AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **44** (2009), Pages 123–152

On the number of nonseparating vertices in strongly connected in-tournaments

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Abstract

A digraph without loops, multiple arcs and directed cycles of length two is called an in-tournament if the set of in-neighbors of every vertex induces a tournament. A local tournament is an in-tournament such that the set of out-neighbors of every vertex induces a tournament as well.

Let $p \geq 2$ be an integer and let T be a strongly connected tournament such that every vertex has at least p positive neighbors and at least p negative neighbors. In 2006, Kotani showed that T has at least k vertices $x_1, x_2, \ldots, x_k$, where $k = \min\{|V(D)|, 4p-2\}$, such that $T - x_i$ $(i = 1, 2, \ldots, k)$ is strongly connected. One year later, Meierling and Volkmann proved that the same proposition is valid for the class of local tournaments. In this paper we shall generalize the result to the class of in-tournaments, thereby generalizing Kotani's as well as Meierling's and Volkmann's results.

1 Terminology and introduction

All digraphs mentioned here are finite without loops and multiple arcs. For a digraph D, we denote by $V(D)$ and $E(D)$ the *vertex set* and *arc set* of D, respectively. The number $|V(D)|$ is the *order* of the digraph D. The subdigraph induced by a subset A of $V(D)$ is denoted by $D[A]$. By $D-A$ we denote the digraph $D[V(D)-A]$. If $A = \{x\}$ is a single vertex, then we write $D - x$ instead of $D - \{x\}$.

If $xy \in E(D)$, then y is a *positive neighbor* of x and x is a *negative neighbor* of y, and we also say that x *dominates* y, denoted by $x \to y$. If A and B are two disjoint subdigraphs of a digraph D such that every vertex of A dominates every vertex of B in D, then we say that A *dominates* B, denoted by $A \to B$. The *outset* $N^+(x)$ of a vertex x is the set of positive neighbors of x. More generally, for arbitrary subdigraphs A and B of D, the outset $N^+(A, B)$ is the set of vertices in B to which there is an arc from a vertex in A. The *insets* $N^-(x)$ and $N^-(A, B)$

are defined analogously. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called *outdegree* and *indegree* of x, respectively. The *minimum outdegree* $\delta^+(D)$ and the *minimum indegree* $\delta^-(D)$ of D are given by $\min\{d^+(x) \mid x \in V(D)\}$ and $\min\{d^-(x) \mid x \in V(D)\}$, respectively. Analogously, we define the *maximum outdegree* $\Delta^+(D) = \max\{d^+(x) \mid x \in V(D)\}$ and the *maximum indegree* $\Delta^-(D) = \max\{d^-(x) \mid x \in V(D)\}$ of D. In addition, let $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ be the *minimum degree* and $\Delta(D) = \max\{\Delta^+(D), \Delta^-(D)\}$ be the *maximum degree* of D.

Throughout this paper, directed cycles and paths are simply called *cycles* and *paths.* A cycle of order k is a *k-cycle.* If C is a cycle of a digraph D with order $|V(D)|$, then C is called a *Hamiltonian cycle.* Let $C = x_1x_2 \ldots x_kx_1$ be a cycle of D with order k. Then $C[x_i, x_j]$, where $1 \le i, j \le k$, denotes the subpath $x_ix_{i+1} \ldots x_j$ of C with *initial vertex* x_i and *terminal vertex* x_j. The notations for paths are defined analogously.

A digraph D is *vertex pancyclic* if every vertex of D belongs to cycles of lengths $3, 4, \ldots, |V(D)|$.

We speak of a *connected digraph* if the underlying graph is connected. A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D, there is a path from x to y. A *strong component* of D is a maximal induced strong subdigraph of D. The *strong component digraph* $SC(D)$ of a digraph D is obtained by contracting the strong components of D into vertices and deleting any parallel arcs obtained in this process. In other words, if $D_1, D_2, \ldots, D_p$, where $p \ge 1$, are the strong components of D, the vertex set of $SC(D)$ is $V(SC(D)) = \{v_1, v_2, \ldots, v_p\}$ and the arc set of $SC(D)$ is $E(SC(D)) = \{v_iv_j \mid N^+(D_i, D_j) \neq \emptyset\}$. Note that $SC(D)$ is acyclic and thus has an acyclic ordering, that is the strong components of D can be labeled $D_1, D_2, \ldots, D_p$ such that there is no arc from D_j to D_i unless $j < i$. We call this ordering the *acyclic ordering of the strong components* of D. The strong components corresponding to vertices with in-degree (out-degree) zero in $SC(D)$ are called *initial (terminal) strong components* of D.

A vertex x is called a *nonseparating vertex* of a strong digraph D if $D - x$ is strong. If D is a digraph such that each of its vertices is nonseparating, we call D a *2-connected* digraph.

