining cases with $v \equiv 2 \pmod 6$, we construct the minimum leave as follows. For $\lambda = 3$, take the union of the leaves for $\lambda = 1$ and $\lambda = 2$. For $\lambda = 4$, we take the union of the leaves of two $MMPT(v, 2)$'s. For $\lambda = 5$ we take the union of an $MMPT(v, 1)$ whose leave contains the edges $\{\{a, b\}, \{c, d\}\}$ and an $MMPT(v, 4)$

whose leave contains $\{\{a,c\},\{a,c\},\{b,c\},\{b,c\}\}$ and then add the triple $\{a,b,c\}$; the resulting leave consists of $\frac{v-4}{2}$ disjoint edges and a 3-star.

In summary, we have the following leave graphs of $MMPT(v,\lambda)$'s:

	$v \pmod 6$					
λ	0	1	2	3	4	5
0	$v\{K_1\}$	$v\{K_1\}$	$v\{K_1\}$	$v\{K_1\}$	$v\{K_1\}$	$v\{K_1\}$
1	$\frac{v}{2}\{K_2\}$	$v\{K_1\}$	$\frac{v}{2}\{K_2\}$	$v\{K_1\}$	$K_{1,3} \cup \frac{v-4}{2}\{K_2\}$	$(v-4)\{K_1\} \cup C_4$
2	$v\{K_1\}$	$v\{K_1\}$	$(v-2)\{K_1\} \cup 2K_2$	$v\{K_1\}$	$v\{K_1\}$	$(v-2)\{K_1\} \cup 2K_2$
3	$\frac{v}{2}\{K_2\}$	$v\{K_1\}$	$3K_2 \cup \frac{v-2}{2}\{K_2\}$	$v\{K_1\}$	$K_{1,3} \cup \frac{v-4}{2}\{K_2\}$	$v\{K_1\}$
4	$v\{K_1\}$	$v\{K_1\}$	$(v-3)\{K_1\} \cup 2K_{1,2}$	$v\{K_1\}$	$v\{K_1\}$	$(v-4)\{K_1\} \cup C_4$
5	$\frac{v}{2}\{K_2\}$	$v\{K_1\}$	$K_{1,3} \cup \frac{v-4}{2}\{K_2\}$	$v\{K_1\}$	$K_{1,3} \cup \frac{v-4}{2}\{K_2\}$	$(v-2)\{K_1\} \cup 2K_2$

The leaves of $MMPT(v,\lambda)$'s are not unique in general for $v \equiv 2 \pmod 3$.

2.3.2 Minimum maximal partial triple systems

At the other extreme, we ask: what is the minimum number of triples in a maximal partial triple system of order v and index λ? Equivalently, how many edges can a triangle-free λ-leave on v vertices have?

Novák (1974) determined this minimum in the case $\lambda = 1$.

A well known theorem in graph theory, Turán's theorem, ensures that the maximum number of edges in a triangle-free multigraph on v vertices with edge-multiplicity at most λ is achieved (uniquely) by $\lambda K_{\lfloor v/2\rfloor,\lceil v/2\rceil}$. The following simple lemma follows:

Lemma 2.6 *Let $v \geq 6$ and $\lambda \geq 1$. Let $v_1 = \lfloor v/2 \rfloor$ and $v_2 = \lceil v/2 \rceil$. Then if $\lambda \equiv 0$ mod $gcd(v_1 - 2, 6)$ and $\lambda \equiv 0$ mod $gcd(v_2 - 2, 6)$, the minimum $MPT(v,\lambda)$ has leave isomorphic to $\lambda K_{v_1,v_2}$.*

Proof: By Turán's theorem, we need only show that $\lambda K_{v_1,v_2}$ is a λ-leave under the conditions of the lemma. The congruence conditions on λ ensure that triple systems of index λ and orders v_1 and v_2 exist; taking the union of these on disjoint sets of elements gives the $PTS(v,\lambda)$ whose leave is $\lambda K_{v_1,v_2}$. □

In general, $\lambda K_{v_1,v_2}$ may have the wrong vertex degrees and may have the wrong number of edges modulo three. Nevertheless, all of the cases are handled by variations of the simplest case described. Novák (1974) shows that the following procedure constructs an $mMPT(v,1)$ for all v:

1. Let n_1 be the nearest integer to $\frac{v}{2}$ for which a $TS(n_1,1)$ exists. Let $n_2 = v - n_1$.
2. Let B_1 be the triples of a $TS(n_1,1)$ on a set X of n_1 elements.
3. Let B_2 be the triples of an $MMPT(n_2,1)$ on a set Y of n_2 elements disjoint from X.
4. Let B_3 be a set of triples obtained by adding, to each pair on Y in the leave of B_2, an element of X so that no pair with one element from X and one from Y is covered more than once.

5. Then $(X \cup Y, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is an $mMPT(v)$.

Novák's proof first determines the maximum number of edges in a triangle-free graph with all vertex degrees of the same parity as $n - 1$ (a candidate to be a leave), establishing in the process a bound on the maximum number of edges in a leave. Then the construction given is shown, by analysing the various cases modulo 12, to realize the minimum.

To state Novák's theorem explicitly:

Theorem 2.7 *A minimum maximal partial triple system of order v ($mMPT(v)$) has $(v^2 + d(v))/12$ triples, where*

$$
\begin{aligned}
d(v) &= -2v + 36 && \text{if } v \equiv 0 \text{ or } 8 \pmod{12}\\
&= -2v && \text{if } v \equiv 2 \text{ or } 6 \pmod{12}\\
&= -2v + 4 && \text{if } v \equiv 4 \pmod{12}\\
&= -2v + 16 && \text{if } v \equiv 10 \pmod{12}\\
&= -1 && \text{if } v \equiv 1 \text{ or } 5 \pmod{12}\\
&= 3 && \text{if } v \equiv 3 \pmod{12}\\
&= 11 && \text{if } v \equiv 7 \text{ or } 11 \pmod{12}\\
&= 15 && \text{if } v \equiv 9 \pmod{12}.
\end{aligned}
$$

2.3.3 The spectrum of maximal partial triple systems

At this point, we know the maximum and minimum possible numbers of edges in a λ-leave of a maximal partial triple system. What values between the maximum and the minimum can be realized?

Severn (1984) addressed this question for $\lambda = 1$, and obtained strong partial results; we explore these in this section. Let $m(v)$ denote the minimum number of triples in an $MPT(v)$ (as determined in Novák's theorem), and let $M(v)$ be the maximum number of triples. Severn proves the following:

Theorem 2.8 *For v odd, if $m(v) \leq t \leq M(v)$ then there exists an $MPT(v, 1)$ having exactly t triples* except *when $v \equiv 1, 3 \pmod 6$ and $t = M(v) - 1$.*

Theorem 2.9 *For v even, $m(v) \leq t \leq M(v)$, there is an $MPT(v, 1)$ having t triples*

1. *if $t \equiv \frac{v-2}{2} \bmod 2$; or*

2. *if $t \geq m(v) + s(v)$.*

Moreover, if $t \not\equiv \frac{v-2}{2} \pmod 2$ and $t \leq m(v) + h(v)$, no $MPT(v, 1)$ exists.

The functions $h(v)$ and $s(v)$ are determined by the congruence class of v modulo 12 as follows:

$v \pmod{12}$	$h(v)$	$s(v)$
0	$(v - 18)/6$	$(2v - 18)/6$
2	$(v + 4)/6$	$(2v + 2)/6$
4	$(v + 2)/6$	$(2v - 2)/6$
6	$v/6$	$(2v + 6)/6$
8	$(v - 14)/6$	$(2v - 10)/6$
10	$(v - 4)/6$	$(2v - 2)/6$

Severn conjectured that, if v is even, $t \not\equiv \frac{v-2}{2} \bmod 2$, and $m(v) + h(v) < t < m(v) + s(v)$, that there is *no* $MPT(v, 1)$ having exactly t triples. Very recently, Colbourn, Rosa and Znám (1991?) settled the remaining cases, showing that the answer is indeed negative, *except* when $v \equiv 8 \pmod{12}$ and the number of triples is $m(v) + s(v) - 2$.

2.4 Leaves of small degree

Until this point, for sufficiency we have been concerned primarily with the number of edges in a leave, and have not considered the structure of the leave. Short of characterizing λ-leaves, a natural objective is to identify strong sufficient conditions. We first recall the following conjecture of Nash-Williams (1970):

Conjecture 2.10 *Let G be a graph with maximum degree k that has $v \geq 4k + 2$ vertices, and that meets the parity condition on degrees and the congruence condition on number of edges modulo three. Then G is a 1-leave.*

The constants chosen are such that the density condition must be met, and Graham's construction shows that they cannot be reduced without violating the density condition.

A weaker formulation of Nash-Williams's conjecture is that for each k there exists an integer n_k such that if G has maximum degree k, has more than n_k vertices and meets the parity and congruence conditions, it is a 1-leave. Remarkably, little is known even about n_3, although Nash-Williams's conjecture would imply $n_3 \leq 13$.

Recently, Gustavsson (1991) has proved the weaker form of Nash-Williams's conjecture:

Theorem 2.11 *Every n-vertex m-edge graph G having all vertex degrees of the same parity as $n - 1$, $m \equiv 0 \pmod 3$, and minimum degree at least $n(1 - 10^{-24})$ has an edge-partition into triangles.*

In other words, $n_k \leq 10^{24}k$. Gustavsson's method employs a key lemma that every partial latin square of order n in which each symbol occurs at least $n(1 - 10^{-7})$ times has a completion to a latin square. The asymptotic result of Theorem 2.10 is a substantial breakthrough in determining leaves of small degrees, but for fewer than 10^{24} vertices it provides no information.

