On the existence of three MOLS with equal-sized holes

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Abstract

We consider sets of three MOLS (mutually orthogonal Latin squares) having holes corresponding to missing sub-MOLS that are disjoint and spanning. We show that three MOLS with n holes of size h exist for $h \geq 2$ if and only if $n \geq 5$, with 61 possible exceptions (h, n).

1 Introduction

Let $P = \{S_1, \ldots, S_n\}$ be a partition of a set S, where $n \geq 2$. A *partitioned incomplete Latin square* (or PILS) having partition P is an $|P| \times |P|$ array L, indexed by S, which satisfies the following properties:

1. a cell of L either contains a symbol from S or is empty
2. the subarrays indexed by $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are called *holes*)
3. the elements occurring in row (or column) s of L are precisely those in $S \backslash S_i$, where $s \in S_i$.

The *type* of L is the multiset $\{|S_1|, \ldots, |S_n|\}$. We use the notation $1^{u_1}2^{u_2}\ldots$ to describe a type, where there are precisely u_i occurrences of i, for $i = 1, 2, \ldots$.

Suppose L and M are PILS having the same partition P. We say that L and M are *orthogonal* if their superposition yields every ordered pair in $S^2 \backslash (\cup S_i{}^2)$. Several PILS are orthogonal if every pair is. The term "orthogonal PILS" is abbreviated to OPILS. (The term HMOLS is also used in the literature and it has the same meaning as OPILS. HMOLS is an abbreviation for "holey mutually orthogonal Latin squares".)

OPILS have been studied by several researchers. Some applications to the construction of other types of designs are as follows: Room frames [9], Howell designs [22], conjugate orthogonal arrays invariant under K_4 [15], Steiner pentagon systems [17], idempotent Schroeder quasigroups [8], perfect Mendelsohn designs [3] [4] and 2−perfect M−cycle systems [16].

The maximum number of OPILS of type T will be denoted $N(T)$. The numbers $N(T)$ have been studied mainly in the case where $T = h^n$. It is easy to see that $N(h^n) \leq n - 2$. We are interested in the situations when $N(h^n) \geq 3$. The following results have been proved.

Theorem 1.1 *1. [1], [4] $N(1^n) \geq 3$ if $n \geq 5$, $n \neq 6, 10, 18, 22, 26$.*

2. [18], [26], [14] If $h = 2$ or 10, then $N(h^n) \geq 3$ if $n \geq 5$, except possibly for $n = 15, 42, 44, 48, 52, 54$, or 80, or for even $n \leq 38$.

3. [19], [14] $N(3^n) \geq 3$ if $n \geq 5$, except possibly for $n = 15, 28, 30, 32, 34, 38, 44, 48$ or 58, or for even $n \leq 24$.

4. [19], [14] $N(4^n) \geq 3$ if $n \geq 5$, except possibly for $n = 14, 18, 22$ or 28.

5. [14] $N(6^n) \geq 3$ if $n \geq 5$, except possibly for $n = 6$, 7, 11, 19, 23, 27, 31, 32, 34, 38, 39, 42, 47, 51, 58, 59 or 62, or for even n, $10 \leq n \leq 24$.

6. [14] If $h \geq 5$, $h \neq 6, 10$, then $N(h^n) \geq 3$ if $n \geq 5$, except possibly for $n = 6, 10, 14, 18, 22, 26, 30, 34$ or 38.

For the sake of completeness, we also mention the following lower bounds on $N(h^n)$ which have been proved in [10] and [27].

Theorem 1.2 *1. For any $h \geq 2$, $N(h^n) \geq 2$ if and only if $n \geq 4$.*

2. $N(2^{17}) \geq 4$ and $N(2^n) \geq 6$ for $n \in \{29, 37, 41, 53, 61, 73, 89, 97\}$.

In this paper, we show that $N(h^n) \geq 3$ for $h \geq 2$ if and only if $n \geq 5$, with 61 ordered pairs (h, n) as possible exceptions, listed in Table 1.

We shall assume that the reader is familiar with the standard terminology of group-divisible designs (GDDs), mutually orthogonal Latin squares (MOLS) and transversal designs (TDs) (see, for example, [5] and [30]). Of course, a TD(k, n) is equivalent to $k - 2$ MOLS of order n. Further, a TD(k, n) is equivalent to a TD$(k-1, n)$ in which the blocks can be partitioned into parallel classes; such a TD is called *resolvable.*

We shall make use of some results concerning existence of MOLS (for a list of lower bounds up to order 10000, see [6]). For existence of three and four MOLS, the following results are known.