A digraph is *semicomplete* if for any two different vertices x and y, there is at least one arc between them. A *tournament* is a semicomplete digraph without 2-cycles. A digraph D is *locally semicomplete* if $D[N^+(x)]$ as well as $D[N^-(x)]$ are semicomplete for every vertex x of D.

We speak of an *r-regular tournament* T if $\delta(T) = \Delta(T) = r$. Similarly, an *almost-regular tournament* is a tournament such that $|\Delta(T) - \delta(T)| \le 1$.

Throughout this paper all subscripts are taken modulo the corresponding number.

Local tournaments were introduced by Bang-Jensen [1] in 1990. In transferring the general adjacency only to vertices that have a common positive or negative neighbor, local tournaments form an interesting generalization of tournaments.

In 1993, Bang-Jensen, Huang and Prisner [2] introduced a further generalization of local tournaments, the class of *in-tournaments*, in claiming adjacency only for vertices that have a common positive neighbor.

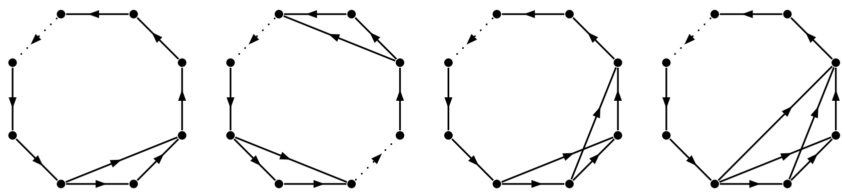

Figure 1: The locally semicomplete digraph with exactly one nonseparating vertex (left); Local tournaments with exactly two nonseparating vertices.

From the well-known result of Moon [9] that a tournament T is strongly connected if and only if it is vertex pancyclic, it follows immediately that a strongly connected tournament T of order greater or equal four contains at least two nonseparating vertices. This was formulated and proved by Korvin [4] in 1967.

Corollary 1.1 (Korvin [4] 1967)**.** *If T is a strong tournament with $|V(T)| \geq 4$, then T contains at least two nonseparating vertices.*

In 1975, Las Vergnas [6] determined all strongly connected tournaments with exactly two nonseparating vertices.

Theorem 1.2 (Las Vergnas [6] 1975)**.** *A strong tournament T on n vertices has at least three nonseparating vertices, unless T is isomorphic to Q_n, where Q_n is the tournament of order n consisting of a path $x_1x_2 \ldots x_n$ and all arcs x_ix_j for $i > j+1$.*

In 1990, Bang-Jensen [1] proved that every strongly connected locally semicomplete digraph that is not a cycle has at least one nonseparating vertex.

Theorem 1.3 (Bang-Jensen [1] 1990)**.** *Let D be a strong locally semicomplete digraph that is not a cycle. Then D has a nonseparating vertex.*

Four years later, Guo and Volkmann showed that every strongly connected locally semicomplete digraph on $n \geq 4$ vertices has at least two nonseparating vertices if it has at least $n+2$ arcs and determined the digraph depicted in Figure 1 to be the only locally semicomplete digraph with exactly one nonseparating vertex.

Theorem 1.4 (Guo & Volkmann [3] 1994)**.** *Let D be a strong locally semicomplete digraph.*

(a) If D has at least $|V(D)|+2$ arcs, then D has at least two nonseparating vertices;

(b) The digraph D has exactly one nonseparating vertex if and only if D is isomorphic to the digraph depicted in Figure 1;

(c) Every vertex of D is a separating vertex if and only if D is a cycle.

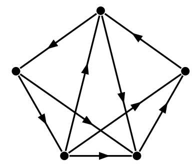

Figure 2: A local tournament with exactly two nonseparating vertices.

In 2008, Meierling and Volkmann [7] characterized the strongly connected local tournaments on $n \geq 4$ vertices and at least $n+2$ arcs with exactly two nonseparating vertices to be either isomorphic to Q_n, to one of the digraphs depicted in Figure 1 or to the digraph depicted in Figure 2 (if $n = 5$).

Theorem 1.5. *Let D be a strong local tournament on n vertices with at least $n+2$ arcs. Then D has exactly two nonseparating vertices if and only if D is isomorphic to Q_n as defined in Theorem 1.2, to one of the digraphs depicted in Figure 1 or to the digraph depicted in Figure 2 (if $n = 5$).*

Concerning in-tournaments little is known about the number of nonseparating vertices. In their initial article [2], Bang-Jensen, Huang and Prisner proved the existence of a nonseparating vertex if the in-tournament in question is strongly connected and not a cycle. It is the analog of Theorem 1.3 for in-tournaments.

