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On minimum vertex covers of generalized Petersen graphs

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Abstract

For natural numbers n and k ($n > 2k$), a generalized Petersen graph $P(n,k)$, is defined by vertex set $\{u_i, v_i\}$ and edge set $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k}\}$; where $i = 1, 2, \ldots, n$ and subscripts are reduced modulo n. Here first, we characterize minimum vertex covers in generalized Petersen graphs. Second, we present a lower bound and some upper bounds for $\beta(P(n,k))$, the size of minimum vertex cover of $P(n,k)$. Third, in some cases, we determine the exact values of $\beta(P(n,k))$. Our conjecture is that $\beta(P(n,k)) \leq n + \lceil \frac{n}{5} \rceil$, for all n and k.

1 Introduction and preliminaries

For the definition of basic concepts not given here, one may refer to a text book in graph theory, for example [11]. A set Q of vertices of a graph $G = (V, E)$ is called a vertex cover, if each edge in E has at least one endpoint in Q. A vertex cover with minimum size in a graph G is called a minimum vertex cover of G and its size is denoted by $\beta(G)$. It is well-known that the vertex-cover problem is an NP-complete problem [6, p. 1006]. Therefore, many attempts are made to find lower and upper bounds, and exact values of $\beta(G)$ for special classes of graphs.

This paper is aimed toward studying the vertex cover problem for the class of generalized Petersen graphs. In a generalized Petersen graph $P(n, k)$, as defined in the abstract, let $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. We call two vertices u_i and v_i the twin of each other and refer to the edge between them as a spoke. Also, by the twin of S, where S is a subset of V, we mean the set that contains twins of all members of S. Moreover, the edges with both endpoints in U are called U-edges and the edges with both endpoints in V are called V-edges. A maximal subset of consecutive vertices of V in a vertex cover Q is called a strip of Q (two vertices of V are consecutive if they have circular consecutive subscripts). We define the size of a strip by the number of its vertices. Note that the size of a strip may be equal to 1. Also, we call a strip odd if it has an odd size. Finally, we call the set of m consecutive twins, namely $\{u_{i+1}, u_{i+2}, \ldots, u_{i+m}\} \cup \{v_{i+1}, v_{i+2}, \ldots, v_{i+m}\}$, an m-sector of the generalized Petersen graph. In Figure 1 a vertex cover is shown for $P(16, 5)$. For example, the set $\{v_{16}, v_1, v_2, v_3, v_4\}$ is a strip.

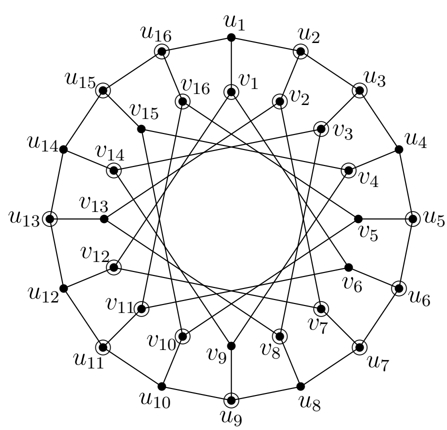

Figure 1: A vertex cover of P(16,5).

We now introduce some more definitions and conventions that will be used in the following sections. First, *from this point on* for brevity, we use cover instead of vertex cover. Second, we show covers by lower case letters, because we consider them as a function on $U \cup V$, which defines the selected and unselected vertices. In addition, by $|c|$ we mean the number of vertices selected by a cover c. Third, we adopt the convention that all subscripts of vertices of a generalized Petersen graph, when they are out of the range $\{1, 2, \ldots, n\}$, to be reduced modulo n to belong to this set. Fourth, in any generalized Petersen graph a cover which selects all V-vertices, besides some U-vertices, is called a trivial cover, and it is not a minimum cover. We denote by $\mathcal{C}(P(n,k))$ the set of all non-trivial covers of $P(n,k)$. Without loss of correctness *from this point on* by a cover we mean a non-trivial cover. This assumption is needed for Lemma 1, part (a).

It seems that Watkins [10] was the first who introduced this class of generalized Petersen graphs and conjectured that they have a Tait coloring, apart from $P(5,2)$. This conjecture later was proved in [5]. Since then this class of graphs has been studied widely because of its interesting traits. There are papers discussing topics such as tough sets, labeling problems, wide diameters, and coloring of generalized Petersen graphs. For example papers [1], [3] and [9] are about the hamiltonian character of generalized Petersen graphs. Also crossing numbers of this class are studied in papers such as [7] and [8]. Recently vertex domination of generalized Petersen graphs has been studied [2].