Table 1: Existence of three OPILS of type h^n

h	possible exceptions n
2	$6,8,10,12,14,15,16,18,20,22,24,28,30,32,34,38,42,44,48,52$
3	$6,8,10,12,14,16,18,20,22,24,28,32,34$
5	$6,10,18,22$
6	$7,10,11,12,14,16,18,19,20,23$
10	$6,8,10,15,16$
11	$6,10$
$14,17,18$	6
23	$6,10,18,22$

Lemma 1.3 *[5],[13] There exist three MOLS of order n if $n \geq 4$, $n \neq 6, 10$.*

Lemma 1.4 *There exist four MOLS of order n if $n \geq 5$, $n \neq 6$, 10, 14, 18, 22, 26, 30, 34 or 42.*

Proof: For most values of n, the result can be found in [5] and [13]. See [1] for the two cases $n = 28$ and 52. Finally, the two cases $n = 38$ and 44 were done by D. Todorov (private communication from P. Schellenberg). □

We also require the idea of incomplete transversal designs [7]. Informally, an incomplete TD$(k,n)-$ TD(k,m) denotes a TD(k,n) "missing" a sub-TD(k,m). We observe that an incomplete TD$(k,n)-$ TD$(k,0)$ and an incomplete TD$(k,n)-$ TD$(k,1)$ exists if and only if a TD(k,n) exists. Also an incomplete TD$(k,n) - \cup_{j=1}^{s}$TD$(k,m_j)$is equivalent to $k-2$ OPILS of type $\{m_1,\ldots,m_s\}$ if $\sum_{j=1}^{s} m_j = n$.

The following obvious construction comes from the direct product of MOLS.

Lemma 1.5 *Suppose there exists a TD(k,n) and a TD(k,m). Then there exists an incomplete TD$(k,mn)-$ TD(k,m).*

For small values of m, the following is known.

Lemma 1.6 *There exists an incomplete TD$(5,n)-$ TD$(5,m)$ in the following cases:*

1. *if $m = 2$, $n \geq 8$ and $n \neq 12$, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28 or 31*
2. *if $m = 7$, $n \geq 28$ and $n \neq 30,34,38,41,44,45,46$ or 48*
3. *if $m = 8$, $n \geq 32$ and $n \neq 46$*
4. *if $m = 9$, $n \geq 36$ and $n \neq 38,42$ or 50*

Proof: See [20], [26], [32], [34] and [33]. □

We shall also make use of a few special examples.

Lemma 1.7 *There exists an incomplete $TD(5,n)-TD(5,m)$ if $(m,n) = (12,3)$, $(18,4)$, $(21,4)$, $(22,4)$ or $(30,5)$.*

Proof: See [23], [24], [28] and [31]. □

2 Constructions

In this section, we give several recursive constructions for OPILS, most of which are previously known. These include two Wilson-type constructions [29], a group-divisible design construction, and a "filling in holes" construction. We also describe a new construction based on k−frames.

The following four constructions were used in [26].

Lemma 2.1 [10] *Suppose $(\mathbf{X}, \mathcal{G}, \mathcal{A})$ is a group-divisible design, and let $w : \mathbf{X} \to \mathbf{Z}^+ \cup \{0\}$. Let k be a positive integer. Suppose that there are k OPILS of type $w(A)$ for every block $A \in \mathcal{A}$. Then there are k OPILS of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

Lemma 2.2 [7] *Suppose there is a $TD(k+1,t)$ and let $u_i \geq 0$ for $1 \leq i \leq t-1$. For $1 \leq i \leq t-1$, suppose there is an incomplete $TD(k, m+u_i)-TD(k,u_i)$. Then $N(m^t u^1) \geq k-2$, where $u = \sum_{i=1}^{t-1} u_i$.*

Lemma 2.3 [7] *Suppose there is a $TD(k+\ell,t)$ and let $u_i \geq 0$ for $1 \leq i \leq \ell-1$. For $1 \leq i \leq \ell-1$, suppose there is an incomplete $TD(k, m+u_i)-TD(k,u_i)$. If $\ell < t+1-k$, then suppose there exists a $TD(k,m)$. Then $N(m^{t-1}(m+u)^1) \geq k-2$, where $u = \sum_{i=1}^{\ell-1} u_i$.*

Lemma 2.4 [26] *Suppose there are k OPILS of type $\{t_1, \ldots, t_n\}$, and let $\epsilon \geq 0$. For $1 \leq i \leq n$, suppose there are k OPILS of type $T_i \cup \{\epsilon\}$, where $t_i = \sum_{t \in T_i} t$. Then $N((\cup T_i) \cup \{\epsilon\}) \geq k$.*

Combining Lemmas 2.2 and 2.3, we get

Lemma 2.5 *Suppose there is a $TD(k+\ell,t)$, let $u_i \geq 0$ for $1 \leq i \leq \ell-1$ and let $v_j \geq 0$ for $1 \leq j \leq t-1$. For $1 \leq i \leq \ell-1$ and $1 \leq j \leq t-1$, suppose there is an incomplete $TD(k, m+u_i+v_j)-TD(k,u_i)-TD(k,v_j)$. Then $N(m^{t-1}(m+u)^1 v^1) \geq k-2$, where $u = \sum_{i=1}^{\ell-1} u_i$ and $v = \sum_{j=1}^{t-1} v_j$.*

As a new generalization of Lemma 2.2, we present the following.