Theorem 1.6 (Bang-Jensen, Huang and Prisner [2] 1993)**.** *Let D be a strong in-tournament that is not a cycle. Then D has a nonseparating vertex.*

Six years later, Tewes [10] gave a sufficient criterion for strongly connected in-tournaments to have at least two nonseparating vertices.

Theorem 1.7 (Tewes [10] 1999)**.** *Let D be a strong in-tournament of order $n \geq 4$ with $\max\{\delta^+(D), \delta^-(D)\} \geq 2$. Then D contains distinct vertices x_1, x_2 such that $D - x_i$ is strong for $i = 1, 2$.*

In 2006, Kotani [5] investigated how many nonseparating vertices a tournament with minimum degree greater or equal two has at the least, and proved several variants of Corollary 1.1.

Theorem 1.8 (Kotani [5] 2006)**.** *Let T be a strong tournament and let $p \geq 2$ be an integer. If $\delta(T) \geq p$, then T has at least $k = \min\{|V(T)|, 4p-2\}$ vertices $x_1, x_2, \ldots, x_k$ such that $T - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Theorem 1.9 (Kotani [5] 2006)**.** *Let T be a strong tournament and let $p \geq 2$ be an integer. If $\delta(T) \geq p$ and $|V(T)| \geq 4p$, then T has at least $k = 4p-1$ vertices $x_1, x_2, \ldots, x_k$ such that $T - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Theorem 1.10 (Kotani [5] 2006)**.** *Let T be a strong tournament and let $p \geq 3$ be an integer. If $\delta(T) \geq p$ and $|V(T)| \geq 4p+1$, then T has at least $k = 4p$ vertices $x_1, x_2, \ldots, x_k$ such that $T - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Inspired by this article, Meierling and Volkmann [8] considered the same question for the class of local tournaments and proved generalizations of Kotani's results. They showed that the conclusion of Theorem 1.8 is valid for local tournaments and characterized the classes of local tournaments that do not fulfill the conclusions of Theorems 1.9 and 1.10 (cf. Definition 2.1, Figure 3 and Corollaries 6.7-6.9).

In this paper we will show that the conclusion of Theorem 1.8 is also true for the class of in-tournaments and characterize the class of in-tournaments that do not fulfill the conclusions Theorems 1.9 and 1.10, thereby generalizing Kotani's as well as Meierling's and Volkmann's results.

2 The exceptional cases

In this section we introduce the classes of local tournaments and in-tournaments that satisfy the preconditions of Theorem 1.9 and Theorem 1.10, but not the respective conclusions (cf. Figure 3).

Definition 2.1. *Let $p \geq 2$ be an integer and let T_1 and T_2 be two $(p-1)$-regular tournaments. Furthermore, let u, v be two vertices such that $u, v \notin V(T_i)$ for $i = 1, 2$.*

We define $\mathcal{T}^*_{loc}$ as the set of all local tournaments D with vertex set

$$V(D) = V(T_1) \cup V(T_2) \cup \{u, v\}$$

and arc set

$$\begin{aligned} E(D) = &E(T_1) \cup E(T_2) \cup \{uw \mid w \in V(T_1)\} \cup \{wv \mid w \in V(T_1)\} \cup \\ &\{vw \mid w \in V(T_2)\} \cup \{wu \mid w \in V(T_2)\} \cup A, \end{aligned}$$

where $A \in \{\{uv\}, \{vu\}, \emptyset\}$.

Let $p \geq 3$ be an integer. Let T_1 be defined as above and let T_0 be a tournament on $2p$ vertices such that $p-1 \leq \delta(T_0), \Delta(T_0) \leq p$. Moreover, let u, v be two vertices such that $u, v \notin V(T_i)$ for $i = 0, 1$. We define $\mathcal{T}^{**}_{loc}$ as the set of all local tournaments D with vertex set

$$V(D) = V(T_0) \cup V(T_1) \cup \{u, v\}$$

and arc set

$$\begin{aligned} E(D) = &E(T_0) \cup E(T_1) \cup \{uw \mid w \in V(T_1)\} \cup \{wv \mid w \in V(T_1)\} \cup \\ &\{vw \mid w \in V(T_0)\} \cup \{wu \mid w \in V(T_0)\} \cup A, \end{aligned}$$

where $A \in \{\{uv\}, \{vu\}, \emptyset\}$.

