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On the connectivity of the direct product of graphs*

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Abstract

In this note we show that the edge-connectivity $\lambda(G \times H)$ of the direct product of graphs G and H is bounded below by $\min\{\lambda(G)|E(H)|, \lambda(H)|E(G)|, \delta(G \times H)\}$ and above by $\min\{2\lambda(G)|E(H)|, 2\lambda(H)|E(G)|, \delta(G \times H)\}$ except in some special cases when G is a relatively small bipartite graph, or both graphs are bipartite. Several upper bounds on the vertex-connectivity of the direct product of graphs are also obtained.

1 Introduction

Let G and H be undirected graphs without loops or multiple edges. The *direct product* $G \times H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, two vertices (x, y) and (v, w) being adjacent in $G \times H$ if and only if $xv \in E(G)$ and $yw \in E(H)$. The direct product is clearly commutative and associative.

Weichsel observed almost half a century ago that the direct product of two graphs G and H is connected if and only if both G and H are connected and not both are bipartite graphs [15]. Many different properties of direct product of graphs have been studied since (unfortunately it appears under various different names, such as cardinal product, tensor product, Kronecker product, categorical product, conjunction etc.). The study encompasses for instance structural results [2, 3, 6, 9, 10, 11], hamiltonian properties [1, 12], and above all the well-known Hedetniemi's

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conjecture on chromatic number of direct product of two graphs (for a comprehensive picture see the book on graph products [8] and survey on Hedetniemi's conjecture [17]). Open problems in the area suggest that a deeper structural understanding of this product would be welcome. Curiously, Weichsel's result has been so far the only one that considers connectivity of direct products.

Vertex- and edge-connectivity have been recently considered for two other commutative graph products, namely for the Cartesian product [13, 16] and for the strong product [4, 14]. In all cases explicit formulae have been obtained by which the (vertex- or edge-) connectivity of a product of graphs is expressed in terms of corresponding graph invariants of factor graphs. Our aim in this paper is to obtain similar results for the direct product of graphs. As it turns out the situation is more complex than with the Cartesian or the strong product, which is in part due to the fact that the direct product of two bipartite graphs is already disconnected.

Let us recall some basic definitions. For a connected graph G, a set S of vertices of G is called *separating* if $G - S$ is not connected. The *vertex-connectivity* $\kappa(G)$ of a graph G is the size of a minimum separating set in G (except if G is a complete graph K_n, when $\kappa(G)$ is defined to be $n-1$). Similarly a set S of edges of G is called *separating* if $G - S$ (the graph obtained from G by deletion of edges from S) is not connected. The *edge-connectivity* $\lambda(G)$ of a graph G is the size of a minimum separating set (of edges) in G. The well-known theorem of Whitney states that $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G)$ denotes the minimum degree of vertices in G. We will say that a separating set of vertices (resp. edges) S in a graph G is a *κ-set* (resp. *λ-set*) in G if $|S| = \kappa(G)$ (resp. $|S| = \lambda(G)$).

In the next section we prove our main theorem, a lower and an upper bound for $\lambda(G \times H)$ where G and H are arbitrary connected graphs. We also present examples of pairs of graphs that achieve each of the expressions from both bounds. In the last section we present some partial results on the vertex-connectivity of direct product of graphs, which show that the problem of finding a nice meaningful formula is difficult.

2 Edge-connectivity of the direct product

The fundamental result of Weichsel suggests that the connectivity of direct product of graphs is in some way related to the distance of factor graphs from being bipartite. Indeed we will make use of the following invariant.

For a graph G the minimum cardinality of a set of edges E such that $G - E$ is a bipartite graph is denoted by $\lambda_b(G)$. Clearly $\lambda_b(G) = 0$ if and only if G is bipartite. This invariant was introduced in [7], and studied also in [5] under the name *bipartite edge frustration* of a graph.