What can be said about small degree for small orders? Of course, characterizing leaves with maximum degree 1 is trivial; there remains the case of maximum degree two. Rogers (1985) first considered this case, showing that for order nine, every quadratic (2-regular) graph except $C_4 \cup C_5$ is a 1-leave. Colbourn and Rosa (1986) established the general theorem:

Theorem 2.12 *Let G be a graph on $v \equiv 1$ (mod 2) vertices with every vertex of degree 0 or 2, and having 0 (mod 3) edges if $v \equiv 1, 3$ (mod 6), and 1 (mod 3) edges if $v \equiv 5$ (mod 6). Then G is a 1-leave unless $v = 7$ and $G = 2\{C_3\} \cup K_1$, or $v = 9$ and $G = C_4 \cup C_5$.*

Colbourn and Rosa (1985) and Franek, Mathon and Rosa (1989) enumerate small partial triple systems with quadratic leaves.

2.5 Subgraphs of leaves

In view of the apparent difficulty in establishing strong sufficient conditions, one might ask whether certain subgraphs are forbidden from appearing in any leave. Colbourn (1987b) addresses this question, and shows in a strong sense that every graph meeting the basic necessary conditions appears as the subgraph of a leave. In particular, he shows:

Theorem 2.13 *There is a polynomial $p(n)$ such that, for any n-vertex graph G having 0 (mod 3) edges:*

- *if G has all vertex degrees even, there is a number $x \leq p(n)$ for which G together with x isolated vertices is a 1-leave; and*
- *if G has all vertex degrees odd, there are numbers $x \leq p(n)$ and $y \leq 2$ such that G, together with x disjoint edges and y disjoint 3-stars, is a 1-leave.*

This establishes that there are no forbidden induced subgraphs, or forbidden subgraphs, for leaves. Theorem 2.13 follows directly from Gustavsson's theorem (2.11), with $p(n) \leq 10^{24}n$.

2.6 Work Points

1. Is there an efficient algorithm for checking the density condition on leaves?
2. (Nash-Williams's conjecture) If $G = (V, E)$ is an n-vertex m-edge graph with all vertex degrees even, $m \equiv 0$ mod 3, and the minimum degree δ_G in G is at least $3\frac{n-1}{4}$, then G has an edge-partition into triangles (equivalently, if $\overline{G}$ has all vertices of degree congruent to $n-1$ mod 2, maximum degree $\Delta_{\overline{G}} \leq \frac{n-1}{4}$, and a number of edges congruent to $\binom{n}{2}$ mod 3, then $\overline{G}$ is a 1-leave.)
3. Is it NP-complete to determine whether a graph is a 1-leave of a *maximal* partial triple system?
4. Extend Novák's theorem to higher index; that is, determine the number of triples in an $mMPT(v, \lambda)$.
5. Extend the theorem of Severn (1984), as completed by Colbourn, Rosa and Znám (1991?), to higher index; that is, determine the spectrum of numbers of triples in an $MPT(v, \lambda)$.

3 Excesses

There is a natural duality between packing triples on a set so that every pair appears at most λ times, and covering the pairs by triples so that no pair appears in fewer than λ triples. A *covering by triples* of order v and index λ, or $CT(v, \lambda)$ is a set V

of v elements, and a collection (multiset) of 3-element subsets of V (*triples* as usual) for which every unordered pair of elements appears in *at least* λ of the triples.

The *excess* of a $CT(v, \lambda)$ is a multigraph whose vertex set is the element set of the covering, and in which each pair $\{x, y\}$ appears as an edge s times exactly when it appears in $\lambda + s$ triples of the covering.

A multigraph is a *λ-excess* if it is the excess of some $CT(v, \lambda)$.

A covering by triples $CT(v, \lambda)$ is *minimal*, and denoted $MCT(v, \lambda)$ if every triple contains at least one pair that occurs in only λ triples. A minimal covering with the fewest triples is *minimum*, and denoted $mMCT(v, \lambda)$; one with the most triples is *maximum*, and denoted $MMCT(v, \lambda)$.

3.1 Necessary Conditions

Let M be a v-vertex multigraph. When is M a λ-excess? We examine basic necessary conditions. As with leaves, we obtain parity conditions on degrees, and a congruence condition on the number of edges.

Lemma 3.1 *Let M be a v-vertex multigraph. If M is a λ-excess, all vertices of M have degree $\equiv \lambda(v-1)$ mod 2, and the number of edges in $M \equiv -\lambda\binom{v}{2}$ mod 3.*

Proof: If M is a λ-excess, the multigraph M' obtained by adding λ additional edges between every pair of vertices of M has an edge-partition into triangles. Thus M' has all vertex degrees even, and since $M = M' \setminus \lambda K_v$, M and λK_v have the same parity of vertex degrees, namely $\lambda(v-1)$ mod 2. By the same token, M' has 0 (mod 3) edges and hence the number of edges of M plus the number of edges of λK_v is 0 (mod 3), giving the congruence condition of the lemma. □

When $v \equiv 0, 1$ mod 3, the parity and congruence conditions for a graph to be a 1-excess are the same as those for a graph to be a 1-leave. Consider the graph $K_{n,n-2}$ with $n \equiv 3 \pmod 6$. It is a 1-leave — in fact, it is a 1-leave with the maximum number of edges when $2n - 2 = v \equiv 4 \pmod{12}$ (by Theorem 2.7). Is it a 1-excess?

Suppose that $K_{n,n-2}$ is a 1-excess of a $CT(2n-2, 1)$ $(V, \mathcal{B})$. Partition the elements of the covering into two classes of sizes n and $n-2$ using the bipartition of the putative excess $K_{n,n-2}$. Call an edge *inside* if it has its endpoints in the same class, *cross* otherwise. There are then $\binom{n}{2} + \binom{n-2}{2}$ inside edges of triples of $\mathcal{B}$, and since every cross edge appears in two triples, there are $2n(n-2)$ cross edges. Since any triple can use at most two cross edges and requires at least one inside edge, this requires that

$$\binom{n}{2} + \binom{n-2}{2} \geq n(n-2),$$

which in turn requires that $n \leq 3$.

This argument can be generalized along the same lines as for leaves:

Lemma 3.2 *Let M be a v-vertex multigraph. Suppose that the vertices of M can be partitioned into sets of sizes s and $v - s$ so that c of the m edges of M cross between* *the two classes of the partition. Then if M is a λ-excess,*

$$2\left[\lambda\binom{s}{2}+\binom{v-s}{2}+m-c\right]\geq \lambda s(v-s)+c.$$

Proof: The left hand side is twice the count of inside edges in the required covering, and the right hand side is the number of cross edges. □

A multigraph is a 0-excess if and only if it has an edge-partition into triangles. Holyer (1981a) shows that deciding whether a graph has an edge-partition into triangles is NP-complete, and hence:

Lemma 3.3 *Deciding whether a multigraph is a 0-excess is NP-complete.* □

3.2 Minimum Excesses

What is the fewest edges that a λ-excess can have? Fort and Hedlund (1958) settle this problem for $\lambda = 1$, and Engel (1981) settled it for all λ. We consider this problem in some detail here.

First we note that if $\lambda \geq$ gcd($v-2$,6), the minimum excess of a $CT(v,\lambda)$ is the same as that of a $CT(v,\lambda-gcd(v-2,6))$, since we can take the union of the covering with smaller index and a triple system of index gcd($v-2$,6) to obtain the covering with larger index. Moreover, a minimum excess of a $CT(v,0)$ is void (has no edges). It remains to treat the following nontrivial cases:

	v (mod 6)			
λ	0	2	4	5
1	odd degrees 0 (mod 3) edges	odd degrees 2 (mod 3) edges	odd degrees 0 (mod 3) edges	even degrees 2 (mod 3) edges
2		even degrees 1 (mod 3) edges		even degrees 1 (mod 3) edges
3		odd degrees 0 (mod 3) edges		
4		even degrees 2 (mod 3) edges		
5		odd degrees 1 (mod 3) edges		

First we consider the cases for index one. When $v \equiv 5 \pmod 6$, the minimum excess is a 2-cycle (two copies of an edge); to produce a covering with this excess, use the $CT(5,1)$ with triples $\{abc, abd, abe, cde\}$ having excess $\{\{a,b\},\{a,b\}\}$. Now a pairwise balanced design with one block of size five and all other blocks having size three exists for all $v \equiv 5 \pmod 6$; replacing the 5-block by a copy of the $CT(5,1)$ gives a $CT(v,1)$ whose excess is a 2-cycle.

When $v \equiv 2,4 \pmod 6$, take a triple system on $v-1$ elements with index one containing the triple $\{a,b,c\}$; add a new element ∞. Remove the triple $\{a,b,c\}$, and add triples $\{\infty,a,b\},\{\infty,a,c\},\{\infty,b,c\}$. Finally, for all triples $\{a,x,y\}$ with

$\{x, y\} \neq \{b, c\}$, add a triple $\{\infty, x, y\}$. The result is a $CT(v, 1)$ whose excess consists of $\frac{v-4}{2}$ disjoint edges and a 3-star, which is easily seen to be minimum.

When $v \equiv 0 \pmod 6$, we proceed in a similar way starting with a pairwise balanced design on $v - 1$ elements having a unique block of size 5, and triples otherwise. Let $\{a, b, c, d, e\}$ be the 5-block. Add an element ∞, and for all triples $\{a, x, y\}$ add a triple $\{\infty, x, y\}$. Then remove the 5-block, and on $\{\infty, a, b, c, d, e\}$ place the triples $\{\infty ab, \infty ac, bcd, bce, de\infty, dea\}$. The result has excess consisting of $\frac{v}{2}$ disjoint edges.

