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Orphan complexes of neighborhood anti-Sperner graphs

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Abstract

We introduce and study a class of simplicial complexes, the orphan complexes, associated to simple graphs whose family of (open or closed) vertex-neighborhoods are anti-Sperner. Under suitable restrictions, we show that orphan complexes of such graphs are always shellable and provide a characterization of graphs in terms of induced forbidden subgraphs contained in this restricted subfamily.

1 Introduction

The study of graphs with respect to specific structures carried by the family of the neighborhoods of its vertices has long been of interest. Other than graph theoretical works, they have also provided an intriguing source of examples for many problems in topological combinatorics. In this direction, we here offer a study of graphs whose neighborhoods of vertices form anti-Sperner families. Recall that a simple graph G with $V(G) = [n] = \{1, 2, \dots, n\}$ is said to be neighborhood anti-Sperner (NAS) if for every $i \in [n]$, there exists $j \in [n]\backslash\{i\}$ such that $N_G(i) \subseteq N_G(j)$, where $N_G(i)$ denotes the open-neighborhood of the vertex i in G, and the vertex j is called a parent of i. These graphs were introduced by Porter [4], and studied later in Porter and Yucas [5]. More recently, McSorley et al [3] extended the definition of NAS-graphs to CNAS-graphs by considering closed-neighborhoods.

We introduce a simplicial complex, the orphan complex $\mathfrak{O}(G)$, associated to any NAS-graph G by declaring that a subset S of $V(G)$ is a face of $\mathfrak{O}(G)$ if it excludes any parent of vertices that it contains. It follows naturally that the clique complex of any NAS-graph G is a subcomplex of $\mathfrak{O}(G)$, and those whose clique complexes are

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isomorphic to their orphan complexes form an interesting subfamily of NAS-graphs. We call such graphs as orphan-complete NAS-graphs. We prove in Section 4 that the clique complexes of orphan-complete NAS-graphs are always shellable. On the other hand, we note that the topological structure of the independence complex of a NAS-graph is almost trivial. In more details, if G is a NAS-graph and $x, y \in V(G)$ are two vertices with $N_G(y) \subsetneqq N_G(x)$, then $\mathcal{I}(G)$ and $\mathcal{I}(G-x)$ are homotopy equivalent, where $\mathcal{I}(G)$ denotes the independence complex of G. This is due to fact that the link of x in $\mathcal{I}(G)$ is cone with apex y in this case (see [1] for more details). Note also that the graph $G-x$ is still a NAS-graph for such a vertex. Therefore, the independence complex $\mathcal{I}(G)$ of a NAS-graph is homotopy equivalent to the independence complex of a hyperoctahedral graph.

2 Preliminaries

By a simple graph G, we will mean an undirected graph without loops or multiple edges. If G is a graph, $V(G)$ and $E(G)$ (or simply V and E) denote its vertex and edge sets. An edge between u and v is denoted by $e = uv$. If $U \subset V$, the graph induced on U is written G_U, and $G - U$ denotes the graph induced on $V - U$. In particular, we abbreviate $G - \{v\}$ to $G - v$. We denote by $|G|$ and $\|G\|$ the order and size of G, while $d(v)$ denotes the degree of a given vertex $v \in V$. The *open-neighborhood* $N(x)$ of a vertex $v \in V$ is defined to be the set $\{u \in V : uv \in E\}$, while $N[v] = N(v) \cup \{v\}$ is called the *closed-neighborhood* of v in G. For a given graph $G = (V, E)$, we denote the *complement* of G by $\overline{G} = (V, \overline{E})$, where $\overline{E} = (V \times V) \backslash E$.

Throughout C_n and P_n denote the (chordless) cycle and path with n vertices, while K_n and $K_{n_1,\ldots,n_k}$ the complete graph and complete multipartite graph respectively. Finally, we recall that a graph G is said to be H-free for some graph H, if G has no induced subgraph isomorphic to H.