Lemma 2.6 *Suppose there is a $TD(k+1,t)$ and let $u_i \geq 0$ for $1 \leq i \leq t-1$. Let s be a positive integer, and for $1 \leq j \leq s$, let a_j be positive integers such that $m \geq \sum_{j=1}^{s} a_j$. For $1 \leq i \leq t-1$, suppose there is an incomplete $TD(k, m+u_i)-TD(k,u_i) - \cup_{j=1}^{s} \mathrm{TD}(k,a_j)$.*

1. *Suppose there exist $k-2$ OPILS of type $(a_j)^t$, for $1 \le j \le s$. Then $N(m^t u^1) \ge k-2$, where $u = \sum_{i=1}^{t-1} u_i$.*
2. *Suppose $m = \sum_{j=1}^{s} a_j$ and let $\epsilon > 0$. Suppose there exist $k-2$ OPILS of type $(a_j)^t \epsilon^1$, for $1 \le j \le s$. Then $N(m^t(u+\epsilon)^1) \ge k-2$, where $u = \sum_{i=1}^{t-1} u_i$.*

Next, we mention the well-known direct product construction.

Lemma 2.7 [*10*] *Suppose there exist k OPILS of type $\{t_1, \ldots, t_n\}$ and k MOLS of side m. Then there exist k OPILS of type $\{mt_1, \ldots, mt_n\}$.*

Our final construction is a new construction that uses a type of design known as a frame, as defined in [24]. We note that this construction is not necessary to prove the results in this paper, but we include it because we feel it is of independent interest and may be of use in the future.

A GDD in which every block has size m is denoted an m−GDD. Suppose $(\mathbf{X}, \mathcal{G}, \mathcal{A})$ is an m−GDD. A *holey parallel class* is a set of blocks which partition $\mathbf{X} \backslash G$ for some $G \in \mathcal{G}$. If the block set $\mathcal{A}$ can be partitioned into holey parallel classes, then the GDD is called an *m−frame*. The *type* of the frame is the type of the underlying GDD. In the construction, we shall make use of the fact that there are exactly $|G|/(k-1)$ holey parallel classes that partition $\mathbf{X}\backslash G$, for every $G \in \mathcal{G}$ (see [24] and [25]).

Theorem 2.8 *Suppose there exists an m−frame of type $\{t_1, \ldots, t_n\}$ and k MOLS of side m. Then there exist k OPILS of type $\{t_1, \ldots, t_n\}$.*

Proof: Suppose $(\mathbf{X}, \mathcal{G}, \mathcal{A})$ is an m−frame, where $\mathcal{G} = \{G_1, \ldots, G_n\}$. For $1 \le i \le n$, name the parallel classes having G_i for their hole $\mathcal{P}_{ij}$, where $1 \le j \le |G_i|/(m-1)$. For $1 \le i \le n$, partition G_i into disjoint $(m-1)$−subsets, denoted G_{ij}, $1 \le j \le |G_i|/(m-1)$. For $1 \le i \le n$ and $1 \le j \le |G_i|/(m-1)$, let $\phi_{ij} : \{2, \ldots, m\} \to G_{ij}$ be a bijection. Also, for $1 \le h \le k+2$, define $\mathbf{X}_h = \mathbf{X} \times \{h\}$, and let $\mathbf{Y} = \cup_{h=1}^{k+2} \mathbf{X}_h$.

We shall construct an incomplete $\mathrm{TD}(k+2, |\mathbf{X}|) - \cup_{i=1}^{n} \mathrm{TD}(k+2, |G_i|)$, which is equivalent to k OPILS of type $\{t_1, \ldots, t_n\}$. The incomplete TD will have groups $\mathbf{X}_h$, $1 \le h \le k+2$, and the ith missing sub-TD will have groups $G_i \times \{h\}$, $1 \le h \le k+2$ $(1 \le i \le n)$.

The blocks are as follows. For any block B, $B \in \mathcal{P}_{ij}$, construct a resolvable $\mathrm{TD}(k+1, m)$ on point set $\cup_{h=1}^{k+1} B \times \{h\}$, having groups $B \times \{h\}$, $1 \le h \le k+1$. Denote the parallel classes by $\mathcal{Q}_{B\ell}$, $1 \le \ell \le m$. We can take the parallel class $\mathcal{Q}_{B1}$ to consist of the blocks $\{B_x : x \in B\}$, where $B_x = \{(x,1), \ldots, (x, k+1)\}$. For $2 \le \ell \le m$, define a set of blocks $\mathcal{B}_{B\ell}$ by adjoining the point $(\phi_{ij}(\ell), k+2)$ to every block in $\mathcal{Q}_{B\ell}$. Define $\mathcal{B}_B = \cup_{\ell=2}^{m} \mathcal{B}_{B\ell}$ and $\mathcal{B} = \cup_{B \in \mathcal{A}} \mathcal{B}_B$. Then $\mathcal{B}$ is the block set of the desired incomplete TD.