Now we turn to higher index. For index two and $v \equiv 5 \pmod 6$, the minimum number of edges in an excess is four; hence the union of two $CT(v, 1)$'s with minimum excess give a $CT(v, 2)$ with minimum excess (although not all $CT(v, 2)$'s with minimum excess are obtained in this way). For index two and $v \equiv 2 \pmod 6$, the minimum excess also has four edges. To produce a $CT(v, 2)$ whose excess has only four edges, begin with a $CT(v, 1)$ whose excess contains $\frac{v-4}{2}$ disjoint edges and the 3-star $\{\{a, b\}, \{a, c\}, \{a, d\}\}$. Take its union with a $MMPT(v, 1)$ whose *leave* contains the same $\frac{v-4}{2}$ edges and the two further edges $\{\{a, b\}, \{c, d\}\}$. This union has all edges appearing twice *except* that $\{a, c\}$ and $\{a, d\}$ appear three times, and $\{c, d\}$ appears only once. Adding the triple $\{a, c, d\}$ then gives a $CT(v, 2)$ with excess $\{\{a, c\}, \{a, c\}, \{a, d\}, \{a, d\}\}$.

For index three, $v \equiv 2 \pmod 6$, take the union of

1. a $CT(v, 1)$ whose excess is the $\frac{v-6}{2}$ edges E, the edge $\{e, f\}$, and a 3-star $\{\{a, b\}, \{a, c\}, \{a, d\}\}$,
2. an $MPT(v, 1)$ whose leave is E and the edges $\{\{a, d\}, \{b, c\}, \{e, f\}\}$, and
3. a $CT(v, 1)$ whose excess is E, the edge $\{b, c\}$ and the 3-star $\{\{a, d\}, \{a, e\}, \{a, f\}\}$.

The resulting $CT(v, 3)$ has excess E and $\{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{a, f\}\}$.

For index four and $v \equiv 2 \pmod 6$, take the union of

1. a $CT(v, 1)$ whose excess is $(n-4)/2$ edges F and the 3-star $\{\{a, b\}, \{a, c\}, \{a, d\}\}$;
2. a $CT(v, 1)$ whose excess is F and the 3-star $\{\{a, c\}, \{b, c\}, \{c, d\}\}$;
3. a $PTS(v, 1)$ whose leave is F and the edges $\{\{a, b\}, \{c, d\}\}$; and
4. a $PTS(v, 1)$ whose leave is F and the edges $\{\{a, d\}, \{b, c\}\}$.

The resulting $CT(v, 4)$ has excess $\{\{a, c\}, \{a, c\}\}$.

For index five and $v \equiv 2 \pmod 6$, we give a somewhat different construction. Partition the vertices into $s = \lfloor v/4 \rfloor$ classes $V_1, \ldots V_s$ with all classes of size four, except possibly the last which has size four or six. On each class V_i of size four, form a 1-factorization F_{0i}, F_{1i}, F_{2i}; and if the last class has size six, form instead five 1-factors $F_{0s}, \ldots, F_{4s}$. Now form five $MMPT(v, 1)$s $\mathcal{B}_0, \ldots, \mathcal{B}_4$ so that $\mathcal{B}_j$ has leave $\bigcup_{i=1}^{s} F_{ji}$, reducing the first subscript modulo three for the classes of size four. Then on each class of size four, place the triples of a $TS(4, 2)$, and if there is a class of size six, place the triples of an minimum covering $CT(6, 1)$. It is easily verified that the resulting $CT(v, 5)$ has a 1-factor as its excess.

In summary, we have the following excess graphs of $mMCT(v, \lambda)$'s:

λ	$v \pmod 6$					
	0	1	2	3	4	5
0	$v\{K_1\}$	$v\{K_1\}$	$v\{K_1\}$	$v\{K_1\}$	$v\{K_1\}$	$v\{K_1\}$
1	$\frac{v}{2}\{K_2\}$	$v\{K_1\}$	$K_{1,3} \cup \frac{v-4}{2}\{K_2\}$	$v\{K_1\}$	$K_{1,3} \cup \frac{v-4}{2}\{K_2\}$	$(v-2)\{K_1\} \cup 2K_2$
2	$v\{K_1\}$	$v\{K_1\}$	$(v-3)\{K_1\} \cup 2K_{1,2}$	$v\{K_1\}$	$v\{K_1\}$	$(v-3)\{K_1\} \cup 2K_{1,2}$
3	$\frac{v}{2}\{K_2\}$	$v\{K_1\}$	$K_{1,5} \cup \frac{v-6}{2}\{K_2\}$	$v\{K_1\}$	$K_{1,3} \cup \frac{v-4}{2}\{K_2\}$	$v\{K_1\}$
4	$v\{K_1\}$	$v\{K_1\}$	$(v-2)\{K_1\} \cup 2K_2$	$v\{K_1\}$	$v\{K_1\}$	$(v-2)\{K_1\} \cup 2K_2$
5	$\frac{v}{2}\{K_2\}$	$v\{K_1\}$	$\frac{v}{2}\{K_2\}$	$v\{K_1\}$	$K_{1,3} \cup \frac{v-4}{2}\{K_2\}$	$(v-3)\{K_1\} \cup 2K_{1,2}$

3.3 The spectrum for minimal coverings

How many triples can a minimal covering by triples contain? Mendelsohn and Assaf (1987) settle this question for index one (they call minimal coverings by triples *imbrical designs*).

First we determine the maximum possible number of triples. Every triple must contain a pair that appears only in that triple. Define a covering on element set $X \cup \{\infty\}$ with $|X| = v-1$, whose triple set $\mathcal{C}$ contains all triples of the form $\{\infty, x, y\}$ for $x, y \in X$. $\mathcal{C}$ has $\binom{v-1}{2}$ triples.

Lemma 3.4 *The maximum number of triples in an $MCT(v,1)$ is $\binom{v-1}{2}$.*

Proof: Let $\mathcal{B}$ be the triples of any minimal covering, with the maximum number of triples, on $X \cup \{\infty\}$, $|X| = v-1$. If $\mathcal{B} = \mathcal{C}$, the covering has $\binom{v-1}{2}$ triples and we are done. Otherwise, there is a triple in $\mathcal{B}$ not containing ∞, say $\{a, b, c\}$. Since $\mathcal{B}$ is minimal, at least one pair of $\{a, b, c\}$, say $\{a, b\}$, appears only in this triple of $\mathcal{B}$ (and hence there is no triple $\{\infty, a, b\}$ in $\mathcal{B}$ at present). Remove the triple $\{a, b, c\}$ and add the triple $\{\infty, a, b\}$ to form a set $\mathcal{B}'$ of triples. In general, $\mathcal{B}'$ need not be a covering, since $\{a, c\}$ and $\{b, c\}$ may not be covered; if either is not, add a triple containing it together with ∞ to obtain a larger minimal covering. Under the assumption that $\mathcal{C}$ is maximum, therefore, $\mathcal{B}'$ is a maximum minimal covering. $\mathcal{B}'$ shares one more triple containing ∞ with $\mathcal{C}$ than does $\mathcal{B}$. Hence repeating this argument, we transform $\mathcal{B}$ into $\mathcal{C}$, with the result that $\mathcal{B}$ has $\binom{v-1}{2}$ triples if it is maximum. □

To obtain the possible numbers of triples in a minimal covering, we consider a more general *augmentation* operation. Given a minimal covering $\mathcal{B}$ and a fixed element ∞ of the covering, suppose that $\mathcal{B}$ contains a triple $\{a, b, c\}$ not containing ∞. At least one of the pairs in $\{a, b, c\}$ is *essential* in that it appears only in this triple of $\mathcal{B}$ (since $\mathcal{B}$ is minimal); at most all three of the pairs in $\{a, b, c\}$ are essential. Removing the triple $\{a, b, c\}$ and adding triples with ∞ for each essential pair of $\{a, b, c\}$ results in a minimal covering that has zero, one or two more triples than the original. To determine the possible numbers of triples, it would therefore suffice to show that, for every possible number of triples less than the maximum, there is a minimal covering having a triple not containing ∞ and having two essential pairs; for if such a covering exists, augmentation on the specified triple increases the number of triples by one. More precisely, we have:

Lemma 3.5 *Let $\mathcal{B}$ be an $MCT(v,1)$ having b triples. Let ∞ be an element of the covering. Let $T = \{x,y,z\}$ be a triple of $\mathcal{B}$ not containing ∞, and having exactly two essential pairs, $\{x,z\}$ and $\{y,z\}$. Suppose further that $\{\infty,x,y\}$ is a triple of $\mathcal{B}$. Then for all $b \leq b' \leq \binom{v-1}{2}$, there is an $MCT(v,1)$ containing b' triples.*

Proof: If T is the only triple of $\mathcal{B}$ not containing ∞, $\mathcal{B}$ has $\binom{v-1}{2} - 1$ triples, and we apply augmentation to T. Otherwise, there is a triple $T' \neq T$, not containing ∞; applying augmentation to T' gives an $MCT(v,1)$ $\mathcal{B}'$ having b, $b+1$ or $b+2$ triples, and having at least one more triple containing ∞. Moreover, $\mathcal{B}'$ still contains T, and T still has two essential pairs (no augmentation makes $\{x,y\}$ essential, or makes $\{x,z\}$ or $\{y,z\}$ inessential). Then, applying the lemma repeatedly to $\mathcal{B}'$ handles (at least) the numbers of triples $b+2 \leq b' \leq \binom{v-1}{2}$. To obtain $b+1$ triples, we apply augmentation to T in $\mathcal{B}$; and to obtain b triples, we simply take $\mathcal{B}$. □

Theorem 3.6 *An $MCT(v,1)$ with b triples exists if and only if*

$$\lceil \frac{v}{3} \lceil \frac{v-1}{2} \rceil \rceil \leq b \leq \binom{v-1}{2},$$

except for $v = 7$, $b = 8$.