A simplicial complex Δ over a given set X is a subfamily of subsets of X which is closed under inclusions and contains every singleton-set formed by elements of X. The elements of Δ are called *faces*, while maximal faces are called *facets*. For a given vertex x, we denote the *deletion complex* and the *link* of x in Δ by $\mathrm{del}_\Delta(x)$ and $\mathrm{lk}_\Delta(x)$ respectively, that are defined by

$$\mathrm{del}_\Delta(x) := \{F \in \Delta : x \notin F\} \quad \text{and} \quad \mathrm{lk}_\Delta(x) := \{F \in \Delta : x \notin F \text{ and } F \cup \{x\} \in \Delta\}.$$

Specifically, we here only deal with simplicial complexes that are associated to simple graphs. When $G = (V, E)$ is a simple graph, we denote by $\Delta(G)$ the *clique complex* of G which is the simplicial complex on V consisting those subsets of V inducing complete subgraphs of G, whereas the complex $\mathcal{I}(G) := \Delta(\overline{G})$ is called the *independence complex* of G. Any face of $\Delta(G)$ is called a *complete* of G, and in particular, a *clique* of G is an inclusion-maximal complete. We note that $\mathrm{del}_{\Delta(G)}(v) = \Delta(G - v)$ and $\mathrm{lk}_{\Delta(G)}(v) = \Delta(G[N(v)])$ for any $v \in V$.

We next briefly outline some definitions and properties of neighborhood anti-Sperner graphs, and refer readers to [4] and [5] for more details.

For any given two distinct vertices x, y of a simple graph $G = (V, E)$, we say that y is a *parent* of x, if $N(x) \subseteq N(y)$. The set $P_G(x) := \{y \in V\backslash\{x\} : y \text{ is a parent of } x\}$ is called the *parent-set* of x in G. We write just $P(x)$ for $P_G(x)$, when G is clear.

A simple graph $G = (V, E)$ is said to be *neighborhood anti-Sperner* (or a NAS-*graph* for short) if $P(x) \neq \emptyset$ for every $x \in V$.

When G is a connected NAS-graph, it is 2-connected and the girth of G is at most 4. Note also that $xy \notin E$ in a NAS-graph G whenever $x \in P(y)$ or $y \in P(x)$.

Porter and Yucas [5] presented a concrete way of constructing new NAS-graphs from a given family of NAS-graphs and an arbitrary graph. Let G be a graph with $V(G) = [n] = \{1, 2, \ldots, n\}$, and let $H_1, H_2, \ldots, H_n$ be a collection of graphs. The graph $I(H_1, H_2, \ldots, H_n : G)$ is defined to be the graph obtained by replacing each vertex i with a copy of H_i, and if $ij \in E(G)$, then connect all vertices of H_i to all vertices in H_j. In the case where $H_i \cong H_j = H$ for all $i, j \in [n]$, we abbreviate $I(H_1, H_2, \ldots, H_n : G)$ to $I(H; G)$. We recall that if every H_i is the complement of a complete graph, the resulting graph I is the graph obtained from G by *multiplication of vertices* (see [2]). The *mirror-graph* $\mathrm{Mir}(G)$ of G is defined by $I(\overline{K}_2 : G)$. The graphs $I(H_1, H_2, \ldots, H_n : G)$ constitute an interesting family of NAS-graphs provided that each H_i is a NAS-graph.

Theorem 2.1 ([5], Theorem 2.11) *If $H_1, H_2, \ldots, H_n$ are NAS-graphs, and G is any graph with $V(G) = [n]$, then $I(H_1, H_2, \ldots, H_n : G)$ is a NAS-graph.*

In a more recent work, McSorley et al [3], have extended the definition of *NAS*-graphs by considering closed vertex-neighborhoods. A graph G is called *closed neighborhood anti-Sperner* (CNAS) if for every x there is a $y \in V\backslash\{x\}$ such that $N[x] \subseteq N[y]$, and a graph H is called *closed neighborhood distinct* (CND) if $N[x] \neq N[y]$ whenever $x \neq y$. In this case, we denote the set of closed-parents of given vertex x by $P[x]$. McSorley et al [3] present an algorithm to construct all connected CNAS-graphs on a fixed number of vertices from labeled CND-graphs. We prove that the family of NAS-graphs constitutes a subclass of CND-graphs.