Definition 2.2. *Let $p \geq 3$ be an integer, let T_1 be a $(p-1)$-regular tournament and let T_2 be a tournament on $2p$ vertices such that $p-1 \leq \delta(T_2), \Delta(T_2) \leq p$. Furthermore, let u, v be two vertices such that $u, v \notin V(T_i)$ for $i = 1, 2$.*

We define $\mathcal{T}^*_{in}$ as the set of all in-tournaments D with vertex set

$$V(D) = V(T_1) \cup V(T_2) \cup \{u, v\}$$

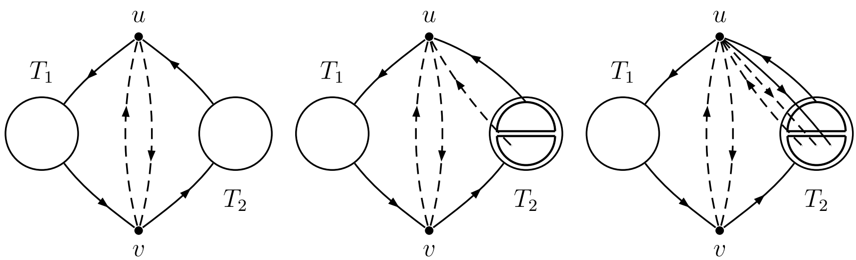

Figure 3: The classes $\mathcal{T}_{loc}^{*}$ / $\mathcal{T}_{loc}^{**}$ (left), $\mathcal{T}_{in}^{*}$ (center) and $\mathcal{T}_{in}^{**}$ (right).

and arc set

$$\begin{aligned} E(D) = &E(T_1) \cup E(T_2) \cup \{uw \mid w \in V(T_1)\} \cup \{vw \mid w \in V(T_2)\} \cup \\ &\{wv \mid w \in V(T_1)\} \cup \{wu \mid w \in V(T_2) \text{ and } d^+(w, T_2) = p - 1\} \cup \\ &A \cup B_1, \end{aligned}$$

where

$$\begin{aligned} A &\in \{\{uv\}, \{vu\}, \emptyset\} \text{ and} \\ B_1 &\subseteq \{wu \mid w \in V(T_2) \text{ and } d^+(w, T_2) = p\}. \end{aligned}$$

In addition, we define $\mathcal{T}_{in}^{**}$ as the set of all in-tournaments D with vertex set

$$V(D) = V(T_1) \cup V(T_2) \cup \{u, v\}$$

and arc set

$$\begin{aligned} E(D) = &E(T_1) \cup E(T_2) \cup \{uw \mid w \in V(T_1)\} \cup \{vw \mid w \in V(T_2)\} \cup \\ &\{wv \mid w \in V(T_1)\} \cup \{wu \mid w \in V(T_2) \text{ and } d^+(w, T_2) = p - 1\} \cup \\ &A \cup B_1 \cup B_2, \end{aligned}$$

where

$$\begin{aligned} A &\in \{\{uv\}, \{vu\}\}, \\ B_1 &\subseteq \{wu \mid w \in V(T_2) \text{ and } d^+(w, T_2) = p\} \text{ and} \\ \emptyset \neq B_2 &\subseteq \{uw \mid w \in V(T_2) \text{ and } d^+(w, T_2) = p\}. \end{aligned}$$

It is easy to check that the classes defined in the examples above do not fulfill Theorems 1.9 and 1.10, respectively.

Remark 2.3. *Let $D \in \mathcal{T}_{loc}^{*} \cup \mathcal{T}_{loc}^{**} \cup \mathcal{T}_{in}^{*} \cup \mathcal{T}_{in}^{**}$ be a digraph. Then u and v are the only separating vertices of D. In addition the following holds.*

(a) If $D \in \mathcal{T}_{loc}^{*}$, then $|V(D)| = 4p$ and $\delta(D) = p$. Hence D does not fulfill the conclusion of Theorem 1.9.

(b) If $D \in \mathcal{T}_{loc}^{**} \cup \mathcal{T}_{in}^{*} \cup \mathcal{T}_{in}^{**}$, then $|V(D)| = 4p + 1$ and $\delta(D) = p$. Hence D does not fulfill the conclusion of Theorem 1.10.

3 Preliminary results

The next results will be frequently used in our proof. The first one is a useful observation about the interaction of a cycle and an external vertex.

Theorem 3.1. **(Bang-Jensen, Huang & Prisner [2] 1993)** *Let D be an in-tournament and let $C = u_1u_2\dots u_su_1$ be a cycle in D. If $|N^+(x,C)| \geq 1$ for a vertex $x \notin V(C)$, then either $x \to C$ or $u_i \to x \to u_{i+1}$ for an integer $1 \leq i \leq s$.*

In 1993, Bang-Jensen, Huang and Prisner [2] generalized the well known result that a tournament is strong if and only if it is Hamiltonian to in-tournaments.

Theorem 3.2 (Bang-Jensen, Huang & Prisner [2] 1993)**.** *An in-tournament is strong if and only if it has a Hamiltonian cycle.*

Under certain assumptions the number of nonseparating vertices of a digraph D can be bounded from below as follows.