We need to verify that two points from different groups and different holes occur in a unique block. So, let (x, h) and (x', h') be two points, where $1 \le h < h' \le k+2$ and x and x' are from different groups of $\mathcal{G}$. There are two cases to consider: (i)

$h' \leq k+1$, and (ii) $h' = k+2$. In case (i), find the unique block B such that $\{x, x'\} \subseteq B$. Then, the pair $\{(x,h),(x',h')\} \subseteq \mathcal{B}_B$. In case (ii), there are unique values i and j such that $x' \in G_{ij}$. Then there is a unique block $B \in P_{ij}$ such that $x \in B$. Finally, define $\ell = (\phi_{ij})^{-1}(x')$. Within the parallel class $Q_{B\ell}$, there is a unique block containing the point (x,h). Hence, $\{(x,h),(x',h')\} \subseteq \mathcal{B}_{B\ell}$. This completes the proof. $\Box$

Example 2.1 *Let* $\mathbf{X} = \{0,1,2,3,4,5,6,7\}$, $\mathcal{G} = \{G_0, \ldots, G_3\}$ *where* $G_i = \{i, i+4\}$ *for* $0 \leq i \leq 3$, *and* $\mathcal{A}$ *is obtained by developing the block* $\{0,1,6\}$ *modulo* 8. *Then* $(\mathbf{X}, \mathcal{G}, \mathcal{A})$ *is a* 3$-$*GDD of type* 2^4. *For* $0 \leq i \leq 3$, *define* $\mathcal{P}_i = \{\{i+1, i+2, i+7\}, \{i+3, i+5, i+6\}\}$ *(where all points are reduced modulo* 8*). Then, we have a* 3$-$*frame. If we apply the construction of Theorem 2.8, we get the following set of two OPILS of type* 2^4.

	6	3	2		7	1	5
6		7	4	3		0	2
3	7		0	5	4		1
2	4	0		1	6	5	
	3	5	1		2	7	6
7		4	6	2		3	0
1	0		5	7	3		4
5	2	1		6	0	4	

	3	1	5		2	7	6
7		0	2	6		3	4
5	4		1	3	7		0
1	6	5		2	0	4	
	2	7	6		3	1	5
6		3	4	7		0	2
3	7		0	5	4		1
2	0	4		1	6	5	

As another application of Theorem 2.8, we can construct three OPILS of type 6^n if $n = 5, 9$ or 13. This follows immediately form the existence of 4$-$frames of these types (see [24] and [21]); and from the existence of three MOLS of order 4. (OPILS of these types have previously been constructed by Lamken in [14], using other methods.)

3 Hole size six

In [14], Lamken proves the existence of three OPILS of type 6^n with 25 possible exceptions. We can remove 15 of these exceptions, leaving 10 values unsettled. We give a complete existence proof here, since it is quite short.

Lemma 3.1 *There exist three OPILS of type 6^n if $n = 5, 9, 13$ or 17.*

Proof: This follows immediately from Theorem 4.3 of [14]. □

Lemma 3.2 *There exist three OPILS of type 6^n if $n = 8$ or 15.*

Proof: There exist $(v,7,1)$–BIBDs for $v = 49$ and 91 [11]. Deleting a point from each BIBD, we get 7–GDDs of types 6^8 and 6^{15}. Apply Lemma 2.1, noting that three OPILS of type 1^7 exist. □

Lemma 3.3 *There exist three OPILS of type 6^6.*

Proof: Apply Lemma 2.2 with $t = 5$, $k = 5$, $m = 6$, $u_1 = u_2 = 1$ and $u_3 = u_4 = 2$. The required incomplete TD(5,7)– TD(5,1) and TD(5,8)– TD(5,2) exist by Lemma 1.4 and Lemma 1.6 respectively. □

Lemma 3.4 *Suppose there exist three OPILS of type 6^n and four MOLS of order q, where $q - 1 \leq 6n \leq 2(q-1)$. Then there exist three OPILS of type 6^{q+n}.*

Proof: Apply Lemma 2.2 with $t = 5$, $k = 5$, $m = 6$, $u_i = 1$ or 2, $1 \leq i \leq q-1$. Incomplete TD(5,7)– TD(5,1) and TD(5,8)– TD(5,2) exist, as noted in the proof of Lemma 3.3. We obtain three OPILS of type $6^q(6n)^1$. Fill in the hole of size $6n$ with OPILS of type 6^n. □

Lemma 3.5 *There exist three OPILS of type 6^m if $21 \leq m \leq 302$, $m \neq 23$.*

Proof: We apply Lemma 3.4 with the values of n indicated in the following table. For each n, three OPILS of type 6^n exist. We get three OPILS of type 6^m for all m, $4n + 1 \leq m \leq 7n + 1$, except when $m - n \in \{6, 10, 14, 18, 22, 26, 30, 34, 42\}$. These possible exceptions are listed in the third column of the table. It is easy to check that all stated values of m are obtained. □

n	interval covered	possible exceptions
5	$21 - 36$	23, 27, 31, 35
6	$25 - 43$	28, 32, 36, 40
8	$33 - 57$	34, 38, 42, 50
9	$37 - 64$	39, 43, 51
15	$61 - 106$	
25	$101 - 176$	
43	$173 - 302$	

Theorem 3.6 *There exist three OPILS of type 6^n if $n \geq 5$, $n \neq 7, 10, 11, 12, 14, 16, 18, 19, 20$ or 23.*

