AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **41** (2008), Pages 63–80

New record graphs in the degree-diameter problem

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Abstract

In 1994, Dinneen and Hafner (*Networks* 24 No. 7, 359–367) published a table of largest orders of graphs of given degree up to 15 and diameter up to 10, known to that date. The table also contained 48 new values found by the authors with the help of computer searches over Cayley graphs of semidirect products of (mostly) cyclic groups. Prior to our work, only relatively few values in the table have been improved; updates have been maintained on the web.

With the help of voltage graphs in combination with random computer search we have substantially improved more than half of the values in (all earlier updates of) the table.

1 Introduction

The problem of finding the largest order $n_{d,k}$ of an undirected graph of a given maximum degree d and a given diameter k has been known for nearly five decades as the *degree-diameter problem.* An obvious upper bound on $n_{d,k}$ is the Moore bound $n_{d,k} \leq 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}$. Non-trivial work is needed to show that equality in this bound holds only if (a) $k = 1$ and $d \geq 1$, or (b) $k = 2$ and $d = 2$, 3, 7 (and, possibly, 57), or (c) $k \geq 3$ and $d = 2$. Apart from these and some other small values of d and k, the orders of the current largest graphs of maximum degree d and diameter k appear to be very far from the Moore bound.

The directed analogue of the problem, which is to find the largest order $n'_{d,k}$ of a digraph of maximum out-degree d and diameter k, has been studied as well; here the directed Moore bound is $n'_{d,k} \leq 1 + d + d^2 + \cdots + d^k$. In sharp contrast to the undirected case, digraphs of order asymptotically matching the order of the directed Moore bound are known. For example, the Kautz digraphs (which are iterated line digraphs of complete digraphs) are of order $d^{k-1}(d+1)$ [14]; other digraphs of similar asymptotic order are also available [15]. The state of the knowledge, however, is different with regard to vertex-transitive and Cayley digraphs, which arise naturally in the context of the directed degree diameter problem and have applications in the study of efficient interconnected networks design. Orders of the current largest vertex-transitive digraphs are very far from the directed Moore bound. For a detailed survey of results concerning both better upper bounds as well as constructions of current largest examples we refer to [11].

Besides explicit constructions that often give infinite classes of relatively large graphs or digraphs of particular degree and diameter, a number of researchers focused on random generation of such (di)graphs for computationally tractable values of d and k. In a sense, this effort culminated in [3] with the publication of a table of the then largest orders of graphs of given degree up to 15 and diameter up to 10. Included in this table were also 48 new values found by the authors with the help of computer search over Cayley graphs of semidirect products of (in most cases) cyclic groups. In the next 12 years only relatively few values in the table have been improved. Updates have been maintained on the web [19, 20]; chronology of the updates and additional explanatory material about the graphs can be found on the same web-sites.

Among the new construction methods surveyed in [11] an important role is played by the *regular covering construction*. The essence of the construction is to "blow up" relatively small *quotient graphs*, possibly with loops, semi edges and multiple edges, into large graphs called *lifts*. The lifting is determined by associating a finite group with the quotient graph in a particular way through a so-called *voltage assignment*, forming a *voltage graph* from the quotient. Lifts turn out to preserve vertex degrees and it is easy to control their diameter by the properties of the voltage graphs. A short summary of details of this construction (for both graphs as well as digraphs) will be given in Section 2.

Voltage graphs have been successfully used in the degree-diameter problem before; see e.g. [10]. A justification of their good potential in a random generation of large graphs of given degree and diameter was indicated in [1, 2]. The method of covering spaces relates well with the Cayley graph generation of [3] since every Cayley graph is a lift of a one-vertex voltage graph. These facts have pre-determined our choice of using voltage graphs in our random generation. Moreover, guided by the success of [3] with Cayley graphs of semidirect products of cyclic groups, we have used the same groups for our voltage assignments. In Section 3 we will briefly outline our random algorithm for generation of suitable voltage assignments on carefully chosen quotients with accompanying explanations regarding notation and lists of voltages. This way we have been able to improve more than half of the current largest orders of graphs of given degree $d \leq 16$ and diameter $k \leq 10$. In Section 4 we present the new graphs in a tabular form. We also present some new vertex-transitive digraphs, improving and adding to the list of known digraphs contained in [8] [5]. Some issues that have emerged from our search are discussed in the final section.

2 Voltage graphs and lifts

The theory of voltage graphs originated in the early seventies of the 20th century in the context of re-visiting the proof of the Map Color Theorem. The development of the theory, which can be regarded as a discrete version of the well known theory of covering spaces in algebraic topology, begun with the paper [6] and culminated in the monograph [7] that laid foundations of modern topological graph theory. We outline the essentials just for (undirected) graphs; digraphs will be mentioned in the conclusion of this section.

Let Γ be a finite, undirected graph, possibly with loops and multiple edges. We also allow semiedges, that is, edges with just one end-vertex and with the other end dangling. To facilitate the description of voltages, we will think of the (undirected) edges of Γ that are not semiedges as pairs of oppositely directed edges, called *darts*. A semiedge admits, by definition, just one direction (into the unique incident vertex). The number of elements in the set D of all darts of Γ is therefore twice the number of all edges of Γ minus the number of semiedges. If e is a dart, then e^{-1} will denote the dart reverse to e; in the case of a semi-edge we set $e^{-1} = e$ by definition.

