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On counting n-element trellises having exactly one pair of noncomparable elements

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Abstract

A trellis is a pseudo ordered set any two of whose elements have a least upper bound and a greatest lower bound. In this paper a formula for the number of n-element trellises having exactly one pair of noncomparable elements is given.

1 Introduction

1.1 Trellis

The concept of a pseudo ordered set was introduced by Fried (see [4]). A reflexive and antisymmetric relation $\trianglelefteq$ on a set A is a *pseudo order* and the pair $\langle A, \trianglelefteq \rangle$ is a *pseudo ordered set* or a *psoset*. Two elements a and b are *non comparable* in A, written $a||b$, if neither $a \trianglelefteq b$ nor $b \trianglelefteq a$ holds in A. A psoset any two of whose elements are comparable is a *tournament*. If B is a subset of a psoset A, an element c in A is an *upper bound* of B if $b \trianglelefteq c$ for all b in B; c is the *least upper bound* of B if c is an upper bound of B and $c \trianglelefteq d$ for any upper bound d of B. The *lower bound* and the *greatest lower bound* of B are defined dually.

A *trellis* is a psoset $\langle T, \trianglelefteq \rangle$ any two of whose elements a and b have a least upper bound c denoted $c = a \vee b$ and a greatest lower bound d denoted $d = a \wedge b$ in T.

The following properties hold immediately (see [9]):

1. $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ — Commutativity
2. $a \wedge (b \vee a) = a = a \vee (b \wedge a)$ — Absorption
3. $a \vee b = a$ and $a \vee c = a$ implies $a \vee (b \vee c) = a$ and
   $a \wedge b = a$ and $a \wedge c = a$ implies $a \wedge (b \wedge c) = a$ — Part preservation.

Thus $\langle T, \wedge, \vee \rangle$ is an algebra.

Conversely, a set T with two commutative, absorptive and part preserving relations $\wedge$ and $\vee$ is a trellis with the pseudo order $\trianglelefteq$ defined by $a \trianglelefteq b$ if $a \wedge b = a$ and $a \vee b = b$. All definitions in this section are based on [9] and [10].

Shashirekha counts n-element trellises by an empirical method for $n < 7$ (see [8]).

1.2 Isomorphism in trellises

Two trellises T_1 and T_2 are *isomorphic* and the bijection $f : T_1 \to T_2$ is an *isomorphism* if and only if $a \trianglelefteq b \Leftrightarrow f(a) \trianglelefteq f(b)$, or equivalently,

$$f(a \vee b) = f(a) \vee f(b) \text{ and } f(a \wedge b) = f(a) \wedge f(b) \quad \text{(see [8])}.$$

A psoset A can be represented by an oriented graph G_A with vertex set A such that a directed segment from a vertex i to a vertex j exists in G_A if and only if $i \triangleleft j$ holds in A (see [8]). Hence two trellises T_1 and T_2 are isomorphic if and only if their corresponding oriented graphs G_{T_1} and G_{T_2} are isomorphic, or equivalently, the adjacency matrices $A(G_{T_1})$ and $A(G_{T_2})$ of the oriented graphs G_{T_1} and G_{T_2} respectively, differ only by a permutation of the rows accompanied by the same permutation of the columns provided the rows and the columns are arranged in the same order (see [3]). For more details regarding trellises see [9], [10] and [8].

All the definitions in the remaining part of this section are based on [5].

1.3 Reduced Ordered Pair Group

Let $X = \{1, 2, 3, \ldots, p\}$ and $X^{[2]} = \{(x, y) : x, y \in X, x \neq y\}$. If A is a permutation group acting on X, then X is the *object set of* A and the *reduced ordered pair group of* A, denoted $A^{[2]}$, acts on $X^{[2]}$ and is induced by A such that for each permutation α in A, there is a permutation α' in $A^{[2]}$ induced by α such that for every pair (i, j) in $X^{[2]}$ the image under α' is given by $\alpha'(i, j) = (\alpha i, \alpha j)$.

If α is a permutation in the symmetric group S_p on p objects, and α' is the permutation in $S_p^{[2]}$ induced by α, then the *converse* of any cycle z' in the disjoint cycle decomposition of α' is that cycle of α' which permutes all ordered pairs (i, j) such that (j, i) is permuted by z'. The cycle z' of α' is *self converse* if (i, j) is permuted whenever (j, i) is.