This paper is organized as follows. In Section 2 we characterize properties of minimum vertex cover of generalized Petersen graphs by defining a new quantity. In Section 3 we introduce a lower bound and some upper bounds for various cases of these graphs, namely:

- $\beta(P(n,k)) \geq n + \frac{(n,k)+1}{2}$, for all odd n, where (n,k) is the greatest common divisor of n and k;
- $\beta(P(n,k)) \leq n + \frac{k+1}{2}$ for all odd k;
- for all $m < k$,
  $\beta(P(n,k)) \leq \frac{n}{m}\beta(P(m, k \pmod{m}))$, where $m \mid n$ and
  $\beta(P(n,k)) \leq \lfloor \frac{n}{m} \rfloor \beta(P(m, k \pmod{m})) + 2k$, otherwise.

We note that most of these bounds are sharp. In Section 4, we determine the exact values of $\beta(P(n,k))$:

- for $k = 1$ and 3;
- when n is even and k is odd;
- when both n and k are odd and $k \mid n$.

Section 5 contains concluding remarks.

2 Properties of minimum vertex cover

Let $S = \{v_i, v_{i+1}, \ldots, v_{i+m}\}$ be a strip of a cover c. By optimally selecting from twins of S, we mean choosing from $u_i, u_{i+1}, \ldots, u_{i+m}$ alternatively beginning with

u_{i+1}. For example, the cover in Figure 1 is optimally selected from twins of the strip $\{v_{10}, v_{11}, v_{12}\}$, but not from twins of the strip $\{v_{16}, v_1, v_2, v_3, v_4\}$. From any cover c we construct a semi-optimal cover, so(c), in the following way: First, we choose all vertices from V selected by c. Second, we choose twins of all vertices of V which are not selected by c. Finally, we optimally select from twins of strips of c. In Lemma 1, we show that so(c) is a cover, whose size is not greater than c.

Lemma 1 *Let c be a cover of the generalized Petersen graph $P(n, k)$; then*

(a) $\mathrm{so}(c)$ *is a cover of* $P(n, k)$*, and*

(b) $|\mathrm{so}(c)| \leq |c|$.

Proof. (a) Note that the vertices selected from V are the same in c and so(c), so all V-edges are covered by so(c). Also, it is evident that all spokes are covered by so(c). We show that all U-edges are also covered by so(c). For, suppose in contradiction that an edge u_iu_{i+1} is not covered by so(c). Then the vertices v_i and v_{i+1} must have been selected by c and are in the same strip, say S. Since twins of S must be optimally selected, at least one of u_i or u_{i+1} must be selected in so(c) (Note that as stated before, we assume that in the covers which we consider, not all vertices of V are selected). This is in contradiction with our assumption.

(b) Note that the only possible difference between c and so(c) are in their selection from twins of their strips. It is enough to show that for any strip $S = \{v_i, v_{i+1}, \ldots, v_{i+m-1}\}$ of c and so(c), the number of selected vertices from twins of S in so(c) is not greater than the number of selected vertices from twins of S in c. To prove this, first, note that so(c) selects exactly $\lceil \frac{m-1}{2} \rceil$ vertices from the twins of S. Second, note that $m-1$ edges between $u_i, u_{i+1}, \ldots, u_{i+m-1}$ must be covered by the twins of S. Since each vertex can cover at most two U-edges, the minimum number of vertices that must be chosen from the twins of S in any cover is $\lceil \frac{m-1}{2} \rceil$. This shows that so($c$) chooses the minimum possible vertices from the twins of S. ∎

Corollary 1 *For a minimum cover c^*, we have $|so(c^*)| = |c^*|$.*

Now we are ready to present our main characterizing theorem. Let $a(c)$ be the number of vertices selected from V by a cover c of a generalized Petersen graph $P(n, k)$, and let $b(c)$ denote the number of odd strips of c. Also, let $d(P(n, k)) = \min_{c \in \mathcal{C}(P(n,k))}(a(c) - b(c))$.