Let G and H be graphs, and $G \times H$ their direct product. For $x \in V(G)$ we let ${}_xH = \{(x, v) \in V(G) \times V(H) \,|\, v \in V(H)\}$ and call it the *H-fiber* with respect to x. On Figure 1 the vertices of the fiber ${}_xH$ are colored black. Clearly the subgraph of $G \times H$ induced by any H-fiber is totally disconnected. Analogously we define the *G-fiber* with respect to a vertex $y \in V(H)$ and we denote it G_y. In this section we

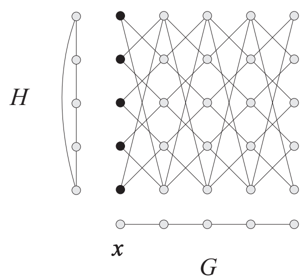

Figure 1: The direct product of a path by a cycle.

prove the following (our main) result.

Theorem 1 *Let G and H be graphs.*

- *Let H be a nonbipartite graph. If G is a nonbipartite graph or $|V(G)| > \frac{\delta(H)}{\lambda_b(H)}$ then $\lambda(G \times H)$ is greater or equal*

$$\min\{\lambda(G)|E(H)|, \lambda(H)|E(G)|, \delta(G \times H)\}$$

and less or equal

$$\min\{2\lambda(G)|E(H)|, 2\lambda(H)|E(G)|, \delta(G \times H)\}\,.$$

- *If G and H are both bipartite, then $\lambda(G \times H) = 0$.*
- *Otherwise (i.e. if G is bipartite and H nonbipartite, with $|V(G)| \leq \frac{\delta(H)}{\lambda_b(H)}$), then $\lambda(G \times H)$ is greater or equal*

$$\min\{2\lambda_b(H)|E(G)|, \lambda(G)|E(H)|, \lambda(H)|E(G)|, \delta(G \times H)\}$$

and less or equal

$$\min\{2\lambda_b(H)|E(G)|, 2\lambda(G)|E(H)|, 2\lambda(H)|E(G)|, \delta(G \times H)\}\,.$$

For the proof of Theorem 1 we need some preliminary observations. (Recall that maximal connected subgraphs of a graph are called *(connected) components*. To simplify the notation we will also call the set of vertices that induces a component C – a component C.)

Lemma 2 *Let G and H be connected graphs not both bipartite. Let S be a λ-set in $G \times H$ and C_1, C_2 connected components of $(G \times H) - S$. If $xy \in E(G)$ is an edge such that $(x,t), (y,t) \in C_i$ for some $t \in V(H)$ and ${}_xH \cup {}_yH \nsubseteq C_i$ then there are at least $\lambda(H)$ edges of the form $(x,a)(y,b)$ from S.*

Proof. Let
$$A_{xy} = \{t \in V(H) \mid (x,t),(y,t) \in C_i\}.$$
Since $A_{xy} \neq \emptyset$ and $A_{xy} \neq V(H)$ the set
$$S_{xy} = \{ab \in E(H) \mid a \in A_{xy}, b \in V(H) - A_{xy}\}$$
is a separating set in H, and so $|S_{xy}| \geq \lambda(H)$. For every edge $ab \in S_{xy}$ at least one of the edges $(x,a)(y,b)$ or $(y,a)(x,b)$ is from S. Since $|S_{xy}| \geq \lambda(H)$, we find that S contains at least $\lambda(H)$ edges with first coordinates x and y. □

For a set of vertices $Y \subset V(G)$ the *open neighborhood* $N_G(Y)$ of Y is the set of vertices $x \notin Y$ such that there exists $y \in Y$ which is adjacent to x. We will write simply $N(Y)$ when the graph is understood from the context.

Lemma 3 *Let G and H be connected graphs not both bipartite. Let S be a λ-set in $G \times H$ and C_1, C_2 connected components of $(G \times H) - S$. Then one of the following occurs:*

(i) There is a G-fiber G_y and an H-fiber ${}_xH$ such that $G_y \cup {}_xH \subseteq C_i$ for some $i \in \{1,2\}$. In this case
$$\lambda(G \times H) = \delta(G \times H).$$

(ii) There is either a G-fiber G_y or an H-fiber ${}_xH$ such that $G_y \subseteq C_i$ or ${}_xH \subseteq C_i$ for some $i \in \{1,2\}$. In this case
$$\lambda(G \times H) \geq \min\{\lambda(G)|E(H)|, \lambda(H)|E(G)|\}.$$