Proof: Necessity follows from the characterization of minimum coverings, and from Lemma 3.4 giving the upper bound. Now it is an easy exercise to verify that the minimum coverings for $\lambda = 1$ with $v \equiv 0,2,4,5 \pmod 6$ all have the property that some triple T has exactly two essential pairs. Let T' be a triple intersecting T in exactly two elements (some such T' exists since T has an inessential pair), and let ∞ be the element in $T' \setminus T$. Then apply Lemma 3.5 to obtain the result.

If $v \equiv 1,3 \pmod 6$, first we handle the cases for $b \geq \frac{v(v-1)}{6} + 2$. Take any triple system of order v and index 1, and replace some triple $\{x,y,z\}$ by $\{\{w,x,y\},\{w,x,z\},\{w,y,z\}\}$ ($w \notin \{x,y,z\}$ is chosen arbitrarily). Then applying Lemma 3.5 using $\infty = y$ and a triple containing $\{w,x\}$ and not containing y gives the required coverings.

For $v \equiv 1,3 \pmod 6$, the cases when $b = \frac{v(v-1)}{6}$ are realized by triple systems, so it remains only to treat the cases when $b = \frac{v(v-1)}{6} + 1$. Mendelsohn and Assaf (1987) call these coverings *failed designs*. The necessary conditions for the excess of such a covering specify that the excess is a 2-regular multigraph on 3 edges, for which there is only one choice — a triangle, say $\{\{x,y\},\{x,z\},\{y,z\}\}$, for which $\{x,y,z\}$ is not a triple.

Now if $v = 7$, there are four blocks through x, say $\{xy1, xy2, xz3, xz4\}$ (the four elements are distinct since every pair not in the excess occurs in only one triple). There are two additional blocks through y, say $\{yz\alpha, yz\beta\}$. But $\{\alpha,\beta\} \cap \{1,2,3,4\} = \emptyset$ since none of the pairs $\{\{y,1\},\{y,2\},\{z,3\},\{z,4\}\}$ are repeated. Hence at least nine elements are required, and no minimal covering on 7 elements with 8 triples exists.

Mendelsohn and Assaf (1987) produce $MCT(v,1)$'s for $v \in \{9,13,15\}$ having $b = \frac{v(v-1)}{6} + 1$ triples. Then since for all $w \geq 2v+1$, $w \equiv 1,3 \pmod 6$, there is a Steiner triple system of order w having a subsystem of order v (the Doyen-Wilson

(1972) theorem), we obtain the required $MCT(w, 1)$ by replacing the subsystem of order v by a failed design of order v. □

The extension to higher index appears not to have been studied.

3.4 Quadratic excesses

A complete characterization of excesses that is computationally tractable appears to be impossible. It is therefore of interest to ask for strong sufficient conditions for a multigraph to be an excess. In analogy with Nash-Williams's conjecture for leaves, Colbourn (1987a) made a similar conjecture for excesses:

Conjecture 3.7 *For every $d \geq 0$, there is an integer m_d for which any multigraph G of maximum degree d, meeting the parity and congruence conditions to be a 1-excess, is a 1-excess if G has more than m_d vertices.*

Little progress has been made on this conjecture. The cases $d \in \{0, 1\}$ are settled by the existence of minimum coverings. For maximum degree 2, any candidate to be an excess must have all vertex degrees equal to 0 or 2, and the candidate multigraph is *quadratic*. Colbourn and Rosa (1987) show that the necessary conditions for a quadratic multigraph to be a 1-excess are sufficient:

Theorem 3.8 *Every quadratic multigraph on $v \equiv 1 \pmod 2$ vertices, and having e edges with $e \equiv 0 \pmod 3$ when $v \equiv 1, 3 \pmod 6$ or with $e \equiv 2 \pmod 3$ when $v \equiv 5 \pmod 6$, is a 1-excess.*

The corresponding problem of determining which quadratic multigraphs are excesses of *minimal* coverings remains open.

3.5 Excesses and leaves: nuclear designs

The similarity between the necessary conditions for a multigraph to be a leave and to be an excess suggests that we may ask for a multigraph to be *both* a leave and an excess. If an n-vertex multigraph is both a λ-leave and a μ-excess, the parity and congruence conditions are equivalent to: $n \equiv 0, 1 \pmod 3$ or $\lambda + \mu \equiv 0 \pmod 3$, and $\lambda \equiv \mu \pmod 2$ when $n \equiv 0 \pmod 2$. These conditions are not sufficient: consider, for example, the 1-leave $K_{6s+1,6s+3}$ — we have seen that this is not a 1-excess.

Using techniques of Opencomb (1984), Colbourn, Hamm and Rosa (1985) established that:

Theorem 3.9 *Every 1-leave on an odd number of elements is a 41-excess.*

Proof: Let G be the 1-leave of an $MPT(v)$, $v \equiv 1 \pmod 2$. Write $v = 8s - t$, $t < 8$. Partition the elements of the MPT into eight sets $S_1, \ldots, S_8$, each of size s or $s - 1$. Now let $\mathcal{B}$ be the set of blocks of a Steiner quadruple system of order 8 on the elements $\{S_1, \ldots, S_8\}$. Each block of the quadruple system selects four of the sets; construct the PTS containing those triples with all three elements within the union of these four sets. The PTS has at most $4s$ elements, and thus by a result of Colbourn and Hamm (1986) has an embedding into a $TS(v,3)$. Repeating this for each of the fourteen blocks of the Steiner quadruple system, and taking the union of the resulting $TS(v,3)$'s gives a $TS(v,42)$. Then removing the triples of the $MPT(v,1)$ with which we started, we obtain a $CT(v,41)$ whose excess is G. $\square$

A similar technique shows that when the number of elements is even, every 1-leave is an 83-excess. Colbourn, Hamm and Rosa (1985) make a strong conjecture in this vein:

Conjecture 3.10 *A 1-leave on v elements is a μ-excess for*

$$\begin{array}{ll} \mu = 1 & v \in \{4,7,9,10\} \\ \mu = 2 & v \equiv 1 \ (\mathit{mod}\ 2),\ v \notin \{7,9\} \\ \mu = 3 & v \equiv 0,4 \ (\mathit{mod}\ 6),\ v \notin \{4,10\} \\ \mu = 5 & v \equiv 2 \ (\mathit{mod}\ 6) \end{array}$$

That these are lower bounds follows from considering the leaves of $mMPT(v,1)$'s.

Mendelsohn, Shalaby and Shen (1991?) consider a related problem. A *nuclear partial triple system* $NPTS(v,\lambda)$ is a $PTS(v,\lambda)$ $(V,\mathcal{B})$ with the property that there exist a $CT(v,\lambda)$ $(V,\mathcal{C})$ and a $PTS(v,\lambda)$ $(V,\mathcal{D})$ for which $\mathcal{B} = \mathcal{C} \cap \mathcal{D}$, and $|\mathcal{B}|$ is the largest possible such intersection. The special case when the nuclear design is the same as the PTS arises precisely when the λ-leave of the PTS is isomorphic to the λ-excess of the CT.

They establish that:

Theorem 3.11 *A nuclear $PTS(v,\lambda)$*

1. *is an $MMPT(v,\lambda)$ when $\lambda(v-1)$ is even; and*
2. *has $\lfloor v/6 \rfloor$ fewer triples than an $MMPT(v,\lambda)$ when $\lambda(v-1)$ is odd.*

3.6 Work Points

1. Is it NP-complete to decide whether a graph (or multigraph) is a λ-excess of a minimal covering of index λ?
2. Determine the necessary and sufficient conditions for a quadratic multigraph to be the excess of a *minimal* covering by triples of index one.
3. Prove (or disprove) Conjecture 3.7.
4. Characterize the graphs for which the density condition for 1-leaves and the density condition for 1-excesses hold simultaneously (and thereby determine a necessary condition for a graph to be both a 1-leave and a 1-excess).
5. Prove or disprove Conjecture 3.10.

4 Neighbourhoods

In this section, we examine another graph-theoretic problem on triple systems. Let $(V, \mathcal{B})$ be a $TS(v, \lambda)$, and let $x \in V$. Consider the multiset of triples

$$\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}.$$

The *neighbourhood* of x, $N_x(\mathcal{B})$, is the *multiset* of pairs $\{\{y, z\} : \{x, y, z\} \in \mathcal{B}_x\}$. The neighbourhood contains the $v - 1$ elements of $V \setminus \{x\}$.

We view the neighbourhood of an element x as a multigraph on the remaining $v - 1$ elements, whose edges are the pairs appearing in triples with x. In general, a multigraph is a *λ-neighbourhood* if it is the neighbourhood of some element in a triple system of index λ. The basic question we address here is: what multigraphs are neighbourhoods?

This neighbourhood problem is a special case of the characterization problem for λ-leaves:

Lemma 4.1 *G is a λ-neighbourhood if and only if G is a λ-regular λ-leave.*

Proof: Suppose that G is the neighbourhood of x in a $TS(v, \lambda)$ $(V, \mathcal{B})$. Consider the partial triple system $(V \setminus \{x\}, \mathcal{B} \setminus \mathcal{B}_x)$; its leave is G.

