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Minimum cycle bases for direct products of K_2 with complete graphs

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Abstract

We construct a minimum cycle basis for the direct product $K_2 \times K_p$ of two complete graphs where $p \geq 2$. For $p > 3$, these bases consists only of squares. This completes the work of R. Hammack, who treated the case $K_p \times K_q$ for $p, q \geq 3$ [*Inform. Process. Lett.* 102 (2007), 214–218.]

1 Introduction

The ideas of cycle bases and minimum cycle bases for graphs go back as far as Kirchhoff's 1847 paper on electrical networks [12]. Recently minimum cycle bases have been applied to various problems, including structural flexibility analysis [10], and search-and-retrieval protocols in chemical information systems [1, 2].

Applications aside, the study of minimum cycle bases is an interesting combinatorial problem on its own, and work has been done (e.g. [9, 13, 14]) in constructing minimum cycle bases in various classes of graphs. Along these lines, some authors have investigated minimum cycle bases for various graph products, the most important of which are the Cartesian product, the strong product, the lexicographical product, and the direct product.

In [7], Imrich and Stadler construct minimum cycle bases for Cartesian and strong products of graphs in terms of the cycle structures of the factors. (It is worth pointing out that their theoretical ideas were applied to the force method of frame analysis in [11].) Berger and Jaradat described the construction of minimum cycle bases for the lexicographical product in [1], [8] and [11].

By contrast, the problem of constructing minimum cycle bases for the direct product appears to be quite subtle and is currently not fully understood. Some partial results were obtained by R. Hammack in [4] and [5]. We are mainly concerned with [4], which describes minimum cycle bases for the direct product $K_q \times K_p$ of two complete graphs with $q, p \geq 3$. However, non-bipartiteness of $K_q \times K_p$ plays a central role, so Hammack's arguments break down in the case $q = 2$ (where $K_2 \times K_p$ is bipartite). The purpose of this note is to present a solution for the missing case $q = 2$.

The remainder of this section lays down some necessary definitions and notation. The *cycle space* $\mathcal{C}(G)$ of a simple graph $G = (V(G), E(G))$ is the vector space over the field $\mathrm{GF}(2) = \{0, 1\}$ where vectors are the subsets X of $E(G)$ for which each vertex of G is incident with an even number of the edges in X. Addition in $\mathcal{C}(G)$ is symmetric difference of sets, that is $X + Y = (X \cup Y) - (X \cap Y)$. The zero vector is the empty set, and scalar multiplication is defined as $1 \cdot X = X$ and $0 \cdot Y = 0$.

A basis of $\mathcal{C}(G)$ is called a *cycle basis* of G. The *length* of a cycle basis $\mathcal{B} = \{C_1, C_2, ..., C_n\}$ is $\ell(\mathcal{B}) = \sum_{i=1}^{n} |C_i|$. Among all cycle bases of G, one with the shortest possible length is called a *minimum cycle basis* or *MCB*.

For more information on cycle bases, the reader is referred to [3], where the following result is found.

Proposition 1. *If G has c components, then* $\dim(\mathcal{C}(G)) = |E(G)| - |V(G)| + c$.

The *direct product* of two graphs G and H, denoted $G \times H$, is the graph whose vertex set is the Cartesian product $V(G \times H) = V(G) \times V(H)$ and whose edge set is $E(G \times H) = \{(w, x)(y, z) : wy \in E(G) \text{ and } xz \in E(H)\}$. (See [6] for a standard reference.)

We denote the complete graph with p vertices as K_p, with $V(K_p) = \{1, 2, \ldots, p\}$. The edges of $K_2 \times K_p$ are precisely the pairs $(1, i)(2, j)$ where $1 \leq i, j \leq p$ and $i \neq j$. Thus, $|E(K_2 \times K_p)| = p(p-1)$. Observe also that $K_2 \times K_p$ is bipartite and is connected for $p > 2$.