(iii) There is no G-fiber G_y and no H-fiber ${}_xH$ such that $G_y \subseteq C_i$ or ${}_xH \subseteq C_i$ for some $i \in \{1,2\}$. In this case one of the following occurs:

(a) There is an edge $xy \in E(G)$ and an edge $uv \in E(H)$ such that
$$(x,t) \in C_1 \iff (y,t) \in C_2 \tag{1}$$

$$(t,u) \in C_1 \iff (t,v) \in C_2 \tag{2}$$

If G and H are both nonbipartite, then
$$\lambda(G \times H) \geq \min\{\lambda(G)|E(H)|, \lambda(H)|E(G)|\}.$$

If one of the factors, say G, is bipartite, then
$$\lambda(G \times H) \geq \min\{\lambda(H)|E(G)|, 2\lambda_b(H)|E(G)|\}.$$

Moreover if $2\lambda_b(H)|E(G)| \leq \lambda(H)|E(G)|$ then $|V(G)| \leq \frac{\delta(H)}{\lambda_b(H)}$.

(b) *There is no edge* $xy \in E(G)$ *such that* (1) *holds, or no edge* $uv \in E(H)$ *such that* (2) *holds. Then*

$$\lambda(G \times H) \geq \min\{\lambda(G)|E(H)|, \lambda(H)|E(G)|\}\,.$$

Proof. *Case (i):* Let $\delta(G) = \delta_1$ and $\delta(H) = \delta_2$. Without loss of generality assume that $\delta_1 \geq \delta_2$ and that there is a G-fiber and an H-fiber contained in C_1. Let

$$X = \{x \in V(G) \,|\, {}_xH \subseteq C_1\} \quad \text{and} \quad Y = \{y \in V(H) \,|\, G_y \subseteq C_1\}\,.$$

Let $v \in N(Y)$ be any vertex from the open neighborhood of Y and let (u, v) be any vertex from C_2 (note that since $v \notin Y$ such a vertex (u, v) exists). If all neighbors of (u, v) are from C_1 then we are done, since then $|S| \geq \deg(u, v) \geq \delta(G \times H)$. Similarly, if every neighbor of (u, v) is either contained in C_1 or has a neighbor in C_1 we find that $|S| \geq \deg(u, v) \geq \delta(G \times H)$. If there is a neighbor $(u', v') \in C_2$ of (u, v) which has all neighbors from C_2 then we find that there are at least δ_1 neighbors of (u', v') which are from $C_2 \cap G_v$. Since $v \in N(Y)$ we find that each of them has at least δ_1 neighbors in $V(G) \times Y$ and hence at least δ_1 neighbors in C_1. Thus $|S| \geq \delta_1^2 \geq \delta_1\delta_2 = \delta(G \times H)$.

Case (ii): Suppose that there is a G-fiber G_t contained in C_i. Thus for every edge $xy \in E(G)$ we have $(x, t), (y, t) \in C_i$ and since ${}_xH \cup_y H \not\subseteq C_i$ we derive by Lemma 2, there are at least $\lambda(H)$ edges from S between ${}_xH$ and ${}_yH$. Hence altogether there are at least $\lambda(H)|E(G)|$ edges from S. If there is an H-fiber contained in C_i, we find analogously that $|S| \geq \lambda(G)|E(H)|$, which implies $|S| \geq \min\{\lambda(G)|E(H)|, \lambda(H)|E(G)|\}$.