Let G be a finite group. A mapping $\alpha :\ D \to G$ will be called a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$, for any dart $e \in D$. Thus, a voltage assignment sends a pair of mutually reverse darts onto a pair of mutually inverse elements of the group. Note that if e is a semiedge, then the voltage condition means that $\alpha(e)$ has order at most two in G. The pair (Γ, α) is the *voltage graph*, which determines the *lift* Γ^α of Γ as follows. Let V be the vertex set of Γ. The vertex set and the dart set of the lift are $V^\alpha = V \times G$ and $D^\alpha = D \times G$. In the lift, (e, g) is a dart from the vertex (u, g) to the vertex (v, h) if and only if e is a dart from u to v in Γ and $h = g\alpha(e)$. The lift itself is considered to be undirected, since (e, g) and $(e^{-1}, g\alpha(e))$ form a pair of mutually reverse darts and therefore give rise to an undirected edge of Γ^α.

The projection $\pi :\ \Gamma^\alpha \to \Gamma$ given by $\pi(e, g) = e$ and $\pi(v, g) = v$ is, topologically, a regular covering of Γ by Γ^α. This is the reason why Γ is often called a (regular) *quotient*. For any vertex v and any dart e of the quotient, the sets $\pi^{-1}(v)$ and $\pi^{-1}(e)$ are called *fibres* above v and e. If e is a semi-edge then (in order not to have semi-edges in the lift) one usually assumes that $\alpha(e)$ has order two in G, in which case $\pi^{-1}(e)$ is a perfect matching on the vertices in $\pi^{-1}(v)$. For any fixed $h \in G$ the mapping $(e, g) \mapsto (e, hg)$ determines an automorphism of the lift Γ^α. This way the voltage group G acts regularly (that is, transitively and freely) on each fibre as a group of automorphisms of the lift. In particular, if the quotient has just one vertex, then G acts regularly on the vertex set of the lift; moreover, if connected, the lift is in such a case a Cayley graph for the group G and the generating set consisting of the voltages assigned to loops and semiedges attached to the single vertex of the quotient. Conversely, all Cayley graphs are lifts of single-vertex graphs.

An example of three coverings and lifts is given in Fig. 1, where the composition of the coverings induced by the voltage assignments on the dumbbell graph in the group Z_5 and on the Petersen graph in Z_2 gives the same lift (the graph of the dodecahedron) as a voltage assignment on the dumbbell graph in the group Z_{10}. We note that a composition of two regular coverings rarely results in a regular covering as in Fig. 1. Necessary and sufficient conditions for this to happen can be found in [12] and for a sufficient condition related to the degree-diameter problem we refer to [1].

It is clear that the degree of a vertex v of the quotient is inherited by all vertices in the fibre above v in the lift. This gives a trivial way to control vertex degrees in

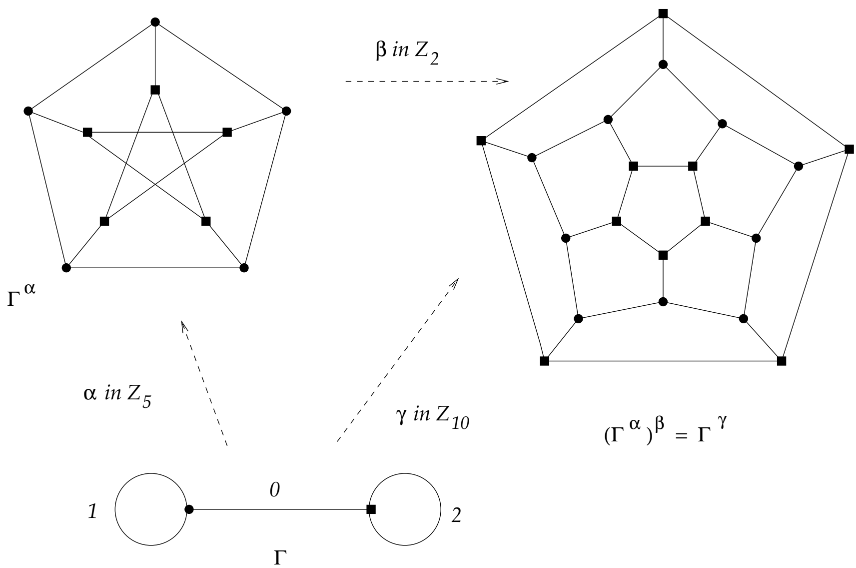

Figure 1: Three examples of lifting, with voltages in cyclic groups.

the lifts. As regards determining the diameter of the lift, by the observation about the regular action of the voltage group on fibres it is sufficient to choose one vertex from each fibre and check all distances in the lift from the chosen vertices.

We also mention the important and well-known fact that for every voltage assignment α on a connected quotient Γ and for any choice of a spanning tree T of Γ there is a voltage assignment β on Γ in the same group with $\beta(e)$ equal to the identity for any dart e in T and such that the lifts Γ^α and Γ^β are isomorphic.

The outlined theory can easily be adapted to digraphs. Indeed, the discussion about turning undirected edges into pairs of oppositely directed darts is vacuous for digraphs. A voltage assignment is simply *any* mapping from directed edges to elements of a group. Concepts such as lift, covering, fibre, and so on are introduced completely analogously to the undirected case and we leave the details to the reader.

3 Generating large graphs

In this section we concentrate on (undirected) graphs. Motivated by the success of the groups used in [3, 8] and by the analysis in [1] we have concentrated on exclusively using semidirect products of cyclic groups as voltage groups in this paper. Recall that if Z_m and Z_n are cyclic groups of orders m and n and if r is an integer such that $1 \leq r \leq n-1$ and $r^n \equiv 1 \pmod{m}$, the elements of the semidirect product $Z_m \rtimes_r Z_n$ of the two groups relative to r are ordered pairs (a, b) where $a \in Z_m$ and $b \in Z_n$, with multiplication given by $(a, b)(c, d) = (a + r^b c, b + d)$.

We now outline our procedure for generating large graphs of given degree and diameter. Given a target degree and diameter, we begin with selecting appropriate candidates for the quotient Γ (of degree equal to the target degree) and the voltage group G. This is followed by running a random process of selecting elements from the group as voltages on edges, loops and semiedges of the quotient, preassigning the identity element to all edges of a fixed spanning tree of the quotient. In the process of selection, (partial) assignments giving multiple adjacencies or loops in the lift are immediately rejected.