Proof: By Lemmas 3.1, 3.2, 3.3 and 3.5, the result is true for $n \leq 302$. Hence, assume $n \geq 303$. Write $n = 5q + a$ where $a \in \{5, 6, 8, 9, 17\}$; then $q \geq 58$ and a TD$(6, q)$ exists. Define $k = 5$, $t = q$ and $m = 30$. For $1 \leq i \leq q - 1$, take $u_i \in \{0, 1, 5\}$ such that $6a = \sum_{i=1}^{q-1} u_i$. Now, apply Lemma 2.2, noting that TD$(5, 30)$ and incomplete TD$(5, 31)-$ TD$(5, 1)$ and TD$(5, 35)-$ TD$(5, 5)$ all exist (this last example comes from Lemma 1.5). We obtain three OPILS of type $30^q(6a)^1$. Fill in the holes with three OPILS of types 6^5 and 6^a, thus producing three OPILS of type $6^{5q+a} = 6^n$. □

4 Hole sizes two, three and ten

First, we deal with hole size three. When n is odd, $n \geq 5, n \neq 15$, three OPILS of type 3^n exist by [19] and [14]. Type 3^{15} can be done as follows.

Lemma 4.1 *There exist three OPILS of type 3^{15}.*

Proof: Use the 7–GDD of type 3^{15} constructed by Baker [2]. Give every point weight 1 and apply Lemma 2.1. □

In the case of an even number of holes, we are able to remove five of Lamken's exceptions from [14], namely orders 30, 38, 44, 48 and 58. We give an alternate proof of the existence result here.

Lemma 4.2 *There exist three OPILS of type 3^n if n is even, $n \geq 26$, $n \neq 28, 32$ or 34.*

Proof: Write $n = 5q + a$ where $a \in \{1, 5, 7, 9, 13\}$ and q is odd. Define $k = 5$, $t = q$ and $m = 15$. If possible, take $u_i \in \{0, 1, 5\}$ for $1 \leq i \leq q - 1$, such that $3a = \sum_{i=1}^{q-1} u_i$. This can be done in the following cases: $n \geq 30$, $n \equiv 0 \bmod 10$; $n \geq 42$, $n \equiv 2 \bmod 10$; $n \geq 54$, $n \equiv 4 \bmod 10$; $n \geq 26$, $n \equiv 6 \bmod 10$; and $n \geq 78$, $n \equiv 8 \bmod 10$. In each case, $t \geq 5$, so there is a TD$(6, t)$. Now, apply Lemma 2.2, noting that TD$(5, 15)$ and incomplete TD$(5, 16)-$ TD$(5, 1)$ and TD$(5, 20)-$ TD$(5, 5)$ all exist (Lemma 1.5). We obtain three OPILS of type $15^q(3a)^1$. Fill in the holes with three OPILS of types 3^5 and 3^a, thus producing three OPILS of type $3^{5q+a} = 3^n$.

We still need to provide constructions for the following cases: $n = 44$; and $38 \leq n \leq 68$, $n \equiv 8 \bmod 10$. For $n \in \{38, 48, 58, 68\}$, we proceed as follows. Write $n = 5t + 3$, where $t \in \{7, 9, 11, 13\}$. Let $k = 5$, $m = 15$ and $\ell = 3$; and define $u_1 = 1$, $u_2 = 5$, $v_1 = v_2 = v_3 = 1$ and $v_j = 0$ if $4 \leq j \leq t - 1$. Apply Lemma 2.5, constructing three OPILS of type $15^{t-1}21^1 3^1$. Filling in holes with three OPILS of types 3^5 and 3^7, we get three OPILS of type $3^{5t+3} = 3^n$.

Finally, we need to consider $n = 44$. Write $44 = 7 \times 5 + 2 + 7$. Let $k = 5$, $m = 15$ and $\ell = 3$; and define $u_1 = 1$, $u_2 = 5$, $v_1 = \ldots = v_4 = 5$, $v_5 = 1$ and $v_6 = 0$. Apply Lemma 2.5, constructing three OPILS of type $15^6 21^2$. Filling in holes with three OPILS of types 3^5 and 3^7, we get three OPILS of type 3^{44}. $\square$

Summarizing previous results, we have

Theorem 4.3 *There exist three OPILS of type 3^n if $n \geq 5$, $n \neq 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 28, 32$ or 34.*

Next, we discuss the case of hole size two. We can construct four new examples.

Lemma 4.4 *There exist three OPILS of type 2^{36} and 2^{54}.*

Proof: For type 2^{36}, apply Lemma 2.2 with $t = 5$, $k = 5$, $m = 14$, $u_1 = u_2 = 1$ and $u_3 = u_4 = 0$. This gives us three OPILS of type $14^5 2^1$. Filling in holes with three OPILS of type 2^7, we get three OPILS of type 2^{36}.