In the other direction, suppose that $G = (X, E)$ is λ-regular and is the leave of a partial triple system $(X, \mathcal{B})$ of index λ; choose an element $x \notin X$, and let $\mathcal{C}$ be all triples of the form $\{\{x, y, z\} : \{y, z\} \in E\}$. Since G is λ-regular, $\mathcal{C}$ contains every pair of the form $\{x, y\}$ with $y \in X$ λ times; moreover, any triple system of index λ containing the triples of $\mathcal{C}$ has G as the neighbourhood of x. To complete the proof, observe that $(X \cup \{x\}, \mathcal{B} \cup \mathcal{C})$ is a $TS(|X| + 1, \lambda)$. □

Interest in neighbourhoods stems primarily from the effort to explore the structure of triple systems. Moreover, the structure of one neighbourhood can often be used to determine properties of the triple system itself. As an example, if the neighbourhood of some element is a multigraph having no regular factor, the triple system is indecomposable.

First we examine the necessary conditions. Suppose that G is a λ-regular multigraph on n vertices. If G is to be a neighbourhood, some $TS(n + 1, \lambda)$ must exist and hence $\lambda \equiv 0 \bmod(\gcd(n - 1, 6))$; this is *equivalent* to the parity and congruence conditions for G to be a λ-leave (Lemma 2.1).

Next we consider the conditions for a λ-regular multigraph to meet the density conditions to be a λ-leave (Lemma 2.2):

Lemma 4.2 *If G is a λ-regular multigraph on $n \geq 8$ vertices, and a $TS(n + 1, \lambda)$ exists, then G meets the density condition to be a λ-leave.*

Proof: Consider any partition of the n vertices of G into classes of sizes s and $n - s$. If the density condition fails, it must fail when all edges of G are inside the classes, and hence we need only check that

$$2\lambda \left(\binom{s}{2} + \binom{n-s}{2} - \frac{n}{2} \right) \geq \lambda s(n - s).$$

Simplifying, we require $3s^2 - 3ns + n^2 - 2n \geq 0$, which holds for all $n \geq 8$, independent of s. $\square$

It is easy to verify that the density condition cannot exclude any multigraph on fewer than four vertices. Thus the density condition only excludes some multigraphs meeting the parity and congruence conditions on 4, 5, 6 and 7 vertices. The only simple graph excluded is the 2-regular 6-vertex graph containing two disjoint triangles.

Colbourn (1987a) made the following conjecture:

Conjecture 4.3 *Let G be a λ-regular n-vertex multigraph, and let $\lambda \equiv 0 \bmod gcd(n-1, 6)$. Then if G meets the density condition to be a λ-leave, G is a λ-neighbourhood.*

Unlike the analogous problems of characterizing leaves and excesses, progress on proving this conjecture has been substantial, as we shall see in the remainder of this section.

4.1 Index Two

The first nontrivial case of the neighbourhood conjecture is for index two. The density condition rules out only two quadratic multigraphs, namely the disjoint union of a triangle and a double edge, and the disjoint union of two triangles.

Colbourn and Rosa (1987a) show that the neighbourhood conjecture holds for index two:

Theorem 4.4 *Let G be a 2-regular multigraph on $n \equiv 0, 2 \pmod 3$ vertices other than $2K_2 \cup K_3$ and $K_3 \cup K_3$. Then G is a neighbourhood in a $TS(n+1, 2)$.*

The strategy of the proof is quite similar to the quadratic leaves theorem, Theorem 2.12. We outline the construction for the case $n \equiv 0, 2 \pmod 6$, the other case being similar. First we describe the construction when the quadratic graph G contains a 2-cycle. Write $n = 2t+2$, and let Q' be the $(2t)$-vertex graph obtained by removing the vertices of one 2-cycle from G. We construct a $TS(2t+3, 2)$ on $\{\infty, A, B\} \cup (Z_t \times \{0, 1\})$ in which N_∞ is isomorphic to G. Without loss of generality, suppose that Q' consists of e even length cycles $C_1, \ldots, C_e$ of lengths $2c_1, \ldots, 2c_s$, respectively, and $2d$ odd length cycles: $L_1, \ldots, L_d$ of lengths $2\ell_1 - 1, \ldots, 2\ell_t - 1$ and $R_1, \ldots, R_d$ of lengths $2r_1 + 1, \ldots, 2r_t + 1$. Let $a_j = \sum_{i=1}^{j-1} c_i$, and $b_j = a_e + \sum_{i=1}^{j-1} (\ell_i + r_i)$.

We then label the vertices of Q' so that each even length cycle C_j is the cycle $((a_j)_0, (a_j+1)_0, \ldots, (a_j+c_j)_0, (a_j+c_j)_1, (a_j+c_j-1)_1, \ldots, (a_j+1)_1, (a_j)_1)$. (When $c_j = 2$, this is a 2-cycle on $\{(a_j)_0, (a_j)_1\}$.) Then odd length cycles are mapped in pairs, L_j and R_j, as follows. L_j is mapped onto $((b_j)_0, (b_j+1)_0, \ldots, (b_j+\ell_j)_0, (b_j+\ell_j-1)_1, (b_j+\ell_j-2)_1, \ldots, (b_j)_1)$; for $k = \ell_j + r_j$, R_j is then mapped onto $(k_0, (k-1)_0, \ldots (k-r_j)_0, (k-r_j-1)_1, (k-r_j)_1, \ldots k_1)$. Having labelled vertices in this way, we take $N_\infty = Q' \cup \{\{A, B\}, \{A, B\}\}$.

Now we choose neighbourhoods for A and B. Form a collection of edges E on $Z_t \times \{0, 1\}$ by taking each edge of the form $\{i_0, i_1\}$ twice, and edges of the form $\{i_0, (i+1)_0\}$, $\{i_1, (i+1)_1\}$ and $\{i_1, (i+1)_0\}$ one each; then remove from this collection the edges in the mapping of Q'. What remains is a 3-regular multigraph R. Now R

can be factored into a 1-factor and a 2-factor; to get the 1-factor, take all edges of the form $\{i_1, (i+1)_0\}$ that remain in R, along with edges of the form $\{i_j, (i+1)_j\}$ required to complete the 1-factor F. Now let the neighbourhood of A, N_A, be $R \setminus F$ together with a 2-cycle on $\{\infty, B\}$. Now N_B is to contain the edges of a 1-factor F, and the edges of an arbitrary 1-factor on $Z_t \times \{0,1\}$ containing no edge of E.

The key observation to make at this point is that no matter what cycle lengths Q' contains, it remains only to partition all edges on the *fixed* multigraph M containing all edges on $Z_t \times \{0,1\}$ *less* those in E into one one-factor F' and a collection of triangles. The neighbourhood of B then contains $F \cup F'$, and the triangles complete the required $TS(2t+3, 2)$. The required partition of M is a problem in mixed differences, since M has the automorphism $i_j \mapsto (i+1)_j$; Colbourn and Rosa (1987a) give solutions for the difference problem.

To complete the case when $n \equiv 0, 2 \pmod 6$, it remains to handle the case when G has no 2-cycle. Suppose that G contains an h-cycle with $h \geq 4$; form G' by replacing the h-cycle by a 2-cycle $\{A, B\}$ and an $(h-2)$-cycle. Now proceed as before, ensuring that some edge of the $(h-2)$-cycle is $\{i_0, i_1\}$. Then a *pullup* of the resulting $TS(2t+3, 2)$ is the replacement of the triples $\{\infty, A, B\}$, $\{\infty, i_0, i_1\}$, $\{A, i_0, (i+1)_0\}$ and $\{B, i_1, (i+1)_0\}$ by the triples $\{\infty, A, i_0\}$, $\{\infty, B, i_1\}$, $\{A, B, i_1\}$ and $\{i_0, i_1, (i+1)_0\}$. After pullup, the neighbourhood of ∞ is isomorphic to G.

The final case arises when G consists of $\frac{n}{3}$ disjoint triangles, and this is easily handled using pairwise balanced designs with the single exception when $G = 2\{K_3\}$.

When $n \equiv 3, 5 \pmod 6$, the strategy is quite similar; as a result of the change in parity, we remove a triangle from G (when one is present), and proceed as above to map the remaining cycles. Pullups can be used in a very similar way to handle the case when the shortest odd cycle (of which there must be at least one) has length at least five.

Theorem 4.4 has an important corollary for larger index:

Corollary 4.5 *Let G be a λ-regular multigraph with λ even, on $n \equiv 0, 2 \pmod 3$ vertices. Then G is a neighbourhood in a $TS(n+1, \lambda)$.*

Proof: Let $s = \frac{\lambda}{2}$. By Petersen's theorem, G has a 2-factorization into s 2-factors $G_1, \ldots, G_s$. By the theorem, each is the neighbourhood of an element in a $TS(n+1, 2)$; their union is a $TS(n+1, \lambda)$ with an element having neighbourhood G. □

4.2 Index Three

For index three, there are three small cubic multigraphs that are eliminated by the density conditions, namely: the disjoint union of two triply repeated edges, a 4-cycle with two opposite edges duplicated, and the unique 3-regular 6-vertex multigraph with connectivity one.

Colbourn and McKay (1987) showed:

Theorem 4.6 *Every n-vertex cubic multigraph meeting the density condition is a neighbourhood in a $TS(n+1, 3)$.*

Their proof relies on partitioning cubic multigraphs into almost regular factors. A *nontrivial path factor* in a graph is a spanning subgraph in which each component is a path of length at least one. We outline the construction for the easiest case, when $n \equiv 0, 2 \pmod 6$.