After a selection has been completed, we determine the diameter of the lift in order to check if the voltage assignment was successful. Computing the diameter requires $O(|\Gamma|^2|G|)$ steps where $|\Gamma|$ is the number of vertices of the quotient and $|G|$ is the order of the group. Obviously, smaller quotients enable faster computations. If, within a predetermined number of attempts, all the generated voltage assignments give lifts of diameter larger than the target diameter, a new group is selected. If, on the other hand, a successful assignment is found, the result is recorded and the procedure stops.

The results of our computations are given in a tabular form in the next section. Our experiments have shown that it was not feasible to run computations on quotients of order 5 or more. All the quotients we have considered have therefore at most 4 vertices. In the tables of results we have used the following notation to describe the quotients:

$B(s, l)$: Bouquet (a single-vertex graph) with s semi edges and l loops.

$D(l, e)$: Dipole (a two-vertex graph) with l loops at each vertex and e edges.

$T(l, e)$: K_3 with l loops at each vertex and e edges joining any two vertices.

$X(l, e)$: K_4 with l loops at each vertex and e edges between any two vertices.

Semiedges have been used only with bouquet quotients and no such quotient will have more than one semiedge, that is, $s = 0$ or 1.

In order to specify the voltage assignments that determine the corresponding large graphs of given degree and diameter we have arranged the voltages in the form of a list of pairs $(a_i, b_i) \in Z_m \rtimes_r Z_n$ arranged in blocks enclosed in square brackets. The following is the explanation of the "grammar" we have used in the lists as given in the tables. With each type of quotients used we give the general structure of the corresponding list of voltages.

Bouquets $B(s, l)$**:** The list of voltages for a bouquet with one semiedge ($s = 1$) and l loops will have the form $[(a_0, b_0)|(a_1, b_1) \dots (a_l, b_l)]$, where (a_0, b_0) is the voltage on the semiedge. In the case of a bouquet with just l loops ($s = 0$) the list will be simply $[(a_1, b_1)(a_2, b_2) \dots (a_l, b_l)]$.

Dipoles $D(l, e)$**:** Assuming the vertices of the dipole are denoted 1 and 2, the list will have the form $[V_1][D_{12}][V_2]$ where the blocks V_i contain the l voltages on the loops on the vertex $i \in \{1, 2\}$ and the block D_{12} lists the e voltages on the darts from 1 to 2.

The graphs $T(l, e)$ **and** $X(l, e)$**:** The corresponding lists of voltages are formed by way of generalisation of the arrangement used for dipoles. That is, we assume that for $t = 3, 4$ the vertex set of K_t is $\{1, 2, \dots, t\}$. The list of voltages for $T(l, e)$ will have the form $[V_1][D_{12}D_{13}][V_2][D_{23}][V_3]$ where V_i is the list of the l voltages on loops attached to the vertex i and D_{ij} are the e voltages on the darts from i to j. Finally, the list of voltages for $X(l,e)$ will have the form $[V_1][D_{12}D_{13}D_{14}][V_2][D_{23}D_{24}][V_3][D_{34}][V_4]$ with the meaning of the symbols similar to the above.

An illustration of two quotients and the corresponding lists is in Figs. 2 and 3, for the table entries regarding degree 7 and diameter 6, and degree 4 and diameter 9, respectively.

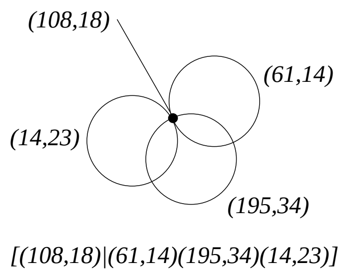

Figure 2: The quotient and the list of voltages for degree 7 and diameter 6.

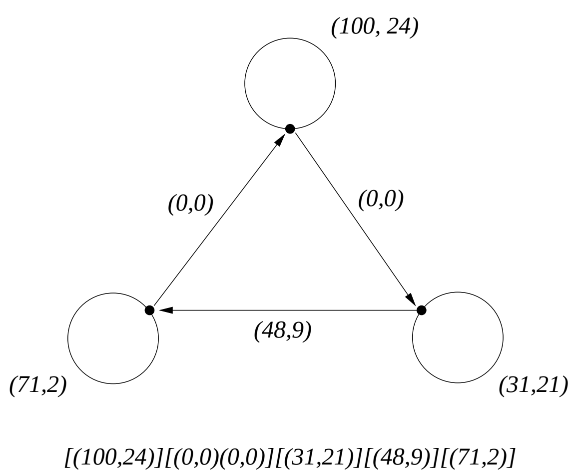

Figure 3: The quotient and the list of voltages for degree 4 and diameter 9.

4 Tables and results

The tables are divided into seven parts. The first two tables are a summary of the orders of all the undirected and directed graphs we found. The third table displays data on the generated graphs of order less than 20,000; their adjacency lists can be downloaded from [17] and [18]. The fourth table describes all the larger graphs we found; the column labeled *%M-B* in the third and fourth tables represents the percentage of the Moore bound reached with the order of the corresponding graphs.

The fifth table displays data on the Cayley digraphs that we found. Finally, the sixth and seventh tables list the complete sets of all non isomorphic Cayley digraphs for some small values of d and k, including many graphs that were not available before. Adjacency lists to these digraphs are linked from [17] and [18]. Tables 6 and 7 were constructed using exhaustive search on all possible groups.

Table 1: Summary of the new largest orders of the undirected graphs presented in Tables 3 and 4.