Case (iii)(a): Let $X \subseteq E(G)$ be the set of edges for which (1) is true. For every edge $xy \in X$ there are at least $\lambda_b(H)$ edges $uv \in E(H)$ such that $(x, u), (x, v) \in C_i$ for some $i \in \{1, 2\}$. Since for every $xy \in X$ and every $uv \in E(H)$ such that $(x, u), (x, v) \in C_i$ there are two edges, namely $(x, u)(y, v)$ and $(x, v)(y, u)$, from S, we find that $|S| \geq 2\lambda_b(H)|X|$. Consider the edges $xy \notin X$. For each $xy \notin X$ there exists a $t \in V(H)$ such that $(x, t), (y, t) \in C_i$ for some $i \in \{1, 2\}$. From Lemma 2 we infer that there are at least $\lambda(H)|E(G)-X|$ such edges in S. Summing up the edges of S coresponding to X and to $E(G) - X$, we get $|S| \geq 2\lambda_b(H)|X| + \lambda(H)|E(G) - X|$. Let $Y \subseteq E(H)$ be the set of edges for which (2) is true. Arguing as above, by interchanging the roles of G and H, we derive $|S| \geq 2\lambda_b(G)|Y| + \lambda(G)|E(H) - Y|$ and therefore

$$|S| \geq \max\{2\lambda_b(H)|X| + \lambda(H)|E(G) - X|, 2\lambda_b(G)|Y| + \lambda(G)|E(H) - Y|\}\,. \quad (3)$$

Note that from $\lambda(H) \leq 2\lambda_b(H)$ or $\lambda(G) \leq 2\lambda_b(G)$ we derive using (3) that

$$|S| \geq \min\{\lambda(H)|E(G)|, \lambda(G)|E(H)|\}.$$

In the case when both G and H are nonbipartite we can indeed show that $\lambda(H) \leq 2\lambda_b(H)$ or $\lambda(G) \leq 2\lambda_b(G)$. Suppose to the contrary that $2\lambda_b(H) < \lambda(H)$ and $2\lambda_b(G) < \lambda(G)$. Since $2\lambda_b(H) < \lambda(H)$ we find that $|S| > 2\lambda_b(H)|X| + 2\lambda_b(H)|E(G) - X| = 2\lambda_b(H)|E(G)|$. By the hand-shaking lemma, $2|E(G)| \geq |V(G)|\delta(G)$ and hence $|S| > \lambda_b(H)|V(G)|\delta(G)$. Since $\delta(G)\delta(H) \geq |S| > \lambda_b(H)|V(G)|\delta(G)$ we infer $\lambda_b(H)|V(G)| < \delta(H)$. Analogously, if $2\lambda_b(G) < \lambda(G)$, we have $\lambda_b(G)|V(H)| < \delta(G)$. Since G and H are not bipartite $\lambda_b(H), \lambda_b(G) \geq 1$, and therefore $|V(G)| < \delta(H)$ and $|V(H)| < \delta(G)$ which is a contradiction.

If G is bipartite and H is nonbipartite, then the expression

$$2\lambda_b(H)|X| + \lambda(H)|E(G) - X|$$

from (3) is at least $\min\{\lambda(H)|E(G)|, 2\lambda_b(H)|E(G)|\}$. Moreover if $2\lambda_b(H)|E(G)| \leq \lambda(H)|E(G)|$ then $2\lambda_b(H) \leq \lambda(H)$ and thus $\lambda_b(H)|V(G)| \leq \delta(H)$ as shown in the previous paragraph.

Case (iii)(b): Suppose there is no edge $uv \in E(H)$ such that (2) holds. Then for every edge $uv \in E(H)$ there is a $t \in V(G)$ such that $(t, u), (t, v) \in C_i$ for some $i \in \{1, 2\}$. Using Lemma 2 again we infer that $|S| \geq \lambda(G)|E(H)|$. If there is no edge $xy \in E(G)$ such that (1) holds, we derive $|S| \geq \lambda(H)|E(G)|$. □

We are now ready to prove Theorem 1.

Proof. First let us verify some upper bounds for $\lambda(G \times H)$. Obviously $\lambda(G \times H) \leq \delta(G)\delta(H)$. Next we show that $\lambda(G \times H) \leq 2\lambda(G)|E(H)|$. It is straightforward to see, that for a λ-set S in G, the set of edges $(x, u)(y, v)$ and $(x, v)(y, u)$, where $xy \in S, uv \in E(H)$ is a separating set of $G \times H$. Hence indeed $\lambda(G \times H) \leq 2\lambda(G)|E(H)|$. Analogously one checks that $\lambda(G \times H) \leq 2\lambda(H)|E(G)|$.