Lemma 4.7 *Let $G = (V, E)$ be a cubic multigraph on $n \equiv 0, 2 \pmod 6$ vertices. Then if $n \geq 8$, G is a 3-neighbourhood.*

Proof: Colbourn and McKay show that every cubic multigraph G has a nontrivial path factor H containing at least one path of length one $\{x, y\}$. Let H' be the complement of H with respect to G; H' has maximum degree two. Form Q' by adding an arbitrary matching on the odd degree vertices of H'. Q' may be a multigraph. Now using the solution for index two, form a $PTS(n+1, 2)$ $\mathcal{B}_1$ in which the neighbourhood of the new element, ∞, is Q'.

Form Q from $H - \{x, y\}$ by adding a matching on the odd degree vertices of H, so that every cycle formed has even length (which is possible since $n-2$ is even). Q is a 2-regular simple graph on $V \setminus \{x, y\}$; in addition, Q has a 1-factorization F_1, F_2 since every cycle length is even. Using the quadratic leaves theorem (Theorem 2.12), form a $PTS(n-1, 1)$ with triples $\mathcal{D}$ on $V \setminus \{x\}$ whose leave is Q; form a collection of triples $\mathcal{B}_2$ as the union of $\mathcal{D}$, $\{\{x, y, z\} : \{y, z\} \in F_1\}$, $\{\{\infty, y, z\} : \{y, z\} \in F_2\}$, and the triple $\{\infty, x, y\}$.

The union $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ has $F_2 \cup Q'$ as the neighbourhood of ∞, but the important fact is that all edges of G that are *not* at present in the neighbourhood of ∞ are in the neighbourhood of x. Let $\mathcal{I}$ be the edges in the neighbourhood of ∞ but not in G, and $\mathcal{O}$ be the edges in G but not in the neighbourhood of ∞. $\mathcal{I}$ and $\mathcal{O}$ are matchings on the same vertex set. Thus, replacing all triples of the form $\{\infty, a, b\}$ for $\{a, b\} \in \mathcal{I}$ by $\{x, a, b\}$, and all triples of the form $\{x, c, d\}$ for $\{c, d\} \in \mathcal{O}$ by $\{\infty, c, d\}$, gives a triple system of order $n+1$ in which element ∞ has neighbourhood G. $\square$

The case when $n \equiv 4 \pmod 6$ is more complicated: the order is not admissible for $\lambda = 1$ or $\lambda = 2$, so neither the solution for index two nor the quadratic leaves theorem can be used in the same way. Nevertheless, Colbourn and McKay (1987) obtain results that are analogous to, but more restricted than, the quadratic leaves theorem and the neighbourhood theorem for index two; then using certain factorizations of cubic multigraphs into nontrivial path factors, they combine these results in a manner quite similar to the easier case outlined here.

4.3 Simple Neighbourhoods

Using the solutions for indices two and three, and the quadratic leaves theorem, Colbourn (1989) obtained the following theorem:

Theorem 4.8 *Let G be a n-vertex λ-regular simple graph, other than two disjoint triangles. Then if $\lambda \equiv 0 \bmod \gcd(n-1, 6)$, G is a λ-neighbourhood.*

We indicate the main ideas of the proof here, showing in the process a stronger result: for $n \geq 8$, $n \equiv 0, 2 \pmod 3$, every λ-regular n-vertex *multigraph* with $\lambda \equiv 0$

mod gcd$(n-1,6)$ is a λ-neighbourhood. First, we observe that we may assume that $\lambda \geq 4$ from the results of previous sections.

Now suppose that the index is even. If $n \equiv 0,2 \pmod 3$, the result is obtained (more generally for multigraphs) in Corollary 4.5. This leaves only $n \equiv 1,4 \pmod 6$. For even index, we have $\lambda \equiv 0 \pmod 6$.

Lemma 4.9 *Let G be a 6-regular simple multigraph on $n \equiv 1 \pmod 6$ edges. Then G is a neighbourhood in a $TS(n+1,6)$.*

Proof: By Petersen's theorem, G has a 2-factorization Q_1, Q_2, Q_3. In addition, since $n \equiv 1 \pmod 3$, there is an element x that does not appear in a triangle in Q_1. For $i = 1,2,3$, let $\{y_i, z_i\}$ be the neighbours of x in Q_i. Let R_i be the result of replacing the edges $\{\{x,y_i\},\{x,z_i\}\}$ in Q_i by the single edge $\{y_i,z_i\}$. Since x is not in a triangle of Q_1, R_1 is simple; R_2 and R_3 may each have a multiple edge. Now since R_1 is a simple quadratic graph with $n-1 \equiv 0 \pmod 3$ edges, there is a $PTS(n,1)$ whose leave is R_1; let $\mathcal{B}_1$ be the triples of this system. For R_2, proceed as follows. If R_2 has no multiple edge, form a $PTS(v,2)$ $\mathcal{B}_2$ by taking the union of a $PTS(n,1)$ with leave R_2 and a $TS(n,1)$. Otherwise, if R_2 has a repeated edge at $\{y_2,z_2\}$, form S_2 by choosing any other edge $\{a,b\}$ of R_2, removing the edge $\{a,b\}$ and one copy of the edge $\{y_2,z_2\}$, and adding the edges $\{\{a,y_2\},\{b,z_2\}\}$ (this cannot introduce repeated edges). Now take the union of a $PTS(n,1)$ with leave S_2 and a $TS(n,1)$ in which the neighbourhood of x contains $\{\{a,b\},\{y_2,z_2\}\}$; produce $\mathcal{B}_2$ by replacing the triples $\{\{x,a,b\},\{x,y_2,z_2\}\}$ by $\{\{x,a,y_2\},\{x,b,z_2\}\}$ in this union. $\mathcal{B}_2$ is a $PTS(v,2)$ whose leave is R_2. Proceed similarly to produce $\mathcal{B}_3$ from R_3. Finally, let $\mathcal{B}_4$ be the triples of a $TS(n,1)$ having the triples $\mathcal{U} = \{\{x,y_1,z_3\},\{x,y_2,z_1\},\{x,y_3,z_2\},\{y_1,y_2,y_3\}\}$. Now the system with triples $\mathcal{C} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$ is a $PTS(n,6)$ with leave $R_1 \cup R_2 \cup R_3$. Then $\mathcal{B} = \mathcal{C} \setminus \mathcal{U} \cup \{\{y_1,z_1,y_3\},\{y_2,z_2,y_1\},\{y_3,z_3,y_2\}\}$ is the collection of triples of a $PTS(n,6)$ with leave $G = Q_1 \cup Q_2 \cup Q_3$. $\square$

Repeated application of Petersen's theorem to factorize the graph then handles all simple graphs with $n \equiv 1 \pmod 6$.

A similar strategy handles the case when $n \equiv 4 \pmod 6$.

For odd index, the strategy is somewhat different. When the index is even, Petersen's theorem ensures that the graph is 2-factorable; for any odd degree λ, however, there are graphs with no regular factor of degree less than λ. Hence we cannot partition the graph as before into regular factors. Nevertheless, we can adapt the ideas used in the case when $\lambda = 3$. We require a graph-theoretic result:

Lemma 4.10 *Let G be a λ-regular n-vertex graph G with λ odd. Then G has a nontrivial path factor containing a path of length one.*

Proof: Add an arbitrary 1-factor F to G to form G', a $(\lambda+1)$-regular graph. Petersen's theorem ensures that G' has a 2-factor Q. Then deleting one edge from each cycle in $Q \setminus F$ is a nontrivial path factor H. If H has any path of length one, we are done. Otherwise, if H has a path of length $\ell > 2$, we can delete one further edge to split the path into paths of lengths 1 and $\ell - 2$, and we are done. Otherwise H has all paths of length 2. Hence it remains only to handle the case when $n \equiv 0 \pmod 6$. For some

2-factorization $Q_1, \ldots, Q_{(\lambda+1)/2}$ of G', consider the edge-partition of G into subgraphs $G_i = Q_i \setminus F$. If any one of the $\{G_i\}$ has a cycle other than a triangle, or a path of length different from two, it is easily seen that the desired path factor exists. Now G has $\lambda n/2$ edges, and hence one of the subgraphs, say G_j, has at least $\frac{\lambda n}{\lambda+1}$ edges. Thus if we have not yet obtained the required path factor, G_j has at most $\frac{1}{\lambda+1}$ paths of length two.

First, whenever G_j contains a 2-path in which the two endvertices are adjacent in G, we add the edge in G_j, turning the 2-path into a triangle. Now consider the neighbours of the vertex on a triangle. If adjacent to any vertex of another triangle, or to the endpoint of a 2-path, the required factor is easily produced. Thus each vertex on a triangle is adjacent only to other vertices on the triangle, and to centers of 2-paths. Similarly, an endvertex of a 2-path can be adjacent only to centers of other 2-paths; otherwise, we can combine two 2-paths with the additional edge to form a 5-path which can in turn be repartitioned as a 1-path and a 3-path.