$d\backslash k$	**4**	**5**	**6**	**7**	**8**	**9**	**10**
4				1,320	3,243	7,575	17,703
5		624		5,516	17,030	57,840	187,056
6	390	1,404		19,383	76,461	307,845	1,253,615
7			11,988	52,768	249,660	1,223,050	6,007,230
8	1,100	5,060		131,137	734,820	4,243,100	24,897,161
9	1,550	8,200		279,616	1,686,600	12,123,288	65,866,350
10	2,286	13,140		583,083	4,293,452	27,997,191	201,038,922
11		19,500		1,001,268	7,442,328	72,933,102	600,380,000
12		29,470		1,999,500	15,924,326	158,158,875	1,506,252,500
13		40,260		3,322,080	29,927,790	249,155,760	3,077,200,700
14		57,837			55,913,932	600,123,780	7,041,746,081
15		76,518		8,599,986	90,001,236	1,171,998,164	10,012,349,898
16					140,559,416	2,025,125,476	12,951,451,931

Table 2: Summary of the new largest orders of the Cayley digraphs presented in Table 5 (italic) and the Cayley digraphs presented in Tables 6 and 7.

$d\backslash k$	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2		20	27			171		
3	27	60	165			*2,041*	*5,115*	*11,568*
4			*465*	*1,378*			*42,309*	*137,370*
5				*3,775*				*1,010,658*
6				*9,020*				

Table 3: The smaller graphs (of order less than 20,000).

<table>
<tr><th rowspan="2">Order</th><th rowspan="2">Degree</th><th rowspan="2">Diameter</th><th colspan="3">Group</th><th rowspan="2">%M-B</th><th rowspan="2">Quotient</th></tr>
<tr><th>m</th><th>n</th><th>r</th></tr>
<tr><td>1,320</td><td>4</td><td>7</td><td>33</td><td>20</td><td>2</td><td>30.18</td><td>$D(1,2)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(21,19)][(0,0)(30,7)][(26,2)]</td></tr>
<tr><td>3,243</td><td>4</td><td>8</td><td>47</td><td>23</td><td>2</td><td>24.71</td><td>$T(1,1)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(8,14)][(0,0)(0,0)][(20,10)][(45,11)][(18,18)]</td></tr>
<tr><td>7,575</td><td>4</td><td>9</td><td>101</td><td>25</td><td>5</td><td>19.24</td><td>$T(1,1)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(100,24)][(0,0)(0,0)][(31,21)][(48,9)][(71,2)]</td></tr>
<tr><td>17,703</td><td>4</td><td>10</td><td>281</td><td>21</td><td>59</td><td>14.99</td><td>$T(1,1)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(160,4)][(0,0)(0,0)][(119,1)][(179,10)][(67,13)]</td></tr>
<tr><td>624</td><td>5</td><td>5</td><td>13</td><td>24</td><td>2</td><td>36.57</td><td>$D(1,3)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(12,1)][(0,0)(9,1)(2,13)][(5,20)]</td></tr>
<tr><td>5,516</td><td>5</td><td>7</td><td>394</td><td>7</td><td>191</td><td>20.20</td><td>$D(1,3)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(376,6)][(0,0)(146,6)(221,2)][(326,2)]</td></tr>
<tr><td>17,030</td><td>5</td><td>8</td><td>131</td><td>65</td><td>3</td><td>15.59</td><td>$D(1,3)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(58,45)][(0,0)(52,1)(105,44)][(31,16)]</td></tr>
<tr><td>390</td><td>6</td><td>4</td><td>13</td><td>15</td><td>3</td><td>41.62</td><td>$D(2,2)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(8,2)(4,14)][(0,0)(6,13)][(4,9)(7,7)]</td></tr>
<tr><td>1,404</td><td>6</td><td>5</td><td>117</td><td>6</td><td>16</td><td>29.95</td><td>$D(1,4)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(4,2)][(0,0)(71,2)(34,3)(100,1)][(52,1)]</td></tr>
<tr><td>19,383</td><td>6</td><td>7</td><td>923</td><td>21</td><td>48</td><td>16.54</td><td>$B(0,3)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(865,19)(330,7)(97,11)]</td></tr>
<tr><td>11,988</td><td>7</td><td>6</td><td>333</td><td>36</td><td>2</td><td>18.35</td><td>$B(1,3)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(108,18)|(61,14)(195,34)(14,23)]</td></tr>
<tr><td>1,100</td><td>8</td><td>4</td><td>55</td><td>20</td><td>2</td><td>34.36</td><td>$B(0,4)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(27,4)(11,12)(9,9)(11,19)]</td></tr>
<tr><td>5,060</td><td>8</td><td>5</td><td>115</td><td>44</td><td>2</td><td>22.58</td><td>$B(0,4)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(14,25)(21,2)(25,7)(29,32)]</td></tr>
<tr><td>1,550</td><td>9</td><td>4</td><td>155</td><td>10</td><td>4</td><td>29.43</td><td>$B(1,4)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(0,5)|(1,7)(52,4)(136,6)(72,1)]</td></tr>
<tr><td>8,200</td><td>9</td><td>5</td><td>205</td><td>40</td><td>7</td><td>19.46</td><td>$B(1,4)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(130,20)|(155,38)(203,14)(87,18)(0,1)]</td></tr>
<tr><td>2,286</td><td>10</td><td>4</td><td>254</td><td>9</td><td>37</td><td>27.87</td><td>$B(0,5)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(253,2)(210,8)(38,5)(111,8)(37,4)]</td></tr>
<tr><td>13,140</td><td>10</td><td>5</td><td>365</td><td>36</td><td>18</td><td>17.80</td><td>$B(0,5)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(263,6)(2,5)(201,20)(325,29)(169,14)]</td></tr>
<tr><td>19,500</td><td>11</td><td>5</td><td>325</td><td>60</td><td>17</td><td>15.95</td><td>$B(1,5)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(234,30)|(179,5)(178,38)(283,1)(285,10)(306,46)]</td></tr>
</table>

Table 4: The larger graphs (of order exceeding 20,000).