Lemma 3.3. *Let D be a strong digraph of order n and let q be an integer such that $1 \leq q \leq n-1$. Suppose that for every subset $X \subseteq V(D)$ with $1 \leq |X| \leq q$ there exists a cycle C of order $n-1$ in D such that $X \subseteq V(C)$. Then D contains at least $q+1$ vertices $x_1, x_2, \dots, x_{q+1}$ such that $D - x_i$ is strong for $i = 1, 2, \dots, q+1$.*

Proof. Let $X = \{x \in V(D) \mid D - x \text{ is strong}\}$ be the set of nonseparating vertices of D. If $|X| \geq q+1$, there is nothing to show. So suppose to the contrary that $|X| \leq q$. By the assumption of this lemma, there exists an $(n-1)$-cycle C in D such that $X \subseteq V(C)$. Let $\{v\} = V(D) - V(C)$. On the one hand this implies that $v \notin X$, but on the other hand C is a Hamiltonian cycle of $D - v$ and thus, $D - v$ is strong. It follows that $v \in X$, a contradiction. □

In the next result we summarize some simple, but useful, lower bounds for the order of digraphs in terms of minimum degree.

Lemma 3.4. *Let D be a digraph of order n without cycles of length two and let $r \geq 1$ be an integer.*

(a) If $d^-(x) + d^+(y) \geq r$ for every arc xy of D, then $n \geq r+1$;

(b) If $\delta^+(D) \geq r$ or $\delta^-(D) \geq r$, then $n \geq 2r+1$;

(c) If $\delta^+(D) \geq r$ and $\Delta^+(D) \geq r+1$ or if $\delta^-(D) \geq r$ and $\Delta^-(D) \geq r+1$, then $n \geq 2r+2$.

The last lemma will be needed in the characterization of the exceptional classes.

Lemma 3.5. *Let $p \geq 3$ and let D be an in-tournament on $2p+1$ vertices with vertex set $W \cup \{u\}$, where $W = \{w_1, w_2, \dots, w_{2p}\}$, such that*

$$p - 1 \leq \delta(D[W]), \Delta(D[W]) \leq p.$$

If $d^+(w, D[W]) = p-1$ implies that $w \to u$ for every vertex $w \in W$, then $D[W]$ is a 2-connected tournament.

Proof. Firstly we will show that $D[W]$ is a tournament. Suppose, without loss of generality, that w_1 and w_2 are not adjacent. It follows that $d^+(w_1, D[W]) = d^+(w_2, D[W]) = p-1$ and thus, $\{w_1, w_2\} \to u$ by our assumption. Using the in-tournament property of D, we conclude that w_1 and w_2 are adjacent, a contradiction. So $D[W]$ is a tournament.

Secondly we will show that $D[W]$ is 2-connected. Suppose, without loss of generality, that w_1 is a separating vertex of $D[W]$ and let $D_1, D_2, \ldots, D_q$ be the strong decomposition of $D[W] - w_1$, where $q \geq 2$. Then there exists a vertex $w \in V(D_1)$ and a vertex $w' \in V(D_q)$ such that

$$d^+(w, D[W]) \geq \frac{|V(D_1)| - 1}{2} + |V(D_2)| + \ldots + |V(D_q)|$$

and

$$d^+(w', D[W]) \leq \frac{|V(D_q)| - 1}{2} + 1.$$

Since $p - 1 \leq \delta(D[W]), \Delta(D[W]) \leq p$, it follows that $q \leq 3$ and $|V(D_i)| = 1$ for $1 \leq i \leq q$, a contradiction to the assumption that $p \geq 3$. So $D[W] - w$ is strong for every vertex $w \in W$ which means that $D[W]$ is 2-connected. $\square$

4 Structure of in-tournaments with high minimum degree

In this section we investigate the structure of in-tournaments that fulfill the assumptions of Theorems 1.9 and 1.10. Throughout this section, let $p \geq 2$ be an integer and let D be a strongly connected in-tournament such that $\delta(D) \geq p$. Let X be a subset of $V(D)$ such that $\emptyset \neq X \neq V(D)$. Obviously D contains a strongly connected induced subdigraph D' such that $V(D) - V(D') - X \neq \emptyset$ (any vertex $x \in X$ satisfies this condition). Assume now that we have chosen a strongly connected subdigraph D' of D under the following conditions:

A. $V(D) - V(D') - X \neq \emptyset$,

B. under condition A: $|V(D') \cap X|$ is maximal and

C. under condition B: $|V(D')|$ is maximal.

Note that, since D' is a strongly connected in-tournament, the digraph D' has a Hamiltonian cycle $C = v_1 v_2 \ldots v_k v_1$ according to Theorem 3.2 (if $k = 1$, then, for the sake of simplicity, the single vertex v_1 will in the following be called a Hamiltonian cycle). Let

$$Y = V(D) - V(C) - X$$

and define

$$\begin{aligned}
X^+ &= \left\{v \in X - V(C) \mid N^-(v,C) \neq \emptyset \text{ and } N^+(v,C) = \emptyset\right\},\\
X^- &= \{v \in X - V(C) \mid v \to C\},\\
\hat{X} &= X - V(C) - X^+ - X^-,\\
Y^+ &= \left\{v \in Y - V(C) \mid N^-(v,C) \neq \emptyset \text{ and } N^+(v,C) = \emptyset\right\},\\
Y^- &= \{v \in Y - V(C) \mid v \to C\} \text{ and}\\
\hat{Y} &= Y - Y^+ - Y^-.
\end{aligned}$$

Note that, according to these definitions,

$$N^+(X^+,C) = N^-(X^-,C) = N^+(Y^+,C) = N^-(Y^-,C) = \emptyset. \tag{1}$$

Using this notation, we prove the following claims.

Claim 1. $\hat{X} = \{x \in X - V(C) \mid N^+(x,C) = N^-(x,C) = \emptyset\}$.

Proof. Suppose that there exists a vertex $x \in \hat{X}$ such that $N^+(x,C) \neq \emptyset$. Then, according to Theorem 3.1, either $x \to C$ or there exists a cycle C' in D such that $V(C') = V(C) \cup \{x\}$. But the first possibility is a contradiction to the definition of $\hat{X}$ and the latter possibility contradicts condition B of the choice of C. So $N^+(\hat{X},C) = \emptyset$.

Suppose that there exists a vertex $x \in \hat{X}$ such that $N^-(x,C) \neq \emptyset$. Note that $N^+(x,C) = \emptyset$ by the observations above. It follows that $x \in X^+$, a contradiction. So $N^-(\hat{X},C) = \emptyset$. □

Analogously to Claim 1, we can prove the following claim.

Claim 2. *If* $|Y| \geq 2$*, then* $N^+(\hat{Y},C) = N^-(\hat{Y},C) = \emptyset$.

Claim 3. $N^+(\hat{X},X^+) = N^+(\hat{X},Y^+) = \emptyset$.

Proof. Suppose that there exists an arc xv such that $x \in \hat{X}$ and $v \in X^+ \cup Y^+$. Then v has negative neighbors both in C and $\hat{X}$. Since D is an in-tournament, this is a contradiction to Claim 1. It follows that $N^+(\hat{X},X^+) = N^+(\hat{X},Y^+) = \emptyset$. □

Using Claim 2, we can prove the following claim analogously to Claim 3.

Claim 4. *If* $|Y| \geq 2$*, then* $N^+(\hat{Y},X^+) = N^+(\hat{Y},Y^+) = \emptyset$.

Claim 5. $N^+(X^+,X^-) = \emptyset$.

Proof. Suppose that there exists an arc x_1x_2 such that $x_1 \in X^+$ and $x_2 \in X^-$. Since $x_1 \in X^+$, we may assume, without loss of generality, that $v_k \to x_1$. But then the cycle

$$C[v_1,v_k]x_1x_2v_1$$

yields a contradiction to condition B of the choice of C. So $N^+(X^+,X^-) = \emptyset$. □

Claim 6. *Let $u, w \in V(D) - V(C)$ be two vertices of D such that $w \to C$ and $N^-(u, C) \neq \emptyset$. If P is a path with initial vertex u and terminal vertex w, then $Y \subseteq V(P)$.*

Proof. We may assume, without loss of generality, that $v_k \to u$. Suppose that $Y \nsubseteq V(P)$. Then the cycle

$$v_k P C[v_1, v_k]$$

yields a contradiction to condition C of the choice of C. So $Y \subseteq V(P)$. □

Claim 7. $|Y^+|, |Y^-| \leq 1$.

Proof. Suppose that $|Y^+| \geq 2$. Let $P = z_0 z_1 \dots z_r$ be a shortest path in D such that $z_0 \in Y^+$ and $N^+(z_r, C) \neq \emptyset$. Then $V(P) \cap Y^+ = \{z_0\}$ and $z_r \in X^- \cup Y^-$, but $V(P) \nsubseteq Y$, a contradiction to Claim 6. □

Claim 8. *If $|Y^+| = |Y^-| = 1$, then $N^+(X^+, Y^-) = N^+(Y^+, X^-) = \emptyset$.*

Proof. Suppose that there exists an arc uw such that $u \in X^+$ and $w \in Y^-$ or $u \in Y^+$ and $w \in X^-$. We may assume, without loss of generality, that $v_k \to u$. Then

$$C[v_1, v_k] u w v_1$$

contradicts condition B of the choice of C. So $N^+(X^+, Y^-) = N^+(Y^+, X^-) = \emptyset$. □