Since the degree is odd, every triangle has at least one edge connecting a vertex of the triangle to a vertex not on the triangle. Hence, there must remain some 2-paths. Moreover, since the 2-paths account for twice as many edges at the centers of 2-paths as at the leaves, an elementary counting argument shows that the number of edges required to account for the degrees in G of the endvertices of the paths in G_j exceeds the degree remaining to be accounted for at the centers of the paths. □

With this lemma in hand, we now address the neighbourhood problem with $n \equiv 0, 2 \pmod 6$ and all odd λ:

Lemma 4.11 *Let G be a λ-regular multigraph on $n \equiv 0, 2 \pmod 6$ vertices, $n \geq 8$, with $\lambda \geq 5$ odd. Then G is a λ-neighbourhood.*

Proof (sketch): This parallels Lemma 4.7 closely. Find a nontrivial path factor H of G having a path $\{x, y\}$ of length 1. Let $R = G - H$. Choose an arbitrary matching M on the vertices of degree $\lambda - 2$ in R, so that $R + M$ is $(\lambda - 1)$-regular; then apply Corollary 4.7 to obtain a $TS(n+1, \lambda-1)$ whose additional point, ∞, has neighbourhood $R + M$. Next form a simple quadratic graph Q from $H \setminus \{x, y\}$ by adding a matching M' on the odd degree vertices so as to form only even length cycles. Let F_1, F_2 be a 1-factorization of Q, and form a $TS(n+1, 1)$ in which the neighbourhood of ∞ is $F_2 \cup \{x, y\}$ and the neighbourhood of x is $F_1 \cup \{\infty, y\}$. Take the union of the two triple systems to form a $TS(n+1, \lambda)$; then interchange the role of x and ∞ in the triples involving the pairs in $N_\infty \setminus G$ and in $G \setminus N_\infty$. □

It remains only to consider the case when $n \equiv 4 \pmod 6$; here, Colbourn (1989) uses a decomposition of the graph into a nontrivial path factor with at least two paths of length one, a spanning subgraph of maximum degree two, and a spanning subgraph with all vertex degrees from $\{\lambda - 4, \lambda - 3\}$; the ingredients are more complicated, but the basic strategy remains the same.

4.4 Neighbourhood Uniform Triple Systems

Until this point, we have been concerned with a single neighbourhood in a triple system. One might instead ask that every neighbourhood be isomorphic; we call such a triple system *neighbourhood uniform.* Naturally every triple system whose automorphism group acts transitively on its elements is neighbourhood uniform; however, the structure of possible neighbourhoods in transitive systems appears not to have been studied.

When one asks for a specific graph to be the neighbourhood in a neighbourhood uniform triple system, little is known. Ducrocq and Sterboul (1980) made the following observation:

Theorem 4.12 *For every $n \equiv 0, 1 \pmod 3$, there exists a simple $TS(n,2)$ in which the neighbourhood of every element is an $(n-1)$-cycle.*

Proof: Ringel (1975) establishes that for every $n \equiv 0, 1 \pmod 3$, there is a nonorientable surface which K_n triangulates. The triples of the simple $TS(v,2)$ are just the faces of this embedding. Now consider the faces incident at a vertex v of K_n; they are necessarily in a cyclic sequence about v in the embedding, and hence the neighbourhood of v is a cycle. Since every vertex other than v is adjacent to v, the neighbourhood is an $(n-1)$-cycle. □

4.5 Double Neighbourhoods for Index One

The neighbourhood problem is trivial for $\lambda = 1$, since for each order n (necessarily even), there is only one isomorphism type of 1-regular multigraph and it is the neighbourhood of any element in any Steiner triple system of order $n+1$.

Colbourn, Colbourn and Rosa (1983) examined a nontrivial extension of the neighbourhood problem in the case $\lambda = 1$. A *double star* of triples based on a pair $\{x, y\}$ of a v-set V is a partial triple system on V so that every triple contains at least one of $\{x, y\}$, and every element $z \in V \setminus \{x, y\}$ appears in exactly one triple with x and exactly one triple with y. Equivalently, we can think of a double star as a simultaneous specification of the neighbourhoods of x and y, and hence as a "double neighbourhood".

Colbourn, Colbourn and Rosa (1983) established that every double star is completable to a Steiner triple system; we give a simpler proof using more recent results:

Theorem 4.13 *Every double star on $n \equiv 1, 3 \pmod 6$ elements can be completed to a Steiner triple system of order n.*

Proof: Let $\mathcal{D}$ be the triples of the double star on $V \cup \{x, y\}$ based at $\{x, y\}$. $\mathcal{D}$ contains a triple $\{x, y, z\}$ containing $\{x, y\}$. Let $\mathcal{C} = \mathcal{D} \setminus \{\{x, y, z\}\}$, and form the set of pairs $E = \{\{b, c\} : \{a, b, c\} \in \mathcal{D}, a \in \{x, y\}\}$. Then the graph $G = (V, E)$ is a quadratic graph with $n-2$ vertices and $n-3$ edges; moreover, every cycle of G has even length. Thus G is the leave of some partial triple system $(V, \mathcal{B})$ of order $n-2$ and index 1 by Theorem 2.12. Then $(V \cup \{x, y\}, \mathcal{B} \cup \mathcal{D})$ is a Steiner triple system of order n containing the double star $\mathcal{D}$. □

The quadratic graph in the double star has been widely studied, primarily as an isomorphism invariant. The isomorphism type of such a graph is determined simply by the number of cycles of each length, and hence the double neighbourhood graph is typically encoded as a list of cycle lengths; this is the *cycle structure* of the pair $\{x, y\}$. The fact that every double star can be completed is equivalent to the fact that, for $n \equiv 1, 3 \pmod 6$, every partition of $n - 3$ into even parts each at least four is a cycle structure for some pair in a Steiner triple system of order n.

4.6 Uniform and Perfect Triple Systems

Rather than prescribing the double neighbourhood at a single pair, one could ask that *all* pairs of vertices have isomorphic double neighbourhood graphs. We call such a Steiner triple system *uniform*. If in addition each double neighbourhood consists of a single cycle, we call the triple system *perfect* (in analogy with "perfect one-factorizations").

Two classes of uniform Steiner triple systems arise from the infinite families of 2-transitive systems: the projective systems and the affine systems. In a projective system of order n, each double neighbourhood is a set of $\frac{n-3}{4}$ 4-cycles, since any three noncollinear elements generate a subsystem of order seven. In an affine system of order n, every double neighbourhood is a set of $\frac{n-3}{6}$ 6-cycles, since every three noncollinear elements generate a subsystem of order nine. More generally, the Hall triple systems, although not 2-homogeneous in general, have the same double neighbourhoods as the affine triple systems; see Bénéteau (1983,1984).

A further class of uniform Steiner triple systems arises from the 2-homogeneous, but not 2-transitive, systems: the Netto systems that exist for $q \equiv 7 \pmod{12}$, q a prime power; see Delandtsheer, Doyen, Siemons and Tamburini (1986). Necessarily, all pairs of elements have isomorphic double neighbourhoods; however, the structure of the double neighbourhood is only partially understood. Robinson (1975) demonstrates that the Netto triple system of order p^α has a double neighbourhood containing a 4-cycle whenever $p \equiv 7 \pmod{24}$, and having no 4-cycle whenever $p \equiv 19 \pmod{24}$. He also examines the distribution of numbers of 4- and 6-cycles in the double neighbourhood.

Beyond these classes for which uniformity is forced by the structure of the automorphism group, little is known. It follows from the classification of 2-homogeneous Steiner triple systems and our remarks above that the only perfect 2-homogeneous systems are the obvious cases for orders seven and nine.

Perfect Steiner triple systems are known to exist for only four orders: 7, 9, 25, and 33. The known perfect Steiner triple system of order 25 is a transitive design acted on by the group $Z_5 \times Z_5$; orbit representatives are: $\{(0,0),(0,1),(1,0)\}$, $\{(0,0),(0,2),(2,1)\}$, $\{(0,0),(1,1),(2,3)\}$, and $\{(0,0),(1,3),(3,3)\}$. The perfect Steiner triple system of order 33 is also transitive; in fact, it is cyclic with starter blocks: $\{0,1,7\}$, $\{0,2,21\}$, $\{0,3,20\}$, $\{0,4,28\}$, $\{0,8,18\}$ and $\{0,11,22\}$. No other transitive Steiner triple system of order at most 27 is perfect.

At the present time, no infinite family of perfect Steiner triple systems is known; nor is any recursive construction known that preserves perfection.

4.7 Quadrilateral-free Steiner triple systems

A 4-cycle in a double neighbourhood of a Steiner triple system is called a *quadrilateral.* An equivalent configuration is the set of four triples:

$$\{\{a,w,x\},\{a,y,z\},\{b,w,y\},\{b,x,z\}\};$$

this is a *Pasch configuration*, *fragment* or *arrow*. When no double neighbourhood of a Steiner triple system contains a quadrilateral, the system is *quadrilateral-free* (or *anti-Pasch*), and is denoted $QFSTS(v)$.

Erdös (1976) conjectured that for every r there is an integer $v_0(r)$ so that for every $v > v_0(r)$, $v \equiv 1,3 \pmod 6$, there is a Steiner triple system on v elements with the property that for $2 \leq j \leq r$, no $j+2$ points carry j triples. For $r = 4$, Erdös's conjecture states that for all $v > v_0(4)$ with $v \equiv 1,3 \pmod 6$, a $QFSTS(v)$ exists.

The unique $STS(7)$ and both nonisomorphic $STS(13)$'s contain quadrilaterals, but Brouwer (1977) conjectures:

Conjecture 4.14 *For $v \equiv 1,3 \pmod 6$, a $QFSTS(v)$ exists except when $v \in \{7,13\}$.*

Brouwer's conjecture remains far from settled, although substantial partial results are available which we outline here. Erdös's more general conjecture apparently remains completely open for $r \geq 6$, while for $r = 5$ Brouwer (1977) gives one infinite family of $QFSTS$ in which no seven points carry five triples.