The two authors arrived at the following results independently. Presented here is a collaborative work. We thank the referee for reading both approaches and suggesting they be combined into a single paper.

2 Results

Finding an MCB for $K_2 \times K_p$ is trivial for $p = 1, 2$ or 3. If p is 1 or 2, then $K_2 \times K_3$ is acyclic, so its cycle space is zero dimensional and there is no MCB. For $p = 3$, it is easy to check that $K_2 \times K_3$ is isomorphic to the hexagon (the cycle on 6 vertices) so its MCB consists only of that single cycle.

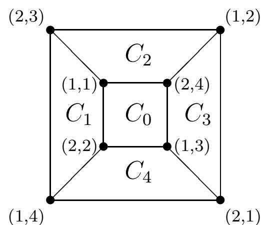

Figure 1

The product $K_2 \times K_4$ is the first nontrivial case. Figure 1 shows $K_2 \times K_4$ embedded in the plane. Observe that it has the structure of a three-dimensional cube. Each square face is bounded by a cycle of length 4 in $\mathcal{C}(K_2 \times K_4)$. Label each of these cycles as follows:

$$\begin{aligned} C_0 &= (1,1)(2,2)(1,3)(2,4)(1,1) \\ C_1 &= (1,1)(2,2)(1,4)(2,3)(1,1) \\ C_2 &= (1,1)(2,4)(1,2)(2,3)(1,1) \\ C_3 &= (1,3)(2,1)(1,2)(2,4)(1,3) \\ C_4 &= (1,3)(2,2)(1,4)(2,1)(1,3) \end{aligned}$$

Proposition 2. *The set* $\mathcal{B}_4 = \{C_0, C_1, C_2, C_3, C_4\}$ *is an MCB for* $K_2 \times K_4$.

Proof. Observe that the quadrangles elements of $\mathcal{B}_4$ are the faces of a planar embedding of the cube and thus constitute a basis. (See [3] Theorem 4.5.1 for verification of this result.) Since the cube contains no odd cycles, and hence no cycles of length less than 4, $\mathcal{B}_4$ must be a minimum cycle basis of $K_2 \times K_4$. ■

We will define an MCB for $K_2 \times K_p$ inductively, beginning with the case $K_2 \times K_4$ addressed in Proposition 2. In preparation for this we define the following cycles that will belong to our MCB. Let $p \geq 5$. For each $1 \leq i \leq p-2$, let S_{1i} be a cycle of form $(1,p)(2,i)(1,\alpha_i)(2,p-1)(1,p)$ where $1 \leq \alpha_i \leq p-2$ and $\alpha_i \neq i$ but α_i is otherwise arbitrary. Likewise, let S_{2i} be a cycle of form $(2,p)(1,i)(2,\alpha_i)(1,p-1)(2,p)$. Figure 2 shows typical cycles S_{1i} and S_{2i}. Let $\mathcal{B}_p = \{S_{1i}, S_{2i} : 1 \leq i \leq p-2\}$.

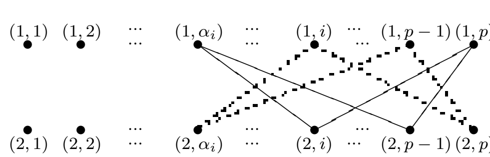

Figure 2

Lemma 2.1. *The set $\mathcal{B}_p$ is linearly independent.*

Proof. For each i, the cycle $S_{1i} \in \mathcal{B}_p$ has an edge $(1,p)(2,i)$ that belongs to no other cycle in $\mathcal{B}_p$. Likewise, each S_{2i} has an edge $(2,p)(1,i)$ which belongs to no other element of $\mathcal{B}_p$. Consequently every nontrivial sum of elements in $\mathcal{B}_p$ contains such edges, and is hence nonzero. Thus $\mathcal{B}_p$ is linearly independent. ■