Claim 9. *Let D be a strong in-tournament and let $C, X^+, X^-, \hat{X}, X, Y^+, Y^-, \hat{Y}, Y$ be defined as above. Let $X - V(C) \neq \emptyset$. If $|Y| \geq 2$, then $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- \neq \emptyset$.*

Proof. Assume to the contrary that $X^- \cup Y^- = \emptyset$. Note that, according to Claim 7, $|Y^+|, |Y^-| \leq 1$ and, according to Claim 1, $N^+(\hat{X}, C) = \emptyset$. Since $|Y| \geq 2$ and $Y^- = \emptyset$, we conclude that $\hat{Y} \neq \emptyset$. Since D is strong, it follows that $N^+(\hat{Y}, C) \neq \emptyset$, a contradiction to Claim 2. So $X^- \cup Y^- \neq \emptyset$. We can analogously show that $X^+ \cup Y^+ \neq \emptyset$. □

Claim 10. *If $X \subseteq V(C)$, then $V(D) - V(C) = \hat{Y}$ and $|\hat{Y}| = 1$.*

Proof. We consider five cases depending on Y.

Case 1. Suppose that $|Y^+| = |Y^-| = 1$ and $\hat{Y} \neq \emptyset$. Since D is strong, there exists an Y^+-Y^--path in D. Let $P = y_0 y_1 \dots y_s$ be a shortest such path. It follows that $V(P) = Y$ by Claim 6. But then, since $y_0 \in Y^+$ and because of the choice of P, it follows that $N^+(y_0) = \{y_1\}$ which contradicts $\delta^+(D) \geq 2$.

Case 2. Suppose that $|Y^+| = |Y^-| = 1$ and $\hat{Y} = \emptyset$. Let $Y^+ = \{y_1\}$ and $Y^- = \{y_2\}$. Then $d^+(y_1), d^-(y_2) \leq 1$, a contradiction.

Case 3. Suppose that $|Y^+| = 1$, $Y^- = \emptyset$ and $\hat{Y} \neq \emptyset$ or $|Y^-| = 1$, $Y^+ = \emptyset$ and $\hat{Y} \neq \emptyset$. By Claim 4 $N^+(\hat{Y}, C) = N^-(\hat{Y}, C) = \emptyset$ and thus, D is not strong, a contradiction.

Case 4. Suppose that $|Y^+| = 1$ and $Y^- = \hat{Y} = \emptyset$ or $|Y^-| = 1$ and $Y^+ = \hat{Y} = \emptyset$. It follows that D is not strong, again a contradiction.

Case 5. Suppose that $Y^+ = Y^- = \emptyset$ and $\hat{Y} \neq \emptyset$. Since D is strong, we conclude that $N^+(\hat{Y}, C) \neq \emptyset$ and $N^-(\hat{Y}, C) \neq \emptyset$. By Claim 4 it follows that $|\hat{Y}| = 1$ which completes the proof of this lemma. □

5 Applications of the structural results

In this section we characterize $\mathcal{T}^*_{loc}$ and $\mathcal{T}^{**}_{loc}$ as the classes of local tournaments that are exceptions for the conclusions of Theorem 1.9 and Theorem 1.10, respectively. Furthermore we characterize $\mathcal{T}^*_{in}$ and $\mathcal{T}^{**}_{in}$ as the classes of in-tournaments that are exceptions for the conclusion of Theorem 1.10.

For the rest of this section let D be a strongly connected in-tournament and let $C, X^+, X^-, \hat{X}, X, Y^+, Y^-, \hat{Y}, Y$ be defined as in Section 4. Furthermore, let $\delta(D) \geq p \geq 2$. Using the preparatory structural results of Section 4, we show the following results.

Lemma 5.1. *Let $X - V(C) \neq \emptyset$. If $|X| = |V(D)| - 1$, then $|X| \geq 4p - 2$.*

Proof. Note that $V(C) \subseteq X$ by condition A of the choice of C. Let $V(D) - X = Y = \{y\}$.

Case 1: Suppose that $X^+ \neq \emptyset$ and $X^- \neq \emptyset$. We define the sets

$$A = \big\{v \in V(D) \mid \text{there exists a path leading from } X^+ \text{ to } v \text{ in } D - y\big\} \text{ and}$$
$$B = \big\{v \in V(D) \mid \text{there exists a path leading from } v \text{ to } X^- \text{ in } D - y\big\}.$$

Note that $A \subseteq X^+ \cup \hat{X}$ and $B \subseteq X^- \cup \hat{X}$. Furthermore, $A \cap B = \emptyset$ by Claim 6. In addition, $N^+(A) - A \subseteq \{y\}$ and $N^-(B) - B \subseteq \{y\}$. Using Lemma 3.4(b), it follows that $|A|, |B| \geq 2p - 1$ and thus,

$$|X| \geq |A| + |B| \geq 4p - 2.$$

Case 2: Suppose that $X^+ = \emptyset$. In order to show that $|V(C)| \geq 2p-1$ we consider the three subcases $Y = \hat{Y}$, $Y = Y^+$ and $Y = Y^-$.