If z_r and z_t are two cycles of lengths r and t respectively, then there are $2rt$ pairs (i, j) in $X^{[2]}$ with i permuted by z_r and j permuted by z_t. These pairs are permuted in $2(r, t)$ cycles of length $[r, t]$ (where (r, t) and $[r, t]$ are the g.c.d. and l.c.m. of r and t respectively). For more details on reduced ordered pair groups, see [5].

1.4 Power Group

If A is a finite permutation group with object set $X = \{1, 2, 3, \ldots, p\}$ and B a finite permutation group with a countable object set Y of at least 2 elements, then the *power group* denoted B^A has the collection Y^X of functions from X into Y as its object set. The permutations of B^A consist of all ordered pairs, written $(\alpha; \beta)$, of permutations α in A and β in B. The *image* of any function f in Y^X under $(\alpha; \beta)$ is given by $((\alpha; \beta)f)(x) = \beta f(\alpha x)$ for each x in X (see [5]).

If $X = \{1, 2, 3, \ldots, p\}$, $Y = \{0, 1\}$ and E_2 is the identity group on n objects, then the power group $E_2^{S_p^{[2]}}$ has the collection $Y^{X^{[2]}}$ of functions from $X^{[2]}$ into Y as its object set. The permutations of $E_2^{S_p^{[2]}}$ consist of all ordered pairs, written $(\alpha; \beta)$, of permutations α in $S_p^{[2]}$ and β, the identity permutation on Y. The image of any function f in $Y^{X^{[2]}}$ under $(\alpha; \beta)$ is given by $((\alpha; \beta)f)(x) = \beta f(\alpha(i, j)) = f(\alpha i, \alpha j)$ for each $x = (i, j)$, $i \neq j$ (see [5]).

2 Preliminaries

If a permutation α in S_p splits into j_k cycles of length k for each k from 1 to p, then α is *of the type* $(j) = (j_1, j_2, j_3, \ldots, j_p)$ and the *cycle structure* of α is $1^{j_1} 2^{j_2} \ldots p^{j_p}$. Note that $\sum_{k=i}^{p} k j_k = p$ (see [6]).

Lemma 2.1 *The number of permutations in S_p of the type (j) is* $\frac{p!}{\prod k^{j_k} j_k!}$.

(See [6].)

Lemma 2.2 (Burnside's Lemma) *Let G be a finite permutation group with object set X. Define $x_1 \approx x_2$ in X if and only if there exists an $\alpha \in G$ such that $\alpha(x_1) = x_2$. Then '$\approx$' is an equivalence relation on X and the number of '$\approx$' equivalence classes (or G orbits) thus defined is*

$$\frac{1}{|G|} \sum_{\alpha \in G} \Psi(\alpha),$$

where $\Psi(\alpha)$ is the number of elements x in X such that $\alpha(x) = x$.

(See [1] and [6].)

Lemma 2.3 (Restricted form of Burnside's Lemma) *Let G be a finite permutation group with object set X. Let Y be a subset of X such that Y is a union of orbits of G. If $G\,|_Y$ denotes the set of permutations obtained by restricting those of G to Y, then the number of $G\,|_Y$-orbits is*

$$\frac{1}{|G|}\sum_{\alpha\in G}\Psi(\alpha),$$

where $\Psi(\alpha)$ is the number of elements x in X such that $\alpha_{|_Y}(x) = x$.

(See [5].)

Counting tournaments of order p is due to Davis (see [2], [7] and [5]).

Lemma 2.4 *The number $T(p)$ of tournaments of order p is*

$$T(p) = \frac{1}{p!}\sum_{(j)}{}^{*}\frac{p!}{\prod k^{j_k} j_k!}2^{t(j)},$$

where the asterisk on $\sum$ calls attention to the unconventional summing only over those partitions (j) of p with $j_k = 0$ whenever k is even, and where

$$t(j) = \frac{1}{2}\left(\sum_{m,n=1}^{p} j_m j_n(m,n) - \sum_{k=1}^{p} j_k\right).$$

(See [5].)

3 n-Element trellises having exactly one pair of noncomparable elements

Note that if C is the collection of all n-element trellises having exactly one pair of noncomparable elements and T is a trellis in C, then up to isomorphism $1||2$, $1\vee 2 = 3$, $1\wedge 2 = 4$ hold in T and the subtrellis $Z = \{5, 6, \ldots, n\}$ is a tournament.