For type 2^{54}, apply Lemma 2.2 with $t = 7$, $k = 5$, $m = 14$, $u_1 = u_2 = 4$, $u_3 = u_4 = 1$ and $u_5 = u_6 = 0$ (an incomplete TD(5,18)− TD(5,4) exists by Lemma 1.6). We get three OPILS of type $14^7 10^1$. Filling in holes with three OPILS of types 2^5 and 2^7, we get three OPILS of type 2^{54}. $\square$

Lemma 4.5 *There exist three OPILS of type 2^{26} and 2^{80}.*

Proof: For type 2^{26}, apply Lemma 2.6 with $t = 5$, $k = 5$, $m = 10$, $u_1 = u_2 = 1$, $u_3 = u_4 = 0$, $s = 1$ and $a_1 = 2$. We need an incomplete TD(5,10)− TD(5,2), an incomplete TD(5,11)− TD(5,2)− TD(5,1), and three OPILS of type 2^5. This gives us three OPILS of type $10^5 2^1$. Filling in holes with three OPILS of type 2^5, we get three OPILS of type 2^{26}.

For type 2^{80}, apply Lemma 2.6 with $t = 9$, $k = 5$, $m = 14$, $u_1 = \ldots = u_8 = 4$, $s = 14$ and $a_1 = \ldots = a_{14} = 1$. An incomplete TD(5,18)− TD(5,4) − 14 TD(5,1) is equivalent to three OPILS of type $1^{14} 4^1$. Let $\epsilon = 2$ and observe that there are three OPILS of type $1^9 2^1$. We get three OPILS of type $14^9 34^1$. Filling in holes with three OPILS of types 2^7 and 2^{17}, we get three OPILS of type 2^{80}. $\square$

Theorem 4.6 *There exist three OPILS of type 2^n if $n \geq 5$, $n \neq 6, 8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 28, 30, 32, 34, 38, 42, 44, 48$ or 52.*

Proof: Combine Theorem 1.1 and Lemmas 4.4 and 4.5. $\square$

Finally, we consider the case of hole size ten. We can remove all but five of the possible exceptions from [14]. We give a complete proof of our existence results, since it is quite short.

Lemma 4.7 *There exist three OPILS of type* 10^n *if* n *is even,* $n \geq 12$, $n \neq 16$.

Proof: Apply Lemma 2.6 with $t = n - 1$, $k = 5$, $m = 10$, $u_i = 0$ or 1 such that $\sum_{i=1}^{t-1} u_i = 10$, $s = 1$ and $a_1 = 2$. We need an incomplete TD(5,10)– TD(5,2), an incomplete TD(5,11)– TD(5,2)– TD(5,1). Three OPILS of type 2^t exist by Theorem 4.6 since t is odd and $t \neq 15$. This gives us three OPILS of type 10^n. □

Theorem 4.8 *There exist three OPILS of type* 10^n *if* $n \geq 5$, $n \neq 6$, 8, 10, 15 *or* 16.

Proof: If n is even, apply Lemma 4.7. If n is odd, $n \neq 15$, take three OPILS of type 2^n and apply the direct product construction (Lemma 2.7) with $m = 5$. □

5 Other hole sizes

In this section, we discuss the hole sizes not yet considered, namely $h \geq 2$, $h \neq 2$, 3, 6 or 10.

Lemma 5.1 *Suppose* $h \geq 2$, $h \neq 2, 3, 6$ *or* 10, *and* $n \geq 5$, $n \neq 6$, 10, 18, 22, *or* 26. *Then there exist three OPILS of type* h^n.

Proof: Take three OPILS of type 1^n and apply the direct product construction (Lemma 2.7) with $m = h$. □

In dealing with the remaining cases, $n = 6$, 10, 18, 22, 26, we shall require some results on incomplete transversal designs TD$(5, v)$– TD$(5, u)$. The following recursive construction is Proposition 3.4 of [7].

Lemma 5.2 *Suppose there is a TD*$(k+1, t)$, *a TD*(k, m) *and a TD*$(k, m+1)$, *and let* $0 \leq s \leq t$. *Then an incomplete TD*$(k, mt+s)$– *TD*(k, s) *exists. Further, if a TD*(k, s) *exists, then the following exist: an incomplete TD*$(k, mt+s)$– *TD*(k, t), *an incomplete TD*$(k, mt+s)$– *TD*(k, m) *(if* $s \neq t$*) and an incomplete TD*$(k, mt+s)$– *TD*$(k, m+1)$ *(if* $s \neq 0$*).*

We obtain the following corollary.

Corollary 5.3 *Suppose there is a TD*$(6, t)$ *and a TD*$(5, s)$, *where* $0 \leq s \leq t$. *Then an incomplete TD*$(5, 4t+s)$– *TD*$(5, t)$ *exists.*

The following direct construction comes from [23].

Lemma 5.4 *If* $h = 3a + 1$ *is a prime exceeding 7, then there is an incomplete TD*$(5, 4a+1)$– *TD*$(5, a)$.

Finallly, we use a singular direct product construction of Horton. The following is Theorem 4 of [12].

Lemma 5.5 *Suppose there exist $k-2$ OPILS of type 1^n, a $TD(k, v-w)$ and an incomplete $TD(5,v)$– $TD(5,w)$. Then an incomplete $TD(k, n(v-w)+w)$– $TD(k,n)$ exists.*

We can now proceed to the construction of three OPILS of type h^n when $n = 6, 10, 18, 22, 26$. Our main tool will be Lemma 2.2. We shall split the problem into three parts according to the congruence class of h modulo 3.