Order	Degree	Diameter	Group			%M-B	Quotient
			m	n	r		
57,840	5	9	1205	48	22	13.23	$B(1,2)$
Voltages	[(1165,24)|(491,41)(849,15)]						
187,056	5	10	3897	48	506	10.70	$B(1,2)$
Voltages	[(765,24)|(3312,37)(1861,44)]						
76,461	6	8	2317	33	79	13.04	$B(0,3)$
Voltages	[(1871,31)(1722,26)(1817,23)]						
307,845	6	9	6841	45	122	10.50	$B(0,3)$
Voltages	[(5722,40)(5970,12)(3528,44)]						
1,253,615	6	10	22793	55	72	8.55	$B(0,3)$
Voltages	[(21654,34)(11504,49)(22520,44)]						
52,768	7	7	1649	32	19	13.46	$B(1,3)$
Voltages	[(1088,16)|(96,31)(586,24)(1521,21)]						
249,660	7	8	6935	36	23	10.61	$B(1,3)$
Voltages	[(1197,18)|(3556,7)(6634,3)(4597,19)]						
1,223,050	7	9	24461	50	4828	8.66	$B(1,3)$
Voltages	[(11895,25)|(7818,43)(15052,37)(7103,48)]						
6,007,230	7	10	66747	90	233	7.09	$B(1,3)$
Voltages	[(6422,45)|(51477,13)(22951,35)(38463,52)]						
131,137	8	7	1847	71	11	11.94	$B(0,4)$
Voltages	[(816,49)(1527,32)(601,19)(358,70)]						
734,820	8	8	12247	60	658	9.56	$B(0,4)$
Voltages	[(6983,39)(7376,26)(410,27)(2413,16)]						
4,243,100	8	9	42431	100	274	7.88	$B(0,4)$
Voltages	[(24880,85)(444,18)(36184,37)(32820,65)]						
24,897,161	8	10	341057	73	2103	6.61	$B(0,4)$
Voltages	[(310448,53)(177761,43)(68031,31)(35532,28)]						
279,616	9	7	4369	64	22	10.37	$B(1,4)$
Voltages	[(3553,32)|(3696,46)(1800,41)(3639,11)(724,38)]						
1,686,600	9	8	23425	72	182	7.81	$B(1,4)$
Voltages	[(21675,36)|(12166,13)(15523,61)(20107,57)(13014,23)]						
12,123,288	9	9	168379	72	2242	7.02	$B(1,4)$
Voltages	[(70579,36)|(133512,24)(107217,34)(59314,52)(66721,49)]						
65,866,350	9	10	346665	190	239	4.77	$B(1,4)$
Voltages	[(246390,95)|(152624,64)(161470,78)(295436,184)(158475,173)]						

Order	Degree	Diameter	Group m	Group n	Group r	%M-B	Quotient
583,083	10	7	3811	153	7	9.75	$B(0,5)$
Voltages	[(978,115)(2539,74)(2553,53)(1953,41)(1425,113)]						
4,293,452	10	8	63139	68	1378	7.97	$B(0,5)$
Voltages	[(7668,40)(16311,5)(15555,20)(7399,52)(22796,21)]						
27,997,191	10	9	1036933	27	3979	5.78	$B(0,5)$
Voltages	[(930549,20)(912214,23)(656652,6)(849847,11)(527163,8)]						
201,038,922	10	10	531849	378	3620	4.61	$B(0,5)$
Voltages	[(45789,159)(82505,117)(47344,316)(287530,344)(443446,190)]						
1,001,268	11	7	27813	36	149	8.19	$B(1,5)$
Voltages	[(14097,18)|(16956,20)(26380,7)(10013,21)(24861,34)(5702,4)]						
7,442,328	11	8	310097	24	26987	6.08	$B(1,5)$
Voltages	[(304732,12)|(224872,15)(62519,10)(223381,8)(93496,13) (183245,19)]						
72,933,102	11	9	368349	198	113	5.96	$B(1,5)$
Voltages	[(28382,99)|(27703,196)(62272,94)(308948,123)(32154,32) (62308,149)]						
600,380,000	11	10	341125	440	157	4.91	$X(1,3)$
Voltages	[(31691,411)][(0,0)(99550,185)(60165,251)(0,0)(231334,349) (313099,87)(0,0)(112517,278)(160342,268)][(157467,433)] [(323365,254)(16622,346)(326002,374)(269993,262)(83914,219) (25515,415)][(36341,105)][(131753,366)(210675,185) (324856,343)][(93609,232)]						
29,470	12	5	421	70	27	15.24	$B(0,6)$
Voltages	[(207,57)(411,49)(245,61)(280,18)(87,38)(238,36)]						
1,999,500	12	7	33325	60	122	8.55	$B(0,6)$
Voltages	[(21898,16)(22824,31)(13770,38)(29676,55)(2489,28)(13071,29)]						
15,924,326	12	8	723833	22	1330	6.19	$B(0,6)$
Voltages	[(551696,18)(429671,12)(471582,13)(663978,15) (286237,7)(420363,16)]						
158,158,875	12	9	421757	375	1472	5.58	$B(0,6)$
Voltages	[(97451,218)(406305,64)(63300,170)(156195,321) (294559,78)(106616,372)]						
1,506,252,500	12	10	602501	625	793	4.83	$X(3,2)$
Voltages	[(195900,98)(405903,173)(375979,531)][(0,0)(62501,596)(0,0) (532725,416)(0,0)(416901,163)][(262471,430)(248176,337) (422271,491)][(199371,123)(476984,595)(274309,120) (298662,494)][(218686,143)(486428,13)(333671,123)] [(74314,569)(372842,389)][(207063,226)(426030,541)(120275,280)]						