Lemma 3.1 *Let $X = \{1, 2, 3, \ldots, n\}, Y = \{0, 1\}$, $Z = \{5, 6, \ldots, n\}, E_2$ the identity group on Y and $G = \{\beta\alpha : \text{either } \beta \in E_n \text{ or } \beta = (1\,2) \text{ and } \alpha \text{ is a permutation on } Z\}$. Then the power group E_2^G has as its object set the collection C, and its orbits are precisely the isomorphic classes of C.*

Proof: If a permutation α in S_n induces a permutation of trellises in C, then $\alpha(1)||\alpha(2)$, $\alpha(1\vee 2) = \alpha(1)\vee\alpha(2)$ and $\alpha(1\wedge 2) = \alpha(1)\wedge\alpha(2)$. Therefore $((\alpha(1) = 1$ and $\alpha(2) = 2)$ or $(\alpha(1) = 2$ and $\alpha(2) = 1))$, $\alpha(3) = 3$ and $\alpha(4) = 4$. Hence the proof follows.

Theorem 1 *If $Tr_1(n)$ is the number of non isomorphic n-element trellises having exactly one pair of noncomparable elements, then*

$$Tr_1(n) = \frac{1}{p!}\sum_{(j)}^{*} \frac{p!}{\prod k^{j_k} j_k!} 2^{t(j)} \left((12)^{\sum_{k=1}^{p} j_k} + (4)^{\sum_{k=1}^{p} j_k} \right),$$

where $p = n-4$, the asterisk on $\sum$ calls attention to the unconventional summing only over those partitions (j) of p with $j_k = 0$ whenever k is even, and where

$$t(j) = \frac{1}{2}\left(\sum_{m,n=1}^{p} j_m j_n (m,n) - \sum_{k=1}^{p} j_k \right).$$

Proof: The following justification for the expression for $t(j)$ is based on [5]:

For the sake of convenience, consider $X = \{1,2,3,\dots,p\}$ instead of $Z = \{5,6,7,\dots, n\}$. The orbits of the power group $E_2^{S_p^{[2]}}$ correspond to digraphs of order p. On restricting this group to the set F of all functions f which represent p-element tournaments, namely those f for which $f(i,j) \neq f(j,i)$, the restricted form of Burnside's Lemma can be applied. As a result the number of nonisomorphic p-element tournaments can be expressed in terms of the number of functions in F fixed by the permutations in the power group $E_2^{S_p^{[2]}}$. Thus, for each permutation α in S_p of the type $(j) = (j_1, j_2, j_3, \dots, j_p)$, we need to find the number of functions f in F such that $f(i,j) = f(\alpha i, \alpha j)$ for all (i,j) in $X^{[2]}$, or those functions fixed by the permutations α' in the power group induced by α; for the sake of convenience, let us say that they are fixed by α instead of α'. Therefore, if the cycles of α determine the partitions (j) of p, then we need to show that the number of functions in F fixed by α is $2^{t(j)}$.

If f is a tournament fixed by α, then f is constant on the cycles of the induced permutation α'. If $z_k = (1,2,\dots,k)$ is any cycle of even length, then α' has the cycle

$$z' = \left(\left(1, \frac{k}{2}+1\right)\left(2, \frac{k}{2}+2\right)\cdots\left(\frac{k}{2}, \frac{k}{2}+\frac{k}{2}\right)\left(\frac{k}{2}+1, 1\right)\left(\frac{k}{2}+2, 2\right)\cdots\left(\frac{k}{2}+\frac{k}{2}, \frac{k}{2}\right)\right)$$

which is self converse. If f is constant on this self converse cycle, then $f(1, \frac{k}{2}+1) = f(\frac{k}{2}+1, 1)$ which contradicts the fact that f is a tournament. Thus α does not fix any tournament Z and hence it fixes no trellises in C also. Hence the asterisk on the summation sign is justified.

If $z_k = (1,2,\dots,k)$ is any cycle of odd length, then the induced permutation α' has, in its cycle representation, $k-1$ cycles each of length k, namely, $((1,2)(2,3)(3,4)\dots(k,1))$, $((1,3)(2,4)(3,5)\dots(k,2))$, $((1,4)(2,5)(3,6)\dots(k,3)),\dots,$ $((1,k)(2,1)(3,2)\dots(k,k-1))$. Note that the ith cycle is the converse of the $(k-i)$th cycle for each i from 1 to $\frac{k-1}{2}$.