Lemma 5.6 *If $h \equiv 0 \bmod 3$, $h \geq 9$, and $n \in \{6, 10, 18, 22, 26\}$, then there exist three OPILS of type h^n, except possibly when $(h,n) = (18,6)$.*

Proof: First, suppose $h \neq 18, 30$. Write $h = 3a$ and let $t = n-1$, $k = 5$, $m = h$, $u_1 = u_2 = u_3 = a$ and $u_4 = \ldots = u_{t-1} = 0$. Apply Lemma 2.2, noting that the following ingredients exist: a TD$(5, 3a)$, an incomplete TD$(5, 4a)$– TD$(5, a)$ (from Lemma 1.5 if $h \neq 9$, and from Lemma 1.7 if $h = 9$) and a TD$(6, n-1)$. We get three OPILS of type h^n.

For $h = 30$, let $t = n-1$, $k = 5$, $m = h$, $u_1 = u_2 = 8$ and $u_3 = u_4 = 7$ and $u_5 = \ldots = u_{t-1} = 0$. An incomplete TD$(5,38)$– TD$(5,8)$ and incomplete TD$(5,37)$– TD$(5,7)$ exist from Lemma 1.6 and a TD$(6,n-1)$ exists. Apply Lemma 2.2.

Finally, suppose $h = 18$ and $n \geq 10$. Let $t = n-1$, $k = 5$, $m = h$, $u_1 = \ldots = u_4 = 4$ and $u_5 = u_6 = 1$ and $u_7 = \ldots = u_{t-1} = 0$. An incomplete TD$(5,22)$– TD$(5,4)$ exists from Lemma 1.7, and TD$(5,19)$ and TD$(6,n-1)$ exist. Apply Lemma 2.2. The case $h = 18, n = 6$ remains as a possible exception. □

Lemma 5.7 *If $h \equiv 1 \bmod 3$, $h \neq 10$ and $n \in \{6, 10, 18, 22, 26\}$, then there exist three OPILS of type h^n.*

Proof: First, assume $h \neq 55$. Write $h = 3a+1$ and let $t = n-1$, $k = 5$, $m = h$, $u_1 = u_2 = u_3 = a$, $u_4 = 1$ and $u_5 = \ldots = u_{t-1} = 0$, and apply Lemma 2.2. We need the following ingredients: a TD$(5, 3a+1)$, a TD$(5, 3a+2)$, an incomplete TD$(5, 4a+1)$– TD$(5, a)$ and a TD$(6, n-1)$. The TDs with five and six groups come from Lemmas 1.3 and 1.4. The incomplete TD is obtained as follows. If $a \geq 5$, $a \neq 6$, 10, 18, 22, 26, 30, 34, or 42, then there is a TD$(6, a)$, and hence we can apply Corollary 5.3 with $b = 1$ to construct the incomplete TD. When $a = 4$, 6, 10, 22, 26, 34, or 42, then apply Lemma 5.4. When $a = 1$, an incomplete TD$(5,5)$– TD$(5,1)$ clearly exists; and when $a = 2$, an incomplete TD$(5,9)$– TD$(5,2)$ exists from Lemma 1.6. For $a = 30$, apply 5.5 with $k = 4$, $n = 30$, $v = 5$ and $w = 1$.

There remains the case $h = 55$ (i.e. $a = 18$). Here, we apply Lemma 2.2 with $t = n-1$, $k = 5$, $m = h$, $u_1 = u_2 = u_3 = 17$, $u_4 = 4$ and $u_5 = \ldots = u_{t-1} = 0$. We now need a TD$(5,55)$, an incomplete TD$(5,72)$– TD$(5,17)$, an incomplete

TD(5, 59)− TD(5, 4) and a TD(6, $n-1$). If we apply 5.3 with $t = 17$ and $s = 4$, we get an incomplete TD(5, 72)− TD(5, 17). To construct the other incomplete TD, start with a TD(5, 12) and delete a point to produce a $\{5,12\}$−GDD of type $4^{12}11^1$. Give every point weight one and apply 2.1. obtaining three OPILS of type $4^{12}11^1$. Then, fill in holes with three MOLS of sides 4 and 11, but leave one hole of size 4 empty. □

Lemma 5.8 *If $h \equiv 2 \bmod 3$, $h \geq 5$ and $n \in \{6, 10, 18, 22, 26\}$, then there exist three OPILS of type h^n, except possibly for $(h,n) = (5,6)$, $(5,10)$, $(5,18)$, $(5,22)$, $(11,6)$, $(11,10)$, $(14,6)$, $(17,6)$, $(23,6)$, $(23,10)$, $(23,18)$ or $(23,22)$.*