Proposition 2.2 *If G is a NAS-graph, then G is a CND-graph.*

Proof. Assume otherwise that G contains two distinct vertices x and y such that $N[x] = N[y]$. Let $z \in P(x)$ so that $y \in N(x) \subseteq N(z)$ which implies that $z \in N[y] = N[x]$. Therefore, $xz \in E$, a contradiction. □

3 Orphan-complete neighborhood anti-Sperner graphs

Definition 3.1 *Let $G = (V, E)$ be a NAS-graph. A subset $S \subseteq V$ is called an* orphan set *if $S \cap P(x) = \emptyset$ for every $x \in S$. We call the collection $\mathfrak{O}(G)$ of orphan sets of G, the* orphan complex *of G that forms a simplicial complex over V.*

Proposition 3.2 *The clique complex* $\Delta(G)$ *of a NAS-graph* G *is a subcomplex of* $\mathfrak{O}(G)$.

Proof. This follows from the fact that if $x \in P(y)$ for some $x, y \in V$, then $xy \notin E$. □

We say that a NAS-graph G is *orphan-complete* if every orphan set of G is a complete of G, otherwise G is called *orphan-incomplete.* We next provide a characterization of orphan-complete NAS-graphs in terms of forbidden induced subgraphs.

Theorem 3.3 *A NAS-graph* G *is orphan-complete if and only if* G *is* $(2K_2, P_4)$*-free.*

Proof. Suppose that G is orphan-complete while containing an induced $2K_2$ (or resp. P_4) with vertex set $\{a, b, c, d\}$ such that $ab, cd \in E$ and $ac \notin E$. It then easily follows that $\{a, c\} \in \mathfrak{O}(G)$ contradicting to orphan-completeness of G.

For the converse, suppose that G is not orphan-complete. Therefore, there exists an orphan set $\{x, y\}$ such that $xy \notin E$. Since $x \notin P(y)$ and $y \notin P(x)$, we choose $u, v \in V$ satisfying $u \in N(x)\backslash N(y)$ and $v \in N(y)\backslash N(x)$. The set $\{x, y, u, v\}$ induces a $2K_2$ or P_4 depending whether uv is an edge of G or not. □

We remark that the property of being orphan-complete is inherited by all induced NAS-subgraphs. On the other hand, orphan-complete graphs are almost connected, that is, if G is a disconnected orphan-complete graph, then G is isomorphic to the disjoint union of an orphan-complete graph and a copy of $\overline{K_n}$ for some $n \geq 1$.

Proposition 3.4 *If* G *is an orphan-complete NAS-graph, so is* $G[N(v)]$ *for any* $v \in V$.

Proof. It is enough to show that $G[N(v)]$ is a NAS-subgraph, which follows from the fact that if $x \in N(v)$, then $P(x) \subset N(v)$. □

Proposition 3.5 *Let* $H_1, H_2, \ldots, H_n$ *be connected orphan-complete NAS-graphs and let* G *be an arbitrary connected graph with* $V(G) = [n]$. *Then* $I = I(H_1, H_2, \ldots, H_n : G)$ *is an orphan-complete NAS-graph if and only if* $G = K_n$.

Proof. We first verify that I is orphan-complete whenever $G = K_n$. We know from Proposition 2.1 that the graph $I(H_1, H_2, \ldots, H_n : K_n)$ is always a NAS-graph, so it is only left to show that it is orphan-complete. Assume that $x, y \in V(I)$ are given such that $\{x, y\}$ is an orphan set in I. If there exists an $i \in [n]$ such that $x, y \in H_i$, then $xy \in E(I)$, since the graph H_i is orphan-complete. Otherwise, suppose that $x \in V(H_i)$ and $y \in V(H_j)$ for some $i, j \in [n]$ with $i \neq j$. Since $G = K_n$ so that $ij \in E(K_n)$, then we again deduce that $xy \in E(I)$.