Let us now consider three cases with respect to bipartiteness of G and H. If G and H are both bipartite, then $G \times H$ is not connected by [15], and so $\lambda(G \times H) = 0$.

Suppose that G is bipartite and H is nonbipartite. We claim that then

$$\lambda(G \times H) \leq 2\lambda_b(H)|E(G)|.$$

Let $R \subseteq E(H)$ be a set of edges such that $|R| = \lambda_b(H)$ and $(V(H), E(H)-R)$ is bipartite. Then the set of edges $(x, u)(y, v)$ and $(x, v)(y, u)$, for $xy \in E(G), uv \in R$ is a separating set of $G \times H$. Hence $\lambda(G \times H) \leq 2\lambda_b(H)|E(G)|$. If the case (iii)(a) of Lemma 3 occurs and $2\lambda_b(H)|E(G)| \leq \lambda(H)|E(G)|$ then $\lambda(G \times H) = 2\lambda_b(H)|E(G)|$ and $|V(G)| \leq \frac{\delta(H)}{\lambda_b(H)}$. In all other cases $\lambda(G \times H) \geq \min\{\lambda(G)|E(H)|, \lambda(H)|E(G)|, \delta(G \times H)\}$. It follows that

$$\lambda(G \times H) \geq \min\{2\lambda_b(H)|E(G)|, \lambda(G)|E(H)|, \lambda(H)|E(G)|, \delta(G \times H)\}$$

and

$$\lambda(G \times H) \leq \min\{2\lambda_b(H)|E(G)|, 2\lambda(G)|E(H)|, 2\lambda(H)|E(G)|, \delta(G \times H)\}.$$

Otherwise both of the graphs G and H are nonbipartite or $|V(G)| > \frac{\delta(H)}{\lambda_b(H)}$. In any case the upper bound is

$$\lambda(G \times H) \leq \min\{2\lambda(G)|E(H)|, 2\lambda(H)|E(G)|, \delta(G \times H)\},$$

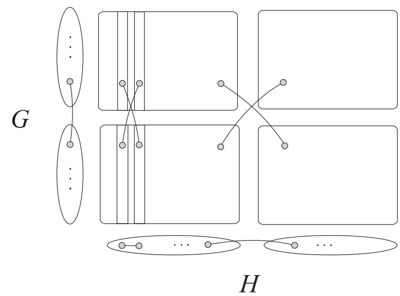

Figure 2: $G = K_{n,n}$ and $H = K_{n,n} \cup \{e\}$.

and the lower bound is

$$\lambda(G \times H) \geq \min\{\lambda(G)|E(H)|, \lambda(H)|E(G)|, \delta(G \times H)\}. \qquad \square$$

We shall now present examples that show each of the bounds from Theorem 1 can be attained. Let G be the complete bipartite graph $K_{n,n}$ and H the graph $K_{n,n}$ with an additional edge ab. Since $H - ab$ is bipartite, we have $\lambda_b(H) = 1$ (see Figure 2). Consider the direct product $G \times H$ and note that deleting all the edges $(x, y)(z, w)$, where $y, w \in \{a, b\}$ results in a disconnected graph. Therefore $\lambda(G \times H) \leq 2\lambda_b(H)|E(G)|$, and it is straightforward to see there are no separating sets with fewer edges, hence $\lambda(G \times H) = 2\lambda_b(H)|E(G)| = 2n^2$.

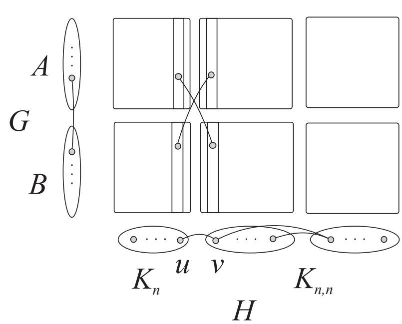

Figure 3: $G = K_{n,n}$ and H is the graph obtained from $K_{n,n} \cup K_n$ by adding bridge uv.