Theorem 1 *For any minimum vertex cover c^* of $P(n, k)$ we have, $a(c^*) - b(c^*) = d(P(n, k))$.*

Proof. Consider an arbitrary cover c of $P(n, k)$. Suppose there are s strips in c with the sizes $m_1, m_2, \ldots, m_s$. Without loss of generality, we assume the first $b(c)$ strips in this list are all odd. In so(c), there are $a(c) = \sum_{i=1}^{s} m_i$ vertices selected from V and $n - a(c) + \sum_{i=1}^{s} \lceil \frac{m_i - 1}{2} \rceil$ vertices selected from U (the last term of this expression shows the number of vertices selected from the twins of strips of c) are selected in $\mathrm{so}(c)$. Thus, we obtain

$$
\begin{aligned}
|\mathrm{so}(c)| &= a(c) + n - a(c) + \sum_{i=1}^{s} \left\lceil \frac{m_i - 1}{2} \right\rceil = n + \sum_{i=1}^{s} \left\lceil \frac{m_i - 1}{2} \right\rceil \\
&= n + \sum_{i=1}^{b(c)} \left\lceil \frac{m_i - 1}{2} \right\rceil + \sum_{i=b(c)+1}^{s} \left\lceil \frac{m_i - 1}{2} \right\rceil = n + \sum_{i=1}^{b(c)} \frac{m_i - 1}{2} + \sum_{i=b(c)+1}^{s} \frac{m_i}{2} \\
&= n + \sum_{i=1}^{s} \frac{m_i}{2} - \frac{b(c)}{2} = n + \frac{a(c) - b(c)}{2}.
\end{aligned} \tag{1}
$$

Now, take the contradictory assumption that $d(P(n,k)) < a(c^*) - b(c^*)$. Clearly, a cover exists with $a(c) - b(c) = d(P(n,k))$. Therefore, by applying Equation (1) we have

$$
\begin{aligned}
a(c) - b(c) < a(c^*) - b(c^*) &\Rightarrow n + \frac{a(c) - b(c)}{2} < n + \frac{a(c^*) - b(c^*)}{2} \\
&\Rightarrow |\mathrm{so}(c)| < |\mathrm{so}(c^*)|.
\end{aligned}
$$

From Corollary 1, we reach the contradictory result that $|\mathrm{so}(c)| < |c^*|$ where $\mathrm{so}(c)$ is a cover by Lemma 1(a). By the definition of $d(P(n,k))$, $d(P(n,k)) \leq a(c^*) - b(c^*)$. Thus, for every minimum cover c^*, we must have $a(c^*) - b(c^*) = d(P(n,k))$. ■

Remark 1 *Note that the converse of Theorem 1 is not necessarily true and a cover c with $a(c) - b(c) = d(P(n,k))$ may not be a minimum one. This fact is clear, because $d(P(n,k))$ imposes a constraint only for the selections from V and not on all vertices.*

Now we use semi-optimal covers to characterize minimum covers in generalized Petersen graphs.

Proposition 1 *In a minimum cover c^* of $P(n,k)$, the maximum size of a strip is at most $2k$ when k is odd, and $2k+1$ when k is even.*

Proof. First, we prove the case where k is odd. Let m be the maximum size of strips of c^* and suppose for a contradiction that $m > 2k$. We show vertices of this strip by $v_i, v_{i+1}, \ldots, v_{i+m-1}$. Since u_{i+k}, v_i, and v_{i+2k} are selected by $\mathrm{so}(c^*)$, there is no need for selection of v_{i+k} in $\mathrm{so}(c^*)$, which is in contradiction with Corollary 1. For even k, the process of the proof is similar. ■

The following lemma is trivial.

Lemma 2 *In a minimum cover c^* of a generalized Petersen graph $P(n,k)$, if a vertex $v \in V$ and its two adjacent vertices in V are selected in c^*, then there exists another minimum cover c of $P(n,k)$ that has the same selection from V as c^* with one exception that it does not select v.*

Theorem 2 *There is a minimum cover of $P(n,k)$ such that the maximum size of its strips is at most $k+1$ when k is odd, and $k+2$ when k is even.*

Proof. Let $\mathcal{C}'(P(n,k))$ be the set of covers such that the maximum size of their strips is smallest among all minimum covers, and let c^* be a cover in $\mathcal{C}'(P(n,k))$ with the smallest number of maximum strips. Let m be the maximum size of strips of c^*. We claim that c^* is a minimum cover satisfying desired condition of the theorem.