Order	Degree	Diameter	Group			%M-B	Quotient
			m	n	r		
40,260	13	5	671	60	82	13.69	$B(1,6)$
Voltages	[(0,30)|(205,45)(482,50)(160,7)(585,17)(405,39)(419,39)]						
3,322,080	13	7	34605	96	143	7.84	$B(1,6)$
Voltages	[(33120,48)|(2596,10)(19715,71)(5625,85)(22987,23) (27099,63)(7180,6)]						
29,927,790	13	8	997593	30	31979	5.88	$B(1,6)$
Voltages	[(729479,15)|(828328,20)(265150,6)(309408,3) (726797,15)(905640,2)(945332,17)]						
249,155,760	13	9	741535	84	2327	4.08	$X(2,3)$
Voltages	[(506416,62)(303593,61)][(0,0)(537895,21)(122479,55)(0,0) (243024,65)(689455,67)(0,0)(686359,39)(556972,42)][(606797,2) (444577,46)][(199750,0)(344951,64)(22322,57) (307674,23)(462208,78)(377024,53)][(620250,81)(562449,56)] [(370941,0)(22279,57)(126774,23)] [(285562,0)(708329,37)]						
3,077,200,700	13	10	4396001	175	2471	4.20	$X(2,3)$
Voltages	[(1050834,82)(4051224,95)][(0,0)(2718431,162)(778139,79)(0,0) (1215107,1)(1384247,148)(0,0)(220738,35)(4207404,106)] [(3515510,153)(3216151,30)][(2974242,84)(1761761,37) (301884,43)(3187698,59)(1881495,51)(370287,8)][(3893317,102) (1456322,122)][(3048922,15)(2001427,86)(841304,131)] [(3285076,63)(2714782,44)]						
57,837	14	5	1483	39	28	13.35	$B(0,7)$
Voltages	[(339,1)(997,37)(867,32)(214,6)(581,32)(316,21)(606,31)]						
55,913,932	14	8	325081	43	6588	5.87	$X(4,2)$
Voltages	[(106430,7)(313773,8)(223714,20)(189186,12)][(0,0)(192573,37) (0,0)(296775,29)(0,0)(300803,40)][(309059,5)(47776,24) (136369,9)(320594,6)][(81013,21)(210541,18)(22155,21) (323259,14)][(98259,35)(265132,29)(311819,11)(157171,18)] [(19417,28)(46745,22)][(36149,1)(125761,8) (25326,32)(42761,24)]						
600,123,780	14	9	3334021	45	1614	4.85	$X(4,2)$
Voltages	[(1246729,33)(2245306,35)(2370184,38)(3311623,8)] [(0,0)(903039,39)(0,0)(1147042,24)(0,0)(2458550,34)] [(1751124,29)(582205,37)(2885098,28)(2542248,0)] [(930208,5)(1991355,16)(863201,19)(904646,33)] [(79860,21)(2911987,23)(3289358,24)(468687,1)] [(1418592,13)(617741,44)][(1081896,41)(2509432,32) (1949080,26)(1516089,35)]						
7,041,746,081	14	10	62316337	113	8864	4.37	$B(0,7)$
Voltages	[(4821549,46)(58679374,110)(29318275,6)(25838660,1) (41798940,9)(25321982,81) (47965471,21)]						

Order	Degree	Diameter	Group			%M-B	Quotient
			m	n	r		
76,518	15	5	1417	54	29	12.33	$B(1,7)$
Voltages	[(650,27)|(218,45)(933,4)(325,32)(168,12)(963,53)(219,16)(32,28)]						
8,599,986	15	7	159259	54	1712	7.07	$B(1,7)$
Voltages	[(6700,27)|(126036,47)(57600,4)(150432,19)(51748,40)(152314,41)(81233,21)(140934,30)]						
90,001,236	15	8	13417	3354	9	5.28	$D(3,9)$
Voltages	[(2398,154)(5487,3145)(5419,960)][(0,0)(7869,2606)(1953,383)(7457,926)(1106,2402)(11049,1954)(9262,168)(7197,2331)(5331,1826)][(6101,2859)(478,2166)(5944,920)]						
1,171,998,164	15	9	2307083	127	30803	4.91	$X(3,3)$
Voltages	[(361079,95)(326285,29)(2120223,7)][(0,0)(468726,21)(1045888,35)(0,0)(426615,10)(2079467,109)(0,0)(557531,26)(1369074,81)][(486204,24)(2072011,82)(901322,37)][(60626,45)(55799,4)(2237110,71)(1416832,68)(1258150,92)(1362776,53)][(1964780,58)(2083678,77)(1955847,30)][(816675,105)(518794,94)(1707609,54)][(1775741,123)(1476939,106)(503929,37)]						
10,012,349,898	15	10	6867181	1458	23984	2.99	$B(1,7)$
Voltages	[(4196252,729)|(1048331,565)(1612817,455)(2771588,1147)(4683982,596)(1023932,784)(4859964,844)(1292623,1291)]						
140,559,416	16	8	924733	38	66209	4.79	$X(2,4)$
Voltages	[(490996,32)(585355,20)][(0,0)(135355,34)(250349,33)(536336,5)(0,0)(678535,20)(314276,18)(904791,36)(0,0)(703279,1)(425301,25)(457434,4)][(849796,37)(643215,25)][(4852,14)(702331,36)(282942,0)(198986,34)(374258,9)(563413,14)(34875,35)(864260,22)][(206530,4)(649398,17)][(647783,14)(175996,35)(315460,21)(627328,20)][(815187,30)(754790,0)]						
2,025,125,476	16	9	2707387	187	3310	4.60	$X(2,4)$
Voltages	[(1294721,172)(2383050,55)][(0,0)(1365324,92)(2484671,138)(1215141,123)(0,0)(1389967,165)(1354321,127)(1346953,57)(0,0)(1916639,7)(2538792,10)(2070390,179)][(405480,172)(418292,174)][(2645580,125)(1546086,141)(2124669,86)(682388,157)(2443933,69)(1998539,172)(155882,13)(1310956,161)][(494047,41)(1354355,39)][(1745234,24)(1086274,36)(573209,135)(437724,124)][(1479853,33)(321564,6)]						
12,951,451,931	16	10	12138193	1067	8428	1.96	$B(0,8)$
Voltages	[(3123830,833)(563013,411)(7468050,376)(5156666,345)(7771080,199)(6588767,930)(8316960,59)(2549080,1041)]						

Table 5: Cayley digraphs.