For the necessary part, let $I = I(H_1, H_2, \ldots, H_n : G)$ be an orphan-complete NAS-graph, and assume for contrary that there exist $i, j \in [n]$ with $i \neq j$ such that $ij \notin E(G)$. If we choose $x_i \in V(H_i)$ and $x_j \in V(H_j)$ arbitrarily, then the set $\{x_i, x_j\}$ is an orphan set in I with $x_i x_j \notin E(I)$, since the sets $N_{H_i}(x_i)$ and $N_{H_j}(x_j)$ are nonempty. This contradicts to orphan-completeness of I; thus, $G = K_n$. □

In the totally disconnected case, we have more options for the host graphs.

Proposition 3.6 *Let G be an arbitrary connected graph over $V(G) = [n]$, and let $\{k_1, k_2, \dots, k_n\}$ be a set of positive integers with $k_i \geq 2$ for each $i \in [n]$. Then $I(\overline{K}_{k_1}, \dots, \overline{K}_{k_n} : G)$ is an orphan-complete NAS-graph provided that either $G = K_n$ or G is an orphan-complete NAS-graph.*

Proof. In case $G = K_n$, if $\{x, y\}$ is an orphan set in I, then there must exist $i, j \in [n]$ with $i \neq j$ such that $x \in V(\overline{K}_{k_i})$ and $y \in V(\overline{K}_{k_j})$; however, we necessarily have $xy \in E(I)$ in this situation. Therefore, $I = I(\overline{K}_{k_1}, \dots, \overline{K}_{k_n} : K_n)$ is orphan-complete.

Assume that $G \neq K_n$ is a connected orphan-complete NAS-graph, and let $\{u, v\}$ be an orphan set in $I = I(\overline{K}_{k_1}, \dots, \overline{K}_{k_n} : G)$. Since the neighborhoods of any two vertices of I contained in the same stable set in I are equal, we again deduce that there exist $r, s \in [n]$ with $r \neq s$ such that $u \in V(\overline{K}_{k_r})$ and $v \in V(\overline{K}_{k_s})$. We claim that the set $\{r, s\}$ is an orphan set in G. Otherwise, suppose without loss of generality that $r \in P_G(s)$, that is, $N_G(s) \subseteq N_G(r)$. Let $w \in N_I(v)$ be given with $w \in V(\overline{K}_{k_t})$ for some $t \in [n] \backslash \{r, s\}$. It follows that $t \in N_G(s)$; hence, $t \in N_G(r)$ so that $w \in N_I(u)$. However, this implies that $u \in P_I(v)$ contradicting to the fact that $\{u, v\}$ is an orphan set. Therefore, $\{r, s\}$ is an orphan set in G, and since G is orphan-complete, we have $rs \in E(G)$ implying that $uv \in E(I)$. □

We note that the graph $I(\overline{K}_{k_1}, \dots, \overline{K}_{k_n} : G)$ could be orphan-complete, though G is not even a NAS-graph. For example, $I(\overline{K}_2 : P_3) = K_{2,4}$ is an orphan-complete NAS-graph. This is also true for regular orphan-complete NAS-graphs. Consider the graph $Q = I(\overline{K}_2, \overline{K}_4, \overline{K}_4, \overline{K}_2 : K_4 - 14)$, where $K_4 - 14$ is the graph obtained by removing the edge 14 from the complete graph on four vertices. Now, the graph Q is an 8-regular orphan-complete NAS-graph. On the other hand, the graphs $I(\overline{K}_2 : P_n)$ or $I(\overline{K}_2 : C_n)$ are not orphan-complete for any $n \geq 4$.

We next show that the orphan complexes of NAS-graphs are always clique complexes of some graphs. Let $G = (V, E)$ be a NAS-graph. We define the *orphan-completion* $C(G) = (V, C(E))$ of G by

$$C(E) := E \cup \{xy \colon \{x, y\} \text{ is an orphan set with } xy \notin E\}.$$

In other words, $C(G)$ is the graph obtained from G by adding successively all non-edges of G that form orphan sets in G. Now, the following is immediate.