For quadrilateral-free STS, from the affine triple systems and Robinson's characterization of quadrilaterals in Netto triple systems, we have:

Lemma 4.15 *A $QFSTS(p^\alpha)$ exists for $p = 3$ or p a prime with $p \equiv 19 \pmod{24}$, and α a nonnegative integer.* □

A more general construction for prime powers was obtained by Brouwer (1977). For prime power order q, a $TS(q,1)$ $\mathcal{S}_C$ can be formed as follows. Let x be a primitive element of $GF(q)$, $t = \frac{q-1}{6}$ and $y = x^{2t}$ so that $y^2 + y + 1 = 0$. Let C be a subset of $GF(q)$ of size t with $0 \notin C$ and if $x^\alpha, x^\beta \in C$ then $\alpha \not\equiv \beta \pmod t$. The system $\mathcal{S}_C$ contains the triples $\{c\{1,y,y^2\} + i : c \in C, i \in GF(q)\}$.

Brouwer (1977) proves that $\mathcal{S}_C$ contains a quadrilateral whenever, for some triple T, $\mathcal{S}_C$ contains the triple $2T$, and in addition $|3T| = 3$. Using this observation, he proved:

Theorem 4.16 *Let $q \equiv 1 \pmod 6$, $q = p^\alpha$ and p prime. Let C be as above. Then*

1. *if $p \in \{7,13\}$, $\mathcal{S}_C$ contains a quadrilateral.*

2. *if $p \notin \{7,13\}$, $\mathcal{S}_C$ contains a quadrilateral if and only if for some $S \in \mathcal{S}_C$, $2S \in \mathcal{S}_C$, where $S \neq 2S$.*

Corollary 4.17 *For $q \equiv 1 \pmod 6$, $q = p^\alpha$, $p \notin \{7,13\}$ a prime, there is a $QFSTS(q)$ whenever $p \equiv 1,3 \pmod 8$ or $\alpha \equiv 0 \pmod 2$.*

Proof sketch: The conditions on p and α are equivalent to the statement that -2 is not a square in $GF(q)$. Choose a set C so that $\mathcal{S}_C$ is invariant under multiplication by -2. $\square$

Brouwer (1977) and Doyen (1981) obtained the first results on $QFSTS$ for orders other than prime powers. First, we recall the Bose construction for Steiner triple systems (Bose (1939,1960)). Let Γ be a commutative idempotent quasigroup of order $g = 2s + 1$ with binary operation $\odot$. Let $V = \Gamma \times Z_3$, and write (x, i) as x_i. Let $\mathcal{B}$ contain the following triples:

1. for $x \in \Gamma$, include the triple $\{x_0, x_1, x_2\}$;
2. for $x, y \in \Gamma$, $x \neq y$, include the triples $\{x_i, y_i, (x \odot y)_{i+1}\}$ for $i \in Z_3$ (reducing subscripts modulo 3 as necessary.

$(V, \mathcal{B})$ is then a Steiner triple system.

Brouwer (1977) and Doyen (1981) established that:

Theorem 4.18 *There exists a $QFSTS(3g)$ whenever g is odd and $g \not\equiv 0$ (mod 7).*

Proof sketch: Form a triple system $(V, \mathcal{B})$ using the Bose construction just described, based on the commutative idempotent quasigroup on Z_g with $x \odot y = \frac{x+y}{2}$. A quadrilateral arises in the Bose construction whenever the quasigroup contains a subquasigroup of order two; since g is odd, none of this type are present. The second type of quadrilaterals requires that $x \odot (x \odot (x \odot y))) = y$ for distinct symbols x, y. By construction, this requires $\frac{7}{8}x + \frac{1}{8}y = y$, or $7(x - y) = 0$; but $g \not\equiv 0$ (mod 7), and hence no quadrilaterals of this second type are present. $\square$

Brouwer (1977) and Griggs, Murphy and Phelan (1991?) extended this result:

Theorem 4.19 *There exists a $QFSTS(v)$ for all $v \equiv 3$ (mod 6) having a parallel class of triples.*

Proof: If $v \not\equiv 0$ (mod 7), the Bose construction yields the desired $QFSTS$ with triples of the form $\{x_0, x_1, x_2\}$ giving a parallel class. There is a $QFSTS(21)$ that has a parallel class of triples; one such is given by the starter blocks

$$\{\{0, 1, 3\}, \{0, 4, 12\}, \{0, 6, 11\}, \{0, 7, 14\}\}.$$

The parallel class is the "short orbit" $\{0, 7, 14\}$ mod 21.

Now suppose that $v \equiv 3$ (mod 6), and that $v = 7u$. Then proceeding inductively there is a $QFSTS(u)$ $\mathcal{B}$ having a parallel class $\mathcal{P}$ on a u-set U. Let $V = Z_7 \times U$. For each $T = \{a, b, c\} \in \mathcal{S} \setminus \mathcal{P}$, $i, j \in Z_7$, include the triple $\{(i, a), (j, b), (i + j, c)\}$ (arithmetic mod 7). For each $T \in \mathcal{P}$, take the triples of the $QFSTS(21)$ with parallel class on $Z_7 \times T$. It is easy to check that the result is a $QFSTS(v)$ with a parallel class. $\square$

Subsequently, Grannell, Griggs and Phelan (1988) established that the Schreiber-Wilson construction also yields $QFSTS$ under certain restrictions. First we recall a restricted form of the construction from Schreiber (1973) and Wilson (1974a) for the case when for every prime divisor p of $n = v - 2$ the order of -2 mod p is singly even. To form an $STS(v)$, write $n = v - 2$ and choose an abelian group Γ of order n with binary operation $\oplus$ and identity 0. Since $n \equiv 1, 5 \pmod 6$, Γ has no elements of order two or three. Now consider all triples of group elements $\{a, b, c\}$ with $a \oplus b \oplus c = 0$. Let $\mathcal{D}$ be the set of such triples in which all three group elements are distinct, and let $\mathcal{T}$ be those in which there are two distinct elements. Finally, let $\mathcal{S}$ be those containing a single element; note that in Γ, $a \oplus a \oplus a = 0$ has the unique solution $a = 0$, since Γ has no elements of order three. Thus $\mathcal{S}$ contains the single triple $\{0, 0, 0\}$.

Now to form a triple system $(\Gamma \cup \{\alpha, \beta\}, \mathcal{B})$,

1. include all triples of $\mathcal{D}$ in $\mathcal{B}$;

2. first replace every unordered pair $\{a, b\}$ by the ordered pair (a, b) if $a \oplus a \oplus b = 0$, and by (b, a) otherwise. Partition the pairs of $\mathcal{T}$ into orbits under the mapping $\phi : z \mapsto z'$ where $z \oplus z' \oplus z' = 0$ (if $(a, b) \in \mathcal{T}$, $a = \phi(b)$). Now consider an orbit of pairs of $\mathcal{T}$ under ϕ. Under the stated restriction that for every prime divisor p of n, the order of -2 mod p is singly even, each orbit of elements from $\mathcal{T}$ is of even length. Suppose that the orbit has length $2k$, and that the pair (a, b) is in the orbit, add to $\mathcal{B}$ the triples $\{\{\alpha, \phi^{2i}(a), \phi^{2i+1}(a)\}, \{\beta, \phi^{2i+1}(a), \phi^{2i+2}(a)\}\}$ for $0 \leq i < k$.

3. include the triple $\{\alpha, \beta, 0\}$ in $\mathcal{B}$.

It is a somewhat tedious case analysis to verify that this system has no quadrilaterals, to obtain the theorem of Grannell, Griggs and Phelan (1988):

Theorem 4.20 *Let $p_1, \ldots, p_s$ be primes so that for $1 \leq i \leq s$, -2 has singly even order mod p_i. Then there exists a $QFSTS(2 + \prod_{i=1}^{s} p_i)$.* $\Box$

Griggs, Murphy and Phelan (1991?) and Stinson and Wei (1991?) observe that the direct product construction for triple systems does not introduce quadrilaterals:

Theorem 4.21 *If there exists a $QFSTS(u)$ and a $QFSTS(v)$, then there exists a $QFSTS(uv)$.*

Stinson and Wei (1991?) establish a second recursive construction:

Theorem 4.22 *If there is a $QFSTS(u)$, $u \equiv 1 \pmod 4$, and u has an odd divisor exceeding 3, then there exists a $QFSTS(3u - 2)$.*

Chee and Lim (1989) give cyclic $QFSTS(v)$ for $v \in \{31, 37\}$.

4.8 Work Points

1. Complete the proof of Conjecture 4.3; for $n \geq 8$, it remains to handle multigraphs with at least one repeated edge on $n \equiv 1 \pmod 3$ vertices.

2. Which graphs are neighbourhoods in *simple* $TS(v, \lambda)$'s? This is open for all $\lambda \geq 2$.

3. Find an infinite family of perfect Steiner triple systems, or prove that there are only finitely many.

4. Characterize the quadratic multigraphs for which some triple system of index two exists with every neighbourhood isomorphic to the specified one.

5. Determine the possible double neighbourhoods in a triple system of index two.

6. Determine the spectrum for quadrilateral-free Steiner triple systems. For $v \leq 300$, the following orders remain unresolved: 55, 79, 85, 103, 115, 127, 151, 157, 175, 187, 199, 223, 229, 247, 253, 259, 271, 295.

7. Prove Erdös's conjecture: for every r, there exists an integer $v_0(r)$ such that if $v > v_0(r)$ and $v \equiv 1, 3 \pmod 6$, there is an $STS(v)$ in which no set of $j+2$ points carries j triples for $2 \leq j \leq r$.

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