Order	Degree	Diameter	Group		
			m	n	r
2,041	3	8	157	13	14
Voltages	[(98,7)(39,3)(100,4)]				
5,115	3	9	341	15	69
Voltages	[(31,12)(317,5)(251,14)]				
11,568	3	10	723	16	44
Voltages	[(714,1)(650,12)(25,13)]				
465	4	5	31	15	28
Voltages	[(12,13)(14,8)(12,11)(29,7)]				
1,378	4	6	53	26	10
Voltages	[(12,4)(8,15)(11,23)(9,25)]				
42,309	4	9	1567	27	193
Voltages	[(73,10)(502,25)(1237,20)(177,6)]				
137,370	4	10	4579	30	160
Voltages	[(1164,21)(3604,25)(4090,24)(1597,4)]				
3,775	5	6	151	25	9
Voltages	[(57,19)(141,20)(81,10)(66,16)(135,8)]				
1,010,658	5	10	15313	66	46
Voltages	[(11099,39)(295,5)(7518,63)(8292,16)(6136,57)]				
9,020	6	6	451	20	84
Voltages	[(52,8)(122,10)(228,6)(36,13)(161,16)(359,15)]				

Table 6: The sizes of the complete sets of all non-isomorphic Cayley digraphs for some small values of d and k found with an exhaustive search. Voltages to all the digraphs that can be constructed with semidirect products are in Table 7 (adjacency lists to these digraphs are linked from [17] and [18]).

Degree	Diameter	Order	Is Arc-Transitive	♯ Distinct Cayley Digraphs
2	4	20	No	3
2	5	27	Yes	2
2	8	171	No	1
3	3	27	No	3
3	4	60	No	18
3	5	165	No	2

Table 7: List of all non-isomorphic Cayley digraphs presented in Table 6 for semidirect products. The additional ten graphs for degree 3 and diameter 4 are not constructed with semidirect products and their adjacency lists are linked from [17] and [18]).

Order	Degree	Diameter	Group		
			m	n	r
20	2	4	5	4	2
Voltages	α_1: [(0,1)(1,0)] α_2: [(0,1)(1,3)] α_3: [(0,3)(1,0)]				
27	2	5	9	3	4
Voltages	α_1: [(1,0)(2,1)] α_2: [(1,0)(2,2)]				
171	2	8	19	9	4
Voltages	α: [(0,4)(1,5)]				
27	3	3	9	3	4
Voltages	α_1: [(0,1)(1,0)(2,1)] α_2: [(0,2)(1,0)(5,2)] α_3: [(1,0)(1,1)(8,1)]				
60	3	4	5	12	2
Voltages	α_1: [(0,1)(1,4)(1,9)] α_2: [(0,1)(1,4)(4,5)] α_3: [(0,1)(1,8)(4,9)] α_4: [(0,2)(1,3)(3,10)] α_5: [(0,2)(1,9)(1,10)] α_6: [(0,3)(1,4)(2,7)] α_7: [(0,3)(1,4)(3,11)] α_8: [(0,7)(1,4)(2,11)]				
165	3	5	11	15	3
Voltages	α_1: [(0,1)(1,3)(1,11)] α_2: [(0,4)(1,12)(9,14)]				

5 Remarks

Graphs and digraphs presented in this paper are a notable improvement of the state of knowledge in the degree-diameter problem.

In the case of graphs, more than half of the values were improved (in fact, 65 out of the 120 values from [19, 20]), in some cases by more than a factor of 3. However, with the smaller graphs the improvements are more modest and vary in size. We will return to commenting on the extent of our improvements at the end of this section, preceded by a discussion on theoretical issues that emerged from our computations.

Most of the graphs listed in our tables are actually Cayley graphs (i.e., regular lifts of bouquets) of semidirect products of cyclic groups. In the literature there appears to be no general lower bound on the order of largest Cayley graphs of degree d and diameter k. However, it follows from [4] that for $d > k$ there are vertex-transitive graphs of degree d, diameter k, and order asymptotically as large as $(d/2)^k$. The fact that most of the graphs of [4] are non-Cayley was proved in [13]. The results for Cayley graphs in our tables suggest that the same lower bound should be provable for Cayley graphs and not just for vertex-transitive graphs. On the other hand, the order of the graphs we have found appears to drift away from the Moore bound relatively quickly.

A very distinct behaviour was noted when considering all the searches done on bouquet quotients (producing Cayley graphs), as some triples of parameters m, n, r for our voltage groups $Z_m \rtimes_r Z_n$ seem to have performed much more successfully than other triples. For example, choosing m and n such that the congruence $r^n \equiv 1 \pmod{m}$ has a large number of solutions (that is, "rich in the r-values") was in general a much more successful strategy compared with the values of m and n that were "poor in the r-values". A perhaps less surprising outcome was that groups $Z_m \rtimes_r Z_n$ with a relatively big centre did very poorly. Thus, most of the groups we have used have a trivial centre, up to a few exceptions with a very small non-trivial centre. A similar but not so distinct behaviour carried through to the groups found for the quotients of order 2, 3 and 4.