Corollary 3.7 $\mathfrak{O}(G) \cong \Delta(C(G))$ *for any NAS-graph G. In particular, G is orphan-complete if and only if $G = C(G)$.*

In general, the graphs $C(G)$ are not even NAS-graphs. As an example, the graph G depicted in Figure 1 is an orphan-incomplete NAS-graph whose completion $C(G)$ is not a NAS-graph, since, for instance, the vertex with label 1 has no parents in $C(G)$.

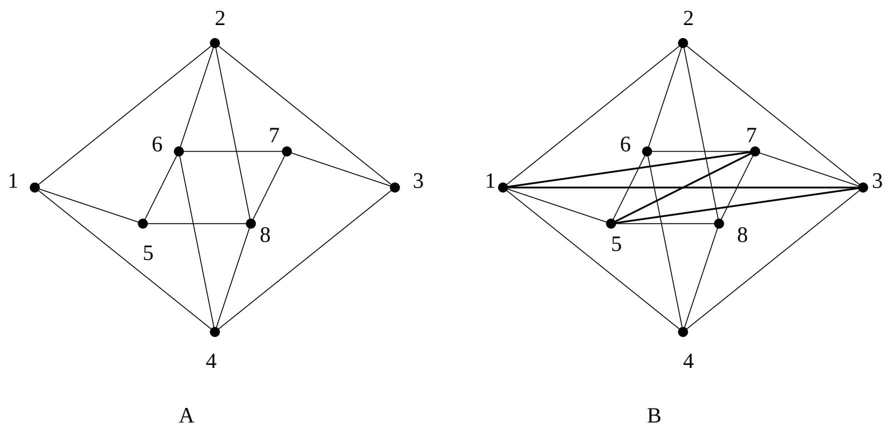

Figure 1: A NAS-graph and its orphan-completion.

4 Shellability of Orphan Complexes

In this section, we show that the orphan complexes of orphan-complete NAS-graphs are shellable. In other words, the clique complex of any such graph is shellable by Corollary 3.7. However, we note that the clique complex of an arbitrary NAS-graph is not in general shellable. For instance, the graph $I(C_4, \overline{K}_2, C_4 : P_3)$ has a nonshellable clique complex.

The notion of shellability has become a fundamental technique in combinatorial topology in order to investigate properties of simplicial complexes. For instance, if a simplicial complex admits a shelling, then it is homotopically a wedge of spheres as well as it has to be a Cohen-Macaulay complex. We recall that a simplicial complex Δ is said to be *shellable* if there exists an order $F_1, \ldots, F_n$ of the facets of Δ such that for every i and l with $1 \leq i < l \leq n$, there is a j with $1 \leq j < l$ and an $x \in F_l$ such that $F_i \cap F_l \subset F_j \cap F_l = F_l \backslash \{x\}$. In our case, a more restrictive shelling type will apply.

Proposition 4.1 ([6]) *Let Δ be a simplicial complex over X. Suppose that $x \in X$ is a vertex such that $del_\Delta(x)$ and $lk_\Delta(x)$ are shellable, and no facet of $lk_\Delta(x)$ is a facet of $del_\Delta(x)$. Then Δ is shellable.*

Let $G = (V, E)$ be a NAS-graph. We call a vertex v of a NAS-graph G an *abortive-vertex* if $G - v$ is a NAS-graph.

Proposition 4.2 *An orphan-complete NAS-graph contains no abortive-vertex if and only if $G \cong Mir(K_n) = K_{2,2,\ldots,2}$ for some $n \geq 2$.*

Proof. If G is the hyperoctahedral graph, every vertex of G has a unique parent so that none of which could be abortive.