First, we prove the theorem when k is odd. By contradiction suppose that $m > k+1$. Let the vertices of a strip with the maximum size be $v_i, v_{i+1}, \ldots, v_{i+m-1}$. The vertex v_{i+2k} cannot be selected by c^*, otherwise if it is selected, then the vertex v_{i+k} satisfies the conditions of Lemma 2, and it means that there is another minimum cover c' that agrees with c^* in choosing from V, except on v_{i+k}. Therefore, in c', the stated strip is divided into two smaller ones and this contradicts the way we have chosen c^* among all covers. Similarly, v_{i+2k+1} must not be selected in c^*, also.

Now, we construct another minimum cover c from $\mathrm{so}(c^*)$ in the following way: c selects the same vertices as $\mathrm{so}(c^*)$ except v_{i+2k} instead of v_{i+k}. Since v_i and u_{i+k} are selected by $\mathrm{so}(c^*)$, c is a cover. In addition, the strip with size m is destroyed in c and we claim that the new strip has size less than m. This contradicts the way we have chosen c^*. To show the claim above, notice that v_{i+m} and v_{i+2k+1} are not selected in c. Therefore, v_{i+2k} is constructed a strip with the size at most $2k-m$, which is less than m.

In the case that k is even, the process of proof is similar. ■

3 A lower bound and some upper bounds

In this section, we present some bounds for $\beta(P(n,k))$. First, we introduce a lower bound.

Proposition 2 *If n is odd then we have $\beta(P(n,k)) \geq n + \frac{(n,k)+1}{2}$.*

Proof. First note that any vertex cover must select at least $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ vertices from U. Second, also we note that the vertices of V are in (n,k) cycles of size $\frac{n}{(n,k)}$, and a vertex cover must choose at least $\lceil \frac{n}{(n,k)}/2 \rceil$ vertices from each of these cycles, so at least $(n,k)\lceil \frac{n}{(n,k)}/2 \rceil = (n,k)(\frac{n}{(n,k)}+1)/2 = \frac{n+(n,k)}{2}$ vertices of V must be included in any vertex cover. This shows that when n is odd, a vertex cover of $P(n,k)$ has at least $\frac{n+1}{2} + \frac{n+(n,k)}{2} = n + \frac{(n,k)+1}{2}$ vertices. ■

Corollary 2 *For all odd n, we have $\beta(P(n,k)) \geq n+1$.*

Next, we find some upper bounds. The following simple bounds are the results of Theorem 1.

Proposition 3 *We have:*

(a) *if* $\frac{n}{(n,k)}$ *is odd, then* $\beta(P(n,k)) \leq n + \frac{n+(n,k)}{4}$*, and*

(b) *if* $\frac{n}{(n,k)}$ *is even, then* $\beta(P(n,k)) \leq n + \frac{n}{4}$*.*

Proof. (a) We find an upper bound by introducing a semi-optimal cover of $P(n,k)$. Since V consists of (n,k) pairwise disjoint cycles of length $\frac{n}{(n,k)}$, we can cover all V-edges by selecting alternatively $\lceil \frac{n}{(n,k)}/2 \rceil$ vertices from each cycle. Now, let c be a cover which consists of all vertices of U and $\lceil \frac{n}{(n,k)}/2 \rceil$ vertices from each cycle of V, chosen alternatively. By the Equation (1) in the proof of Theorem 1, we know that $|\mathrm{so}(c)| = n + \frac{a(c)-b(c)}{2}$. Note that

$$a(c) = (n,k) \left\lceil \frac{\frac{n}{(n,k)}}{2} \right\rceil = (n,k)\frac{\frac{n}{(n,k)}+1}{2} = \frac{n+(n,k)}{2}.$$

Thus, we obtain

$$\beta(P(n,k)) \leq |\mathrm{so}(c)| = n + \frac{a(c)-b(c)}{2} \leq n + \frac{a(c)}{2} = n + \frac{n+(n,k)}{4}.$$

(b) The proof is similar to (a). ∎

We now present some more powerful upper bounds.