If a tournament f is fixed by a permutation containing a cycle of odd length, then $f(1,2) = f(2,3) = \cdots = f(k,1) = 1$ or 0, $f(1,3) = f(2,4) = \cdots = f(k,2) = 1$

or 0, $f(1,4) = f(2,5) = \cdots = f(k,3) = 1$ or $0, \ldots, f(1, \frac{k+1}{2}) = f(2, \frac{k+3}{2}) = \cdots = f(k, \frac{k-1}{2}) = 1$ or 0. Therefore z_k fixes exactly $2^{\frac{k-1}{2}}$ tournaments and the contribution to $t(j)$ due to all the odd cycles of α is $\sum j_k \frac{k-1}{2}$ summed over odd k.

Now we consider two cycles z_m and z_n of α and the pairs in $X^{[2]}$ which have one point in each. Two such cycles induce $2(m,n)$ cycles of ordered pairs in $X^{[2]}$. These latter cycles consist of (m,n) pairs of converse cycles.

Therefore the number of tournaments fixed by the product $z_m z_n$ is $2^{(m,n)}$ and the contribution to $t(j)$ of all such pairs z_m and z_n with $m \neq n$ is $\sum_{m<n} j_m j_n (m,n)$.

The contribution to $t(j)$ of a pair of cycles of the same length k is $\sum \binom{j_k}{2} k$. Thus

$$
\begin{aligned}
t(j) &= \sum j_k \frac{k-1}{2} + \sum\nolimits_{m<n} j_m j_n (m,n) + \sum k \frac{j_k(j_k-1)}{2} \\
&= \frac{1}{2}\left[\sum j_k(k-1) + 2\sum\nolimits_{m<n} j_m j_n(m,n) - \sum k j_k + \sum j_k j_k (k.k)\right] \\
&= \frac{1}{2}\left[\sum_{m,n=1}^{p} j_m j_n (m,n) - \sum j_k\right].
\end{aligned}
$$

To find $Tr_1(n)$, from the note above it suffices to count the number of isomorphic classes of trellises in C. From the lemma above and by Burnside's Lemma, it suffices to find the number of trellises T in C fixed by each permutation in G, where $G = \{\beta\alpha :$ either $\beta \in E_n$ or $\beta = (1\ 2)$ and α is a permutation on $Z\}$.

If T is a trellis in C, then $1||2$, $1 \vee 2 = 3$, $1 \wedge 2 = 4$ hold in T and hence for each i from 5 to n, the submatrix $(a_{i1}, a_{i2}, a_{i3}, a_{i4})$ of its adjacency matrix $A(G_T)$ can be defined by precisely one of the following 12 ordered 4-tuples $(a_{i1}, a_{i2}, a_{i3}, a_{i4})$ given in the table on the next page.

Further, if any cycle $z_k = (i_1, i_2, \ldots, i_k)$ fixes the trellis T, then $a_{i_1,l} = a_{i_2,l} = a_{i_3,l} = \cdots = a_{i_k,l}$ for each l from 1 to 4. Hence for a trellis fixed by a permutation α on Z, the number of choices for defining the $p \times 4$ submatrix $(a_{ij})_{5 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant 4}$ of its adjacency matrix $A\,(G_T)$ is $(12)^{\sum_{k=1}^{p} j_k}$.

a_{i1}	a_{i2}	a_{i3}	a_{i4}
0	0	0	0
0	0	0	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	1
1	1	1	1

Now, if the pemutation (1 2) fixes a trellis T in C, then for each i from 5 to n, the submatrix $(a_{i1}, a_{i2}, a_{i3}, a_{i4})$ of the adjacency matrix $A(G_T)$ can be defined by precisely one of the following four ordered 4-tuples given in the table below.

a_{i1}	a_{i2}	a_{i3}	a_{i4}
0	0	0	0
0	0	0	1
1	1	0	1
1	1	1	1

Hence for a trellis fixed by the permutation (1 2)α in G, where α is a permutation on Z, the total number of choices for defining the submatrix $(a_{ij})_{p\times 4}$, where $i = 5$ to n and $j = 1$ to 4 is $(4)^{\sum_{k=1}^{p} j_k}$. Moreover $a_{34} = 0$ or 1. Thus

$$Tr_1(n) = \frac{2}{2p!}\sum_{(j)}^{*} \frac{p!}{\prod k^{j_k} j_k!} 2^{t(j)} \left((12)^{\sum_{k=1}^{p} j_k} + (4)^{\sum_{k=1}^{p} j_k} \right).$$

Hence the theorem holds.

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