THE COMPLEMENTS OF SOME COVERING DESIGNS AND QUESTIONS ON THE OVERALL SIGNIFICANCE

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Introduction

Define, as in Mills[1], a (v,k,t)-covering to be a collection of k-element subsets (called blocks) of a v-element set, S, such that every t-element subset of S is contained in at least one block. Let the covering number N(v,k,t) be the least number of blocks in any (v,k,t)-covering.

We begin with two well-known designs:

	N(9,6,4)=12	N(9,3,2)=12
1.	4 5 6 7 8 9	1 2 3
2.	2 3 6 7 8 9	1 4 5
3.	2 3 4 5 8 9	1 6 7
4.	2 3 4 5 6 7	1 8 9
5.	1 3 5 6 7 8	2 4 9
6.	1 3 4 7 8 9	2 5 6
7.	1 3 4 5 6 9	2 7 8
8.	1 2 5 6 8 9	3 4 7
9.	1 2 4 6 7 9	3 5 8
10.	1 2 4 5 7 8	3 6 9
11.	1 2 3 5 7 9	4 6 8
12.	1 2 3 4 6 8	5 7 9

Question

Now is it important or trivial to ask:

Does the full complement of a (9,6,4)-cover give a (9,3,2)-cover, or, does the full complement of a (9,3,2)-cover give a (9,6,4)-cover? Are both covers of equal importance and only have a coincidental relationship?

We continue with another two well-known designs:

	N(10,4,3)=30	N(10,6,4)=20
1.	1 2 3 4	5 6 7 8 9 10
2.	1 2 5 8	
3.	1 2 6 9	3 4 5 7 8 10
4.	1 2 7 10	3 4 5 6 8 9
5.	1 3 5 7	2 4 6 8 9 10
6.	1 3 6 10	
7.	1 3 8 9	2 4 5 6 7 10
8.	1 4 5 6	2 3 7 8 9 10
9.	1 4 7 9	
10.	1 4 8 10	
11.	1 5 9 10	2 3 4 6 7 8
12.	1 6 7 8	2 3 4 5 9 10
13.	2 3 5 6	1 4 7 8 9 10
14.	2 3 7 8	
15.	2 3 9 10	
16.	2 4 5 10	1 3 6 7 8 9
17.	2 4 6 7	
18.	2 4 8 9	1 3 5 6 7 10
19.	2 5 7 9	1 3 4 6 8 10
20.	2 6 8 10	1 3 4 5 7 9
21.	3 4 5 9	
22.	3 4 6 8	1 2 5 7 9 10
23.	3 4 7 10	1 2 5 6 8 9
24.	3 5 8 10	1 2 4 6 7 9
25.	3 6 7 9	1 2 4 5 8 10
26.	4 5 7 8	1 2 3 6 9 10
27.	4 6 9 10	1 2 3 5 7 8
28.	5 6 7 10	
29.	5 6 8 9	
30.	7 8 9 10	1 2 3 4 5 6

Question

While these designs are well known, it may not be well known that a (10,6,4)-covering design can be found in the complement of a (10,4,3)-covering design. The question now may not be so trivial because we could ask: does the complement of a (10,4,3)-cover show a simple way of finding a (10,6,4)-cover, or, does the complement of a (10,6,4)-cover provide a simple way of constructing a (10,4,3)-cover because it leaves us with only 10 blocks to find? Or is the relationship between these two designs purely coincidental?

Now suppose we wish to find a covering design for (11,7,5), for which, at the time of writing, the minimum number of blocks is not known. The Schonheim bound[1] indicates a minimum of 32 blocks. We can quickly reconcile this as we know that a (10,6,4)-cover requires 20 blocks and since v, k and t have all been increased by 1, then 11 multiplied by 20 and divided by 7 equals 31.4. Since we cannot use part of a block we must go to the nearest whole number and therefore an (11,7,5)-cover is going to require a minimum of 32 blocks. How do we proceed to construct this design? To simplify the problem we could increase v, k and t so that we look for a (12,8,6)-cover as sometimes it is easier to work with an even number of elements. If we can find a neat way to construct a (12,8,6)-cover, then an (11,7,5)-cover could be taken from such a construction. This is not necessarily the best approach, it is merely a suggestion when faced with a difficult problem for which no hard and fast rules exist. There is no proof either, in this particular case, that it gives the absolute minimum number of blocks, but it does give a very close approximation.

Constructing N(12,8,6)=?

We commence by dividing the 12 elements into two groups of six which we call A and B. As t=6, we place under A and B every possibility where the full value of t could fall.