For the other direction, we first note that the order of G is necessarily even, since otherwise G contains at least one vertex as being a parent of two distinct vertices, any of which is an abortive-vertex. Furthermore, assume that there exist $x, y \in V$ such that $xy \notin E$. Since G is orphan-complete, we have either $x \in P(y)$ or $y \in P(x)$ from which the claim follows. □

Theorem 4.3 *The orphan complex $\mathfrak{O}(G)$ of any orphan-complete NAS-graph G is shellable.*

Proof. We proceed by the induction on the order of G. Since the base case is trivial, assume that the claim holds for any such graph whose order is less than or equal to $n > 2$. Let G be an orphan-complete graph with $|G| = n + 1$. If G contains no abortive vertex, then by Proposition 4.2, it is the hyperoctehadral graph; hence, $\mathfrak{O}(G) = \Delta(G)$ is shellable. So, suppose that v is an abortive-vertex of G. It follows that the graphs $G - v$ and $G[N(v)]$ are orphan-complete NAS-graphs so that they are shellable by the induction hypothesis. Furthermore, if F is a facet of $\mathrm{lk}_{\Delta(G)}(v) = \Delta(G[N(v)])$, then $F \cup \{u\}$ is a face of $\mathrm{del}_{\Delta(G)}(v) = \Delta(G - v)$ for any $u \in P(v)$. In other words, any facet of $\Delta(G[N(v)])$ can not be a facet of $\Delta(G - v)$. Therefore, $\Delta(G)$ is shellable by Proposition 4.1. □

5 Concluding Remarks

We do not know whether the clique complex of the orphan-completion of an arbitrary orphan-incomplete NAS-graph is shellable. To answer such a question, the first step will be to have a characterization of graphs that are the orphan-completion of NAS-graphs.

The construction of the orphan complex may also be extended to CNAS-graphs in a similar way. Namely, call a subset $S \subseteq V$ of a CNAS-graph $G = (V, E)$ an *orphan set* if $S \cap P[x] = \emptyset$ for any $x \in S$. We then call the resulting simplicial complex formed by the orphan sets of G, the *orphan complex* and continue to denote it by $\mathfrak{O}(G)$. It turns out that the independence complex $\mathcal{I}(G)$ of G is a subcomplex of $\mathfrak{O}(G)$ in this case. Similarly, G is called *orphan-complete* if every orphan set of G is a complete of $\overline{G}$. Almost all statements given in Sections 3 and 4 have similar counterparts in case of CNAS-graphs. For instance, a CNAS-graph is orphan-complete if and only if G is (C_4, P_4)-free. On the other hand, the provided shelling technique will not apply in this case, since the link of a vertex does not have to be the independence complex of a CNAS-graph. In other words, for a CNAS-graph G and any vertex v of G, the link of v in $\mathcal{I}(G)$ is $\mathcal{I}(G[V \backslash N[v]])$, while the graph $G[V \backslash N[v]]$ is not in general a CNAS-graph. Furthermore, the clique complexes of CNAS-graphs are not always shellable as in the case of NAS-graphs. For an example, consider the graph H depicted in Figure 5 whose clique complex triangulates a Möbius band; hence, it is nonshellable. We define G to be the graph satisfying $\overline{G} = H \cup \overline{K}_2$. Clearly, G is a connected CNAS-graph with two full-vertices. Moreover, G is orphan-incomplete, since H contains an induced P_4. The clique complex $\Delta(G)$ is the simplicial complex obtained from the clique complex of H with two disjoint vertices are added; thus, it is nonshellable.

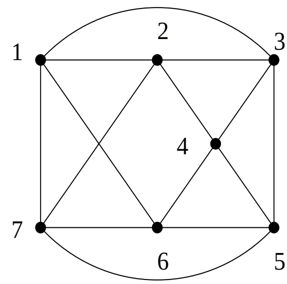

Figure 2

Another possible direction to lead could be the classification of finite partially ordered sets whose comparability graphs are NAS or CNAS-graphs. In view of Theorem 3.3, those posets whose comparability graphs are orphan-complete NAS-graphs form a subclass of interval orders.

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