Proposition 4 *If both n and k are odd, then* $\beta(P(n,k)) \leq n + \frac{k+1}{2}$*.*

Proof. Let c select the vertices of U with odd subscripts, and the vertices of V with even subscripts. It is easy to see that c covers all U-edges and all spokes. Moreover, c covers all V-edges except the ones in which both endpoints have odd subscripts. They are $\frac{k+1}{2}$ edges between the vertices v_{n-i} and v_{n-i+k}, for $i = 0, 2, 4, \ldots, k-1$. Now, if c selects one endpoint of each of these edges, it becomes a cover of size $n + \frac{k+1}{2}$. ∎

Now, we present a recursive upper bound in the following theorem. In this theorem, we find a cover by applying a fixed pattern on m-sectors of generalized Petersen graphs.

Theorem 3 *For a generalized Petersen graph $P(n,k)$, we have*

(a) *if* $m \mid n$*, then* $\beta(P(n,k)) \leq \frac{n}{m}\beta(P(m,r))$*, and*

(b) *if* $m \nmid n$*, then* $\beta(P(n,k)) \leq \left\lfloor \frac{n}{m} \right\rfloor \beta(P(m,r)) + 2k$

where $m < k$ *and* $r \equiv k \pmod m$*,* $m > 2r > 0$*.*

Proof. (a) In this case, we can partition $P(n,k)$ into $\frac{n}{m}$ sectors of size m. Now, we present a suitable pattern of selection from an m-sector and apply this pattern on all m-sectors of $P(n,k)$ such that selected vertices cover all edges of $P(n,k)$. Since

$r \equiv k \pmod{m}$, the ith vertex of an m-sector is adjacent to $(i+r)$th vertex of one of the next m-sectors, where $i+r$ is taken modulo m in the range of $\{1, 2, \ldots, m\}$. By considering this fact, it is not hard to see that we can find the desired pattern by finding a cover of $P(m, r)$. By cutting $P(m, r)$ between the edges (u_1, u_m) and (v_1, v_m), we have an m-sector with a pattern of selection which is defined by the cover of $P(m, r)$.

Applying this pattern repeatedly on m-sectors yields a cover of $P(n, k)$. Obviously, this selection covers all spokes and U-edges. Furthermore, all V-edges must be covered, because otherwise, if an edge $v_i v_{i+k}$ exists which is not covered, this means that the edge $v_{i \pmod{m}} v_{(i+k) \pmod{m}}$ or equivalently $v_j v_{j+r}$ is not covered in $P(m, r)$, where $j \equiv i \pmod{m}$, which is a contradiction.

(b) In this case, we can apply the idea of previous case with some modifications. We partition $P(n, k)$ from u_1 and v_1 into $\lfloor \frac{n}{m} \rfloor$ consecutive m-sectors. Then, we apply the pattern obtained from the cover of $P(m, r)$ to these sectors. Let $r' \equiv n \pmod{m}$. Therefore, an r'-sector is left. For covering $P(n, k)$ completely, it is not hard to see that it suffices to choose $u_{n-r'+1}, \ldots, u_n$ and $v_{n-k+1}, \ldots, v_n, \ldots, v_{n+k-r'}$ in addition to other vertices. Now, we reach the following upper bound for $m < k$:

$$\begin{aligned} \beta(P(n,k)) &\leq \beta(P(m,r)) \left\lfloor \frac{n}{m} \right\rfloor + r' + (n + k - r' - (n - k + 1) + 1) \\ &\leq \beta(P(m,r)) \left\lfloor \frac{n}{m} \right\rfloor + 2k. \end{aligned}$$

■

Remark 2 *As it is clear from the proof of Theorem* 3*(b), the bound can be improved, because to be sure about covering of edges, we blindly selected* $2k$ *vertices (that some of them might have been selected before in their* m*-sectors).*

Corollary 3 *For all even* k*, we have*

(a) if $k-1 \mid n$*, then* $\beta(P(n,k)) \leq n + \dfrac{n}{k-1}$

(b) if $k-1 \nmid n$*, then* $\beta(P(n,k)) \leq n + \left\lfloor \dfrac{n}{k-1} \right\rfloor + 2k.$

Proof. (a) It is enough to let $m = k-1$ in Theorem 3. In this case, we obtain

$$\begin{aligned} \beta(P(n,k)) &\leq \frac{n}{m} \beta(P(m,r)) \\ &= \frac{n}{k-1} \beta(P(k-1,1)) \\ &= \frac{n}{k-1} k = n + \frac{n}{k-1}. \end{aligned}$$

(b) The proof is similar to part (a). ■

4 Some exact values for $\beta(P(n,k))$

In this section, we introduce exact values of $\beta(P(n,k))$ for some n and k.