Proposition 3. *Let $\mathcal{B}_4$ be as described in Proposition 1, and for $p \geq 5$ let $\mathcal{B}_p$ be as described above. Then $\mathcal{B}_p \cup \mathcal{B}_{p-1} \cup ... \cup \mathcal{B}_5 \cup \mathcal{B}_4$ is an MCB for $K_2 \times K_p$.*

Proof. We first use induction on p to show that $\mathcal{B}_p \cup \mathcal{B}_{p-1} \cup ... \cup \mathcal{B}_5 \cup \mathcal{B}_4$ is a basis for $\mathcal{C}(K_2 \times K_p)$ when $p \geq 4$. By Proposition 2, $\mathcal{B}_4$ is a cycle basis for $K_2 \times K_4$. Let $p > 4$ and suppose $\mathcal{B}^* = \mathcal{B}_{p-1} \cup \mathcal{B}_{p-2} \cup ... \cup \mathcal{B}_5 \cup \mathcal{B}_4$ is a basis for $K_2 \times K_{p-1}$.

We now show $\mathcal{B}_p \cup \mathcal{B}^*$ is a basis for $K_2 \times K_p$. This is equivalent to showing that $|\mathcal{B}_p \cup \mathcal{B}^*| = \dim(\mathcal{C}(K_2 \times K_p))$ and that $\mathcal{B}^* \cup \mathcal{B}_p$ is linearly independent.

Observe that $\mathcal{B}_p$ and $\mathcal{B}^*$ are disjoint because each element of $\mathcal{B}_p$ contains an edge incident with $(1,p)$ or $(2,p)$ and no elements of $\mathcal{B}^*$ have such edges. So $|\mathcal{B}_p \cup \mathcal{B}^*|$=$|\mathcal{B}_p|$ + $|\mathcal{B}^*|$. By the inductive hypothesis, $|\mathcal{B}^*| = \dim(\mathcal{C}(K_2 \times K_{p-1}))$=$(p-1)(p-2)$−$2(p-1)+1$=$p^2-5p+5$. By our construction of $\mathcal{B}_p$, it follows that $|\mathcal{B}_p| = 2(p-2)$, so $|\mathcal{B}_p \cup \mathcal{B}^*|$=$2(p-2)+p^2-5p+5 = p^2-3p+1$. From Proposition 1, $\dim(\mathcal{C}(K_2 \times K_p))$=$p^2-3p+1$. So $|\mathcal{B}_p \cup \mathcal{B}^*| = \dim(\mathcal{C}(K_2 \times K_p))$.

Next we show $\mathcal{B}_p \cup \mathcal{B}^*$ is linearly independent. The set $\mathcal{B}^*$ is linearly independent by the inductive hypothesis. By Lemma 2.1 $\mathcal{B}_p$ is linearly independent. We must thus only show that $\text{Span}(\mathcal{B}^*) \cap \text{Span}(\mathcal{B}_p) = \{0\}$. To see this is true, suppose $X \in \text{Span}(\mathcal{B}_p) \cap \text{Span}(\mathcal{B}^*)$. Since $X \in \text{Span}(\mathcal{B}_p)$, either $X = 0$ or X contains an edge of form $(1,p)(2,i)$ or $(2,p)(1,i)$. But since $X \in \text{Span}(\mathcal{B}^*)$, X can have no such edges, so $X = 0$.

This completes the proof that $\mathcal{B}_p \cup \mathcal{B}_{p-1} \cup ... \cup \mathcal{B}_5 \cup \mathcal{B}_4$ is a basis. Observe that all cycles in $\mathcal{B}_p \cup \mathcal{B}_{p-1} \cup ... \cup \mathcal{B}_5 \cup \mathcal{B}_4$ are of length 4. As previously noted, since the smallest possible cycle in a bipartite graph has length 4, this basis is necessarily minimal. ■

Corollary 1. *Any MCB of $K_2 \times K_p$ for $p > 3$ consists only of squares, and its length is $4p^2 - 12p + 4$.*

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