Let G be the complete bipartite graph $K_{n,n}$ and H the graph obtained from the disjoint union of $K_{n,n}$ and K_n by adding a bridge uv between them. Let A and B be the parts of bipartition of G. It is straightforward to see that deleting all the edges $(x, u)(y, v)$, where $x \in A$ and $y \in B$ results in a disconnected graph (see Figure 3).

Hence $\lambda(G \times H) \leq \lambda(H)|E(G)|$. Clearly this is a minimum separating set, therefore $\lambda(G \times H) = \lambda(H)|E(G)|$.

In the case $G = C_{2n}$ and $H = C_{2m+1}$, the edge-connectivity of $G \times H$ is equal to the smallest degree in the product, that is $\lambda(G \times H) = \delta(G \times H) = 4$. If $G = K_n$ and H is the graph obtained from two disjoint copies of K_n by adding an edge, we get $\lambda(G \times H) = 2\lambda(H)|E(G)| = n(n-1)$.

3 Vertex-connectivity

In this section we present various upper bounds on the vertex-connectivity of the direct product of graphs. As in the case of edge-connectivity, some bounds are related to some type of distance of factor graphs from bipartite graphs. We introduce two such concepts.

For a graph G let $\kappa_b(G)$ be the smallest size of a set $S \subseteq V(G)$ such that $G - S$ is a bipartite graph. By $\kappa_b'(G)$ we denote the smallest size of a set $S \subset V(G)$ of endvertices of edges from $F \subset E(G)$ such that $G - F$ is bipartite. In what follows we use $A_G^1(S)$ and $A_G^2(S)$ to denote the two parts of the bipartite graph $G - S$.

Proposition 4 *For any graphs G and H,*

$$\kappa(G \times H) \leq \kappa_b(G)|V(H)| + \kappa_b(H)|V(G)| - \kappa_b(G)\kappa_b(H)\,.$$

Proof. Let $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$ be such that $G - S_1$ and $H - S_2$ are bipartite and $|S_1| = \kappa_b(G)$, $|S_2| = \kappa_b(H)$ respectively. Note that $(S_1 \times V(H)) \cup (V(G) \times S_2)$ is a separating set in $G \times H$. Indeed removing it from $G \times H$ results in a graph isomorphic to $(G - S_1) \times (H - S_2)$ which is clearly disconnected (since it is a direct product of two bipartite graphs). Hence the result follows since the set $(S_1 \times V(H)) \cup (V(G) \times S_2)$ has the desired size. □

Proposition 5 *Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ and let*

$$\begin{aligned} A &= \{u_1v_1, \ldots, u_kv_k\} \subseteq E_1 \\ B &= \{w_1z_1, \ldots, w_\ell z_\ell\} \subseteq E_2 \end{aligned}$$

such that $G' = (V_1, E_1 \setminus A)$ and $H' = (V_2, E_2 \setminus B)$ are bipartite. Let $X_1 \cup Y_1$ be the bipartition of G' and $X_2 \cup Y_2$ the bipartition of H'. Let $C \subseteq V(G \times H)$ be a set such that

(i) For every $i \in \{1, \ldots, k\}$ either $\{u_i, v_i\} \times X_2 \subseteq C$ or $\{u_i, v_i\} \times Y_2 \subseteq C$.

(ii) For every $i \in \{1, \ldots, \ell\}$ either $\{w_i, z_i\} \times X_1 \subseteq C$ or $\{w_i, z_i\} \times Y_1 \subseteq C$.

Then the graph $(G \times H) - C$ is not connected, moreover $\kappa(G \times H) \leq |C|$.

Proof. Suppose that $C \subseteq G \times H$ is a set such that (i) and (ii) is true. We claim that $(X_1 \times X_2) \cup (Y_1 \times Y_2)$ and $(X_1 \times Y_2) \cup (Y_1 \times X_2)$ are (at least) two connected components of $(G \times H) \setminus C$. Since (i) is true there is no edge with one endvertex in $X_1 \times X_2$ and other in $X_1 \times Y_2$. Analogously (ii) implies that there is no edge with one endvertex in $X_1 \times X_2$ and other in $Y_1 \times X_2$. With similar arguments we prove that there is no edge with one endvertex in $(Y_1 \times Y_2)$ and other in $(X_1 \times Y_2) \cup (Y_1 \times X_2)$, hence the result follows. □