Lemma 3 *$P(n,k)$ is a bipartite graph if and only if n is even and k is odd.*

Proof. For odd n, all vertices of U form an odd cycle. In addition, for n and k both even, the cycle $u_1v_1v_{k+1}u_{k+1}u_ku_{k-1}...u_2u_1$ is an odd cycle, C_{k+3}. For even n and odd k, let $X = \{u_i, v_{i+1} \mid i \text{ odd }\}$ and $Y = \{u_i, v_{i+1} \mid i \text{ even }\}$. It is easy to see that this is a bipartition of vertices of $P(n,k)$. ■

Proposition 5 *$\beta(P(n,k)) = n$ if and only if n is even and k is odd.*

Proof. If $\beta(P(n,k)) = n$ then by Corollary 2, n is not odd. To cover U-edges and spokes, any minimum cover c^* must alternatively select $\frac{n}{2}$ vertices from U and V, say without loss of generality with beginning from u_1 and v_2 respectively. Therefore, when k is even, the edge v_1v_{k+1} is not covered by c^*, which is a contradiction. This completes the proof of sufficiency. Conversely, for even n and odd k, by Lemma 3, $P(n,k)$ is a bipartite graph, thus for an arbitrary bipartition, the smaller part is a cover of size at most n. Also notice $P(n,k)$ has a matching of size n, for example the spokes, thus the size of any cover must be at least n. Therefore, we have $\beta(P(n,k)) = n$. ■

Theorem 4 *$\beta(P(n,k)) = n+1$ if and only if n is odd and $k = 1$, or $(n,k) = (5,2)$.*

Proof. *Sufficiency.* Trivially $\beta(P(5,2)) = 6 = 5+1$. Now suppose n is odd and $k = 1$. By Corollary 2, we have $\beta(P(n,k)) \geq n+1$. Consider a cover that selected vertices $\{u_i \mid i \text{ odd }\}$ and $\{v_i \mid i = 1 \text{ or } i \text{ even }\}$. Since this is a cover of size $n+1$, we obtain $\beta(P(n,k)) = n+1$.

Necessity. Suppose $\beta(P(n,k)) = n+1$. Let c^* be an arbitrary minimum cover of $P(n,k)$. If $n = 2k+1$, then $\beta(P(2k+1,k)) = 2k+1+\lceil\frac{2k+1}{5}\rceil$ (see [4]) and therefore, $\lceil\frac{2k+1}{5}\rceil = 1$; and so the only solutions in this case are $(n,k) = (3,1)$ and $(n,k) = (5,2)$. Now suppose $n > 2k+1$. By considering Equation (1) in the proof of Theorem 1 for c^*, we have $a(c^*) - b(c^*) = 2$. Therefore, the size of each strip of c^* must be less than 4. Thus, it is easy to check that all strips have size 1 except exactly one strip with size 2 or 3.

We claim that c^* must select at least one vertex from every pair of consecutive vertices of V. Otherwise, without loss of generality, say the two consecutive unselected vertices are v_{k+1} and v_{k+2}. Clearly, v_1, v_2, v_{2k+1}, and v_{2k+2} are selected by c^*, and they form two strips of size more than one or one strip of size more than three, a contradiction. Therefore, without loss of generality the strip of size greater than one is $\{v_n, v_1, v_2\}$ when n is even, and is $\{v_1, v_2\}$ when n is odd, and c^* must select alternatively from vertices of V beginning with v_2.

If k is even, then both endpoints of edges $v_1v_{k+1}, v_3v_{k+3}, \ldots, v_{k+1}v_{2k+1}$ have odd subscripts. Note that c^* has selected vertices with even subscripts and v_1. So it can cover only one of these $\frac{k}{2}+1$ edges. This leaves $\frac{k}{2}$ uncovered edges, a contradiction. If k is odd (which by Proposition 5 implies that n is also odd), then both endpoints of edges between the vertices v_{n-i} and v_{n-i+k}, for $i = 0, 2, 4, \ldots, k-1$ have odd subscripts. Again, since that c^* has selected vertices with even subscripts and v_1, it can cover only one of these $\lceil \frac{k}{2} \rceil$ edges. Hence, we have $k = 1$. This completes the proof of necessity. ■

By Proposition 5 and Theorem 4, the following is immediate.