In addition, we have noted that certain choices of triples m, n, r have led to "good groups" that seem to work "very quickly" in the sense that results have usually arisen well within the predetermined number of attempts. In contrast, "bad groups" even when showing good prospects during diameter testing (e.g., having the vast majority of vertices within the target diameter distance from each other in a large number of voltage selections) have failed an exhaustive search of all possible voltage assignments. We believe that an explanation of such a behaviour is a challenging problem of independent interest.

A possible line of reasoning to explain certain aspects of the behaviour described above is based on a probabilistic argument which we now outline. For a group G let $\mathrm{Aut}(G)$ be its automorphism group and let $I(G)$ be the set of all involutions of G.

Lemma 1 *Let $s \in \{0, 1\}$ and $l \geq 2$. Assume that there exists a voltage assignment α in a group G on the bouquet $B(s, l)$ with non-involutory elements assigned to the loops, such that the lift has diameter k. Further, let the set of voltages in α be preserved by b automorphisms of G. Then, the probability that a randomly chosen voltage assignment on $B(s, l)$ yields a lift of diameter k is at least*

$$\frac{|\mathrm{Aut}(G)|}{b|I(G)|^s} \binom{|G| - |I(G)| - 1}{l}^{-1}$$

Proof. Since we assume existence of at least one voltage assignment α as above, the reciprocal of the product of the binomial coefficient with $|I(G)|^s$ represents the probability that exactly the voltage assignment α has been selected at random. The action of $\mathrm{Aut}(G)$ on α gives, by our assumption, further $|\mathrm{Aut}(G)|/b$ voltage assignments giving a lift with the same parameters. □

This simple observation suggests that a group with a large automorphism group will have a higher value of the probability estimate given in Lemma 1 and thus will be a favourable candidate for random search. Moreover, this appears to apply to all quotients we have considered.

Specifically, semidirect products of cyclic groups do have relatively large automorphism groups. Indeed, by a result of [16], if the group $G = Z_m \rtimes_r Z_n$ has a trivial centre, then $|\mathrm{Aut}(G)| = m\phi(m)$ where $\phi(m)$ is the Euler totient function. Lemma 1 now has the following consequence.

Lemma 2 *Let $s \in \{0, 1\}$ and $l \geq 2$. Let $G = Z_m \rtimes_r Z_n$ have a trivial centre. Suppose that there exists a voltage assignment α in a group G on the bouquet $B(s, l)$ with non-involutory elements assigned to the loops, such that the lift has diameter k. Further, assume that only the trivial automorphism of G setwise preserves the voltages in α. Then, the probability that a randomly chosen voltage assignment on $B(s, l)$ yields a lift of diameter k is at least*

$$\frac{m\phi(m)}{|I(G)|^s}\binom{|G| - |I(G)| - 1}{l}^{-1}$$

A thorough look at the tables shown in Section 4 and also in [5], [3] and [8] suggests that most successful semidirect products have a large value of m in comparison to n, and a trivial centre, with m having a large value of $\phi(m)$; in addition, it turns out that the voltage assignments therein are indeed setwise preserved by the trivial automorphism only. In many cases, mostly for graphs with small degree, Lemma 2 then gives a relatively large value of the probability estimate. Therefore, a quick random computation will indicate with sufficiently high probability if a suitable set of voltages exists. This suggests a possible explanation to the observation made in [8] and also in our own computations that record graphs are found either quickly, or not at all.

Since the family of semidirect products is relatively rich, Lemma 2 appears to be a useful tool in eliminating groups that will have very small chance to be successful in practice.

We conclude with returning to observations on the size of our improvements. We believe that further small improvements using the voltage constructions method are possible, in particular for graphs of big orders. However, additional improvements by, say, a factor of at least two, seem less likely now by our methods, as orders of most of our bigger graphs are already about 5% of the corresponding Moore bound. When one considers the Lemmas 1 and 2 it is clear that in order for a successful voltage set to be found randomly (assuming at least one such set exists), many random sets must be checked, and with increase in the size of the graphs and groups this will involve computational times that are impractical by today's standards.

Moreover, when considering computational evidence, it appears that with groups of bigger size, less distinct voltage sets exist (here we mean sets that are not mapped to each other by group automorphisms). Thus, it appears likely that for the graphs of orders only about 5% of the Moore bound, the groups are in the range where many such sets are available for us to find. Improvements by a factor of at least two will probably also mean that the number of such distinct sets in the groups will decrease by some factor and the likelihood of finding any such sets will decrease even further. For example, when considering Lemma 2, it is easy to see why even if there was a group of order 40% of the Moore bound for degree 16 and diameter 10 with exactly one successful distinct voltage set, finding it randomly seems unlikely.

Acknowledgments

The authors would like to thank C. Paul Bonnington, Michael J. Dinneen, Paul Hafner and Joseph Heled from the University of Auckland for their support and advice in the writing of this paper.

All the results presented in this paper were independently confirmed by Peter Nelson and Michael Brough, students of University of Auckland, as a part of a summer scholarship project supervised by the authors.

This research was done as a part of the PhD program of Eyal Loz under the supervision of C. Paul Bonnington and Jozef Širáň from the University of Auckland. The first author is supported by the Top Achiever Doctoral Scholarship of the TEC of New Zealand. The second author acknowledges partial support by his RSNZ Marsden Fund Grant and by the VEGA Grant No. 1/2004/05 and the APVV Grants No. 20-000704 and 0040-06.

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[18] Moore Bound database on-line: `http://moorebound.indstate.edu/`

[19] Charles Delorme's LaTeX Table: `http://www.lri.fr/~cd/Or.dvi`

[20] Francesc Comellas's WWW Table:
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(Received 26 Feb 2007; revised 3 Nov 2007)