Proof: Write $h = 3a + 8$ and let $t = n - 1$, $k = 5$, $m = h$, $u_1 = u_2 = u_3 = a$, $u_4 = 8$ and $u_5 = \ldots = u_{t-1} = 0$. In order to apply Lemma 2.2, we need the following: a TD(5, $3a + 8$), an incomplete TD(5, $4a + 8$)− TD(5, a), an incomplete TD(5, $3a + 16$)− TD(5, 8) and a TD(6, $n - 1$). As before, the required TDs with five and six groups exist. The incomplete TDs exist in the following cases. If $a \geq 8$, $a \neq 10$, 14, 18, 22, 26, 30, 34, or 42, then there is an incomplete TD(5, $4a + 8$)− TD(5, a) by Corollary 5.3; and when $a = 7$, an incomplete TD(5, $4a+8$)− TD(5, a) exists by Lemma 1.6. If $a \geq 6$, $a \neq 10$, then there is an incomplete TD(5, $3a+16$)− TD(5, 8) by Lemma 1.6. Hence, we are finished if $h \neq 5$, 8, 11, 14, 17, 20, 23, 26, 38, 50, 62, 74, 86, 98, 110 or 134.

Next, we write $h = 3a + 5$ and let $t = n - 1$, $k = 5$, $m = h$, $u_1 = u_2 = u_3 = a$, $u_4 = 5$ and $u_5 = \ldots = u_{t-1} = 0$. Now, we need a TD(5, $3a + 5$), an incomplete TD(5, $4a+5$)− TD(5, a), an incomplete TD(5, $3a+10$)− TD(5, 5) and a TD(6, $n-1$). As before, the TDs exist. Incomplete TDs exist as follows. If $h = 20$, 26, 38, 50, 62, 74, 86, 98, 110 or 134 , then we construct an incomplete TD(5, $4a + 5$)− TD(5, a) from Corollary 5.3. If $h = 38$, 50, 62, 86, 98, 110 or 134 , then we construct an incomplete TD(5, $3a + 10$)− TD(5, 5) from Lemma 5.2 using the equation $h + 5 = 4t + 7$, where it can be checked in each case that a TD(6, t) exists. If $h = 74$, then instead use the equation $79 = 4 \times 17 + 11$. If $h = 26$, then we first construct three OPILS of type $4^6 6^1$ from Lemma 2.3 with $k = 5$, $\ell = 3$, $t = 7$, $m = 4$, $u_1 = u_2 = 1$ and $u_4 = \ldots = u_6 = 0$. Then, fill in holes using $\epsilon = 1$ in Lemma 2.4, leaving a hole of size 5 empty. If $h = 20$, then an incomplete TD(5, 25)− TD(5, 5) exists from Corollary 5.2. Hence, we can apply Lemma 2.2 in all these cases.

We have yet to do the cases $h = 5$, 8, 11, 14, 17 and 23. When $h = 8$, we let $t = n - 1$, $k = 5$, $m = 8$, $u_1 = u_2 = u_3 = u_4 = 2$ and $u_5 = \ldots = u_{t-1} = 0$. We have a TD(5, 8), an incomplete TD(5, 10)− TD(5, 2) (Lemma 1.6) and a TD(6, $n - 1$). Apply Lemma 2.2.

When $(h,n) = (11,18)$, $(11,22)$, $(11,26)$ or $(23,26)$, we can apply Lemma 2.2 with $t = n - 1$, $k = 5$, $m = h$ and $u_i \in \{0,1\}$ such that $u = h$.

When $h = 14$ and $n = 10$, 18, 22, or 26, we can apply Lemma 2.2 with $t = n-1$, $k = 5$, $m = 14$ and $u_i \in \{0,1,4\}$ such that $u = 14$ (note that an incomplete TD(5, 18)− TD(5, 4) exists by Lemma 1.7). Similarly, when $h = 17$ and $n = 10$, 18, 22, or 26, we can apply Lemma 2.2 with $t = n-1$, $k = 5$, $m = 17$ and $u_i \in \{0,1,4\}$ such that $u = 17$ (an incomplete TD$(5,21)-$ TD$(5,4)$ exists by Lemma 1.7).

Finally, we can do the case $(h,n) = (5,26)$ as follows. Apply Lemma 2.2 with $t = 25$, $k = 5$, $m = 25$ and $u_1 = 5$ and $u_2 = \ldots = u_4 = 0$. An incomplete TD$(5,30)-$ TD$(5,5)$ exists by Lemma 1.7. We get three OPILS of type $25^5 5^1$. Fill in the holes of size 25 with three OPILS of type 5^5, obtaining three OPILS of type 5^{26}. □

Summarizing, we have

Theorem 5.9 *If $h \geq 2$, $h \notin \{2,3,6,10\}$, and $n \geq 5$, then there exist three OPILS of type h^n, except possibly for $(h,n) = (5,6)$, $(5,10)$, $(5,18)$, $(5,22)$, $(11,6)$, $(11,10)$, $(14,6)$, $(17,6)$, $(18,6)$, $(23,6)$, $(23,10)$, $(23,18)$ or $(23,22)$.*

Proof: Combine Lemmas 5.1, 5.6, 5.7 and 5.8. □

We have now discussed all possible hole sizes, so we have our main result.

Theorem 5.10 *If $h \geq 2$ and $n \geq 5$, then there exist three OPILS of type h^n, except possibly for the 61 possible exceptions (h,n) listed in Table 1.*

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