Corollary 4 *$\beta(P(n,k)) \geq n+2$ if and only if none of the following conditions holds:*

(a) n is even and k is odd,

(b) n is odd and $k = 1$,

(c) $n = 5$ and $k = 2$.

Proposition 6 *For n and k both odd, where $k \mid n$, $\beta(P(n,k)) = n + \frac{k+1}{2}$.*

Proof. If $k|n$, $(n,k) = k$. Thus, by Propositions 2 and 4 , we have

$$n + \frac{(n,k)+1}{2} = n + \frac{k+1}{2} \leq \beta(P(n,k)) \leq n + \frac{k+1}{2}.$$

■

By the results above, we have the following precise values.

Proposition 7 $\beta(P(n,1)) = \begin{cases} n & \textit{if } n \textit{ is even} \\ n+1 & \textit{if } n \textit{ is odd.} \end{cases}$

And we recall the following result of [4]:

Proposition 8 $\beta(P(n,2)) = n + \lceil \frac{n}{5} \rceil.$

Proposition 9 $\beta(P(n,3)) = \begin{cases} n & \textit{if } n \textit{ is even} \\ n+2 & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. If n is even, this is implied by Proposition 5. If n is odd, by Corollary 4, $\beta(P(n,3)) \geq n+2$ and by Proposition 4, $\beta(P(n,3)) \leq n+2$. Thus, we obtain $\beta(P(n,3)) = n+2$. ■

5 Concluding remarks

In the two previous sections, we presented bounds and some exact values for the size of minimum covers of generalized Petersen graphs. With careful inspection of these results, we reach the following proposition.

Proposition 10 *For large enough n and for a fixed $k \neq 4$, we have $\beta(P(n,k)) \leq n + \lceil \frac{n}{5} \rceil + O(1)$.*

Proof. This can be proved from the following facts:

1. for an odd k, by Propositions 4 and 5, the statement is correct;
2. for $k = 2$, according to [4], the statement holds;
3. for any even $k > 4$, by Corollary 3, the statement is true.

■

Due to Proposition 10 and our observations for generalized Petersen graphs for small n, we conjecture the following.

Conjecture 1 *For all n and k, $\beta(P(n,k)) \leq n + \lceil \frac{n}{5} \rceil$.*

This conjecture is checked to be true for all $2k < n$, where $n \leq 35$.

Question 1 *It seems that by using ideas similar to the ones used in the proof of Theorems 1 and 2, an algorithm with polynomial complexity for finding minimum covers of generalized Petersen graphs can be found. But we have still not found one.*

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References

[1] B. Alspach, The classification of hamiltonian generalized Petersen graphs, *J. Combin. Theory, Ser. B* 34(3) (1983), 293–312.

[2] J. Ebrahimi B., N. Jahanbakht and E.S. Mahmoodian, Vertex domination of generalized Petersen graph, submitted.

[3] K. Bannai, Hamiltonian cycles in generalized Petersen graphs, *J. Combin. Theory, Ser. B*, 24(2) (1978), 181–188.

[4] M. Behzad, P. Hatami and E.S. Mahmoodian, Minimum vertex cover of generalized Petersen graph $P(n, 2)$, *Bull. Inst. Combin. Applic.*, to appear.

[5] F. Castagna and G. Prins, Every generalized Petersen graph has a Tait coloring, *Pacific J. Math.* 40(1) (1972), 53–58.

[6] T.H. Cormen, C.E. Leiserson, R.L. Rivest and C. Stein, *Introduction to Algorithms*, McGraw-Hill, second edition, 2001.

[7] D. McQuillan and R.B. Richter, On the crossing numbers of certain generalized Petersen graphs, *Discrete Math.* 104(3) (1992), 311–320.

[8] G. Salazar, On the crossing numbers of loop networks and generalized Petersen graphs, *Discrete Math.* 302(1–3) (2005), 243–253.

[9] A.J. Schwenk, Enumeration of hamiltonian cycles in certain generalized Petersen graphs, *J. Combin. Theory, Ser. B* 47(1) (1989), 53–59.

[10] M.E. Watkins, A theorem on Tait colorings with an application to generalized Petersen graphs, *J. Combin. Theory* 6 (1969), 152–164.

[11] D.B. West, *Introduction to graph theory*, Prentice-Hall, 2nd edition, 2001.

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