Corollary 6 *Let G and H be any graphs and $S_1 \subseteq V(G), S_2 \subseteq V(H)$ be endvertices of edges $F_1 \subset E(G)$ and $F_2 \subset E(H)$ respectively, such that $G - F_1$ and $H - F_2$ are bipartite. Furthemore, let $|S_1| = \kappa_b'(G)$ and $|S_2| = \kappa_b'(H)$. Then*

$$\kappa(G \times H) \leq \min_{1 \leq i,j \leq 2} \{|A_G^i(S_1)|\kappa_b'(H) + |A_H^j(S_2)|\kappa_b'(G)\}\,.$$

To demonstrate Proposition 4 and Corollary 6 consider the following example. Let G be the graph obtained from $K_{n,n}$ to which two edges uv, uw with a common endvertex u are added, and let H be $K_{r,s}$. Then $\kappa_b(G) = 1$ and $\kappa_b'(G) = 3$, while $\kappa_b(H) = \kappa_b'(H) = 0$. Suppose $r = s$. Then it is easy to see that $\{u\} \times V(H)$ is a minimum separating set, and so $\kappa(G \times H) = \kappa_b(G)|V(H)| = 2r$. On the other hand if $3r < s$, and we denote by Y the maximal independent set of H with r vertices, then it is not hard to see that $\{u, v, w\} \times Y$ is a separating set of $G \times H$. Moreover, $\kappa(G \times H) = \kappa_b'(G) \cdot |Y| = 3r$.

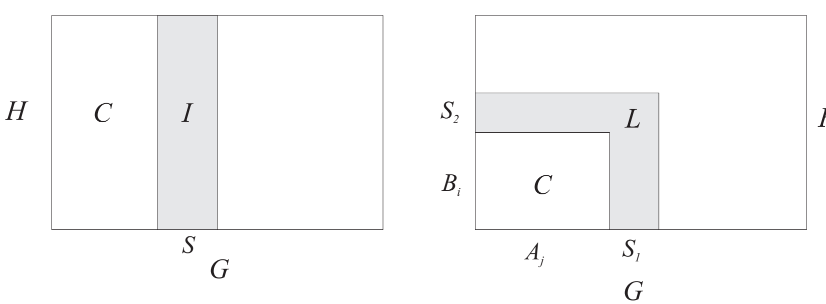

Figure 4: An I-set and an L-set in $G \boxtimes H$. A connected component of $(G \boxtimes H) - I$ and $(G \boxtimes H) - L$ is denoted by C.

A characterization of minimum separating sets in a strong product of graphs from [14] says that every minimum separating set in $G \boxtimes H$ of graphs G and H is either an I-set or an L-set. Let us recall definitions of I-sets and L-sets (see Figure 4). Let S be a separating set in $G = (V_1, E_1)$ or $H = (V_2, E_2)$ and let $I = S \times V_2$ or $I = V_1 \times S$, respectively. Then I is called an *I-set* in $G * H$ (here $*$ denotes any graph product). Let S_1 and S_2 be separating sets in G and H respectively and let $A_1, \ldots, A_k$ be connected components of $G - S_1$ and $B_1, \ldots, B_\ell$ be connected components of $H - S_2$. Then for any $i \leq \ell$ and $j \leq k$

$$L = (S_1 \times B_i) \cup (S_1 \times S_2) \cup (A_j \times S_2)$$

is called an *L-set* in $G * H$ (see Figure 4).

Since $E(G \times H) \subseteq E(G \boxtimes H)$, every separating set in $G \boxtimes H$ is also a separating set in $G \times H$. Thus any I-set or L-set is a separating set in $G \times H$. Since the direct product of two bipartite graphs is disconnected, it is not surprising that in the case when $S = S_1 \times V_2$ and a connected component of $G - S_1$ and H are bipartite, the I-set S is not a minimal separating set (with respect to inclusion). This leads to the following two definitions.