A	B
1 2 3 4 5 6	7 8 9 10 11 12
6	0
0	6
5	1
1	5
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4	2
2	4
3	3

If the full value of t falls in any of the first four possibilities (up to the dividing line), then we would only need six blocks to contain them and these can be:

1. 1 2 3 4 5 6 7 8
2. 1 2 3 4 5 6 9 10
3. 1 2 3 4 5 6 11 12
4. 1 2 7 8 9 10 11 12
5. 3 4 7 8 9 10 11 12
6. 5 6 7 8 9 10 11 12

Of the remaining possibilities where t could fall, they can be covered in 45 blocks, and the 45 blocks can be found in the complement of an N(12,4,3)-covering design. You will note that the 45 blocks use exactly four elements from A and B. This of course means that an N(12,8,6)-cover can be constructed in 51 blocks and where each element is used exactly 34 times.

From the 51 blocks a total of 49 are taken from the complement of the (12,4,3)-covering design.

	N(12,4,3)=57	Complementary blocks
1.	1 2 3 4	5 6 7 8 9 10 11 12*
2.	1 2 3 5	
3.	1 2 3 6	
4.	1 2 7 8	3 4 5 6 9 10 11 12
5.	1 2 9 10	3 4 5 6 7 8 11 12
6.	1 2 11 12	3 4 5 6 7 8 9 10
7.	1 3 7 9	2 4 5 6 8 10 11 12
8.	1 3 8 11	2 4 5 6 7 9 10 12
9.	1 3 10 12	2 4 5 6 7 8 9 11
10.	1 4 5 6	
11.	1 4 7 10	2 3 5 6 8 9 11 12
12.	1 4 8 12	2 3 5 6 7 9 10 11
13.	1 4 9 11	2 3 5 6 7 8 10 12
14.	1 5 7 11	2 3 4 6 8 9 10 12
15.	1 5 8 10	2 3 4 6 7 9 11 12
16.	1 5 9 12	2 3 4 6 7 8 10 11
17.	1 6 7 12	2 3 4 5 8 9 10 11
18.	1 6 8 9	2 3 4 5 7 10 11 12
19.	1 6 10 11	2 3 4 5 7 8 9 12
20.	2 3 7 12	1 4 5 6 8 9 10 11
21.	2 3 8 9	1 4 5 6 7 10 11 12
22.	2 3 10 11	1 4 5 6 7 8 9 12
23.	2 4 5 6	
24.	2 4 7 11	1 3 5 6 8 9 10 12
25.	2 4 8 10	1 3 5 6 7 9 11 12
26.	2 4 9 12	1 3 5 6 7 8 10 11
27.	2 5 7 9	1 3 4 6 8 10 11 12
28.	2 5 8 11	1 3 4 6 7 9 10 12
29.	2 5 10 12	1 3 4 6 7 8 9 11
30.	2 6 7 10	1 3 4 5 8 9 11 12
31.	2 6 8 12	1 3 4 5 7 9 10 11
32.	2 6 9 11	1 3 4 5 7 8 10 12
33.	3 4 5 6	1 2 7 8 9 10 11 12*
34.	3 4 7 8	1 2 5 6 9 10 11 12
35.	3 4 9 10	1 2 5 6 7 8 11 12
36.	3 4 11 12	1 2 5 6 7 8 9 10
37.	3 5 7 10	1 2 4 6 8 9 11 12
38.	3 5 8 12	1 2 4 6 7 9 10 11

39.	3 5 9 11	1 2 4 6 7 8 10 12
40.	3 6 7 11	1 2 4 5 8 9 10 12
41.	3 6 8 10	1 2 4 5 7 9 11 12
42.	3 6 9 12	1 2 4 5 7 8 10 11
43.	4 5 7 12	1 2 3 6 8 9 10 11
44.	4 5 8 9	1 2 3 6 7 10 11 12
45.	4 5 10 11	1 2 3 6 7 8 9 12
46.	4 6 7 9	1 2 3 5 8 10 11 12
47.	4 6 8 11	1 2 3 5 7 9 10 12
48.	4 6 10 12	1 2 3 5 7 8 9 11
49.	5 6 7 8	1 2 3 4 9 10 11 12
50.	5 6 9 10	1 2 3 4 7 8 11 12
51.	5 6 11 12	1 2 3 4 7 8 9 10
52.	7 10 11 12	
53.	8 10 11 12	
54.	9 10 11 12	1 2 3 4 5 6 7 8*
55.	7 8 9 10	1 2 3 4 5 6 11 12*
56.	7 8 9 11	
57.	7 8 9 12	

*Already included in the first six blocks.

If we can cover (12,8,6) in 51 blocks using each element exactly 34 times then we can cover (11,7,5) in 34 blocks. This is no proof, of course, that either of these designs is a minimum cover. If they are not, then a completely different method of construction is needed.

Question

In the three examples given, are all the complementary blocks purely coincidental?

Reference

[1] W.H. Mills, Construction of covering designs, Congressus Numerantium, 73 (1990) 29-36.

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