Let S be a separating set in $G = (V_1, E_1)$ such that a connected component of $G - S$ is bipartite with the bipartition $A \cup B$ and let $H = (V_2, E_2)$ be bipartite with the bipartition $C \cup D$. Let $A' = N_G(A) \cap S$ and $B' = N_G(B) \cap S$. Then the sets

$$(A' \times C) \cup (B' \times D) \subseteq S \times V_2 \quad \text{and} \quad (A' \times D) \cup (B' \times C) \subseteq S \times V_2$$

are called *almost I-sets*. Analogously we define an almost I-set when S is a separating set in H and G is bipartite.

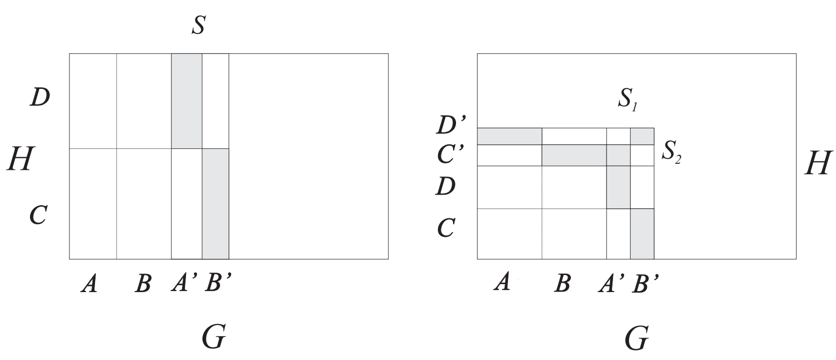

Figure 5: An almost I-set and almost L-set in $G \times H$.

Let S_1 be a separating set in $G = (V_1, E_1)$ such that a connected component of $G - S_1$ is bipartite with the bipartition $A \cup B$ and let S_2 be a separating set in $H = (V_2, E_2)$ such that a connected component of $H - S_2$ is bipartite with the bipartition $C \cup D$. Let $A' = N_G(A) \cap S_1, B' = N_G(B) \cap S_1, C' = N_H(C) \cap S_2$ and $D' = N_H(D) \cap S_2$. Then the sets

$$L_1 = (A' \times D) \cup (A' \times C') \cup (B \times C') \cup (A \times D') \cup (B' \times C) \cup (B' \times D') \qquad (4)$$

and

$$L_2 = (B' \times D) \cup (B' \times C') \cup (A \times C') \cup (A' \times C) \cup (A' \times D') \cup (B \times D') \qquad (5)$$

are called *almost L-sets* (see Figure 5). Note that on Figure 5 intersections $A' \cap B'$ and $C' \cap D'$ are empty; however, in general they can be nonempty.

Proposition 7 *For any graphs G and H every I-set, L-set, almost I-set, and almost L-set in $G \times H$ is a separating set in $G \times H$.*

Proof. Clearly, every I-set and every L-set in the direct product is a separating set. Suppose S is a separating set in G and $A \cup B$ is the bipartition of a connected component of $G - S_1$. Suppose also that H is bipartite with the bipartition $C \cup D$. Consider the set $X = (A \times C) \cup (B \times D)$. It is straighforward to see that all the neighbors of X in $G \times H$ are in $(A' \times D) \cup (B' \times C)$, where A' and B' are defined as above. Similarly all the neighbors of $(A \times D) \cup (B \times C)$ are contained in $(A' \times C) \cup (B' \times D)$.

If S_1 is a separating set in $G = (V_1, E_1)$ such that a connected component of $G - S_1$ is bipartite with the bipartition $A \cup B$ and S_2 is a separating set in $H = (V_2, E_2)$ such that a connected component of $H - S_2$ is bipartite with the bipartition $C \cup D$, then consider the set $X = (A \times C) \cup (B \times D)$ and observe that all the neighbors of X are contained in L_1 as defined in (4). Analogously the neighbors of $(A \times D) \cup (B \times C)$ are contained in L_2 as defined in (5). □

Propositions 4 and 7 and Corollary 6 present different upper bounds on the vertex-connectivity of the direct product of graphs. A more compact (lower and upper) bound has yet to be found. The results and examples of this section indicate this could be very difficult.

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