Subcase 2.1: Suppose that $Y = \hat{Y}$. By (1) and Claim 1, it follows that $N^+(C) - V(C) \subseteq \{y\}$. Using Lemma 3.4(b), we conclude that $|V(C)| \geq 2p - 1$.

Subcase 2.2: Suppose that $Y = Y^+$. By Claim 1, it follows that $N^+(C, \hat{X}) = \emptyset$ and thus, $N^+(C) - V(C) \subseteq \{y\}$. Using Lemma 3.4(b), we conclude that $|V(C)| \geq 2p - 1$.

Subcase 2.3: Suppose that $Y = Y^-$. We derive $N^+(C) \subseteq V(C)$, a contradiction to the assumption that D is strong.

Note that $N^-(X^-, C) = \emptyset$ by definition and $N^-(\hat{X}, C) = \emptyset$ by Claim 1. Hence $N^-(X^- \cup \hat{X}) - (X^- \cup \hat{X}) \subseteq \{y\}$. Using Lemma 3.4(b), it follows that $|X^- \cup \hat{X}| \geq 2p-1$ and thus,

$$|X| \geq |V(C)| + |X^+ \cup \hat{X}| \geq 4p - 2.$$

The case $X^- = \emptyset$ can be solved analogously to Case 2 which completes the proof of this lemma. □

Lemma 5.2. *Let $|X| \leq |V(D)| - 2$ and $X - V(C) \neq \emptyset$.*

(a) If $\delta(D) \geq p \geq 2$, then

(i) $|X| \geq 4p - 1$ or

(ii) $|X| = 4p - 2$ *and* $D \in \mathcal{T}^*_{loc}$.

(b) If $\delta(D) \geq p \geq 3$, *then*

(i) $|X| \geq 4p$ *or*

(ii) $|X| = 4p - 1$ *and* $D \in \mathcal{T}^*_{in} \cup \mathcal{T}^{**}_{in} \cup \mathcal{T}^{**}_{loc}$ *or*

(iii) $|X| = 4p - 2$ *and* $D \in \mathcal{T}^*_{loc}$.

The proof of the above lemma is rather lengthy and is therefore divided in several cases (see below).

Proof of Lemma 5.2. Recall that $|Y| \geq 1$. If $|Y| = 1$, we conclude $|V(C) - X| \geq 1$, since $|X| \leq |V(D)| - 2$. If $|Y| \geq 2$, we derive by Claim 9 that $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- \neq \emptyset$. We will now consider several cases as follows.

1: $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- \neq \emptyset$

1.1: $\hat{Y} = \emptyset$ (Note: this implies $Y^+ \cup Y^- \neq \emptyset$, since $Y \neq \emptyset$)

1.1.1: $X^+ \neq \emptyset$ and $Y^+ \neq \emptyset$

1.1.2: $X^- \neq \emptyset$ and $Y^- \neq \emptyset$

1.1.3: $Y^+ \neq \emptyset$ and $Y^- \neq \emptyset$ (Note: this implies $X^+ = X^- = \emptyset$ by 1.1.1 and 1.1.2 and hence, $\hat{X} \neq \emptyset$)

1.1.4: $Y^+ \neq \emptyset$ and $Y^- = \emptyset$ (Note: this implies $X^+ = \emptyset$ by 1.1.1 and $X^- \neq \emptyset$, since $X^- \cup Y^- \neq \emptyset$)

1.1.5: $Y^+ = \emptyset$ and $Y^- \neq \emptyset$ (Note: this implies $X^+ \neq \emptyset$, since $X^+ \cup Y^+ \neq \emptyset$, and $X^- = \emptyset$ by 1.1.2)

1.2: $|\hat{Y}| = 1$

1.3: $|\hat{Y}| \geq 2$

2: $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- = \emptyset$

3: $X^+ \cup Y^+ = \emptyset$ and $X^- \cup Y^- \neq \emptyset$

Case 1.1.1: Let $Y^+ = \{y\}$. We define the sets

$$A = \big\{v \in V(D) \mid \text{there exists a path leading from } X^+ \text{ to } v \text{ in } D - y\big\} \text{ and}$$
$$B = \big\{v \in V(D) \mid \text{there exists a path leading from } v \text{ to } X^- \cup Y^- \text{ in } D - y\big\}.$$

Note that $A \subseteq X^+ \cup \hat{X}$ and $B - Y^- \subseteq X^- \cup \hat{X}$. Furthermore, $A \cap B = \emptyset$ by Claim 6. Let $A_1, A_2, \ldots, A_q$ be an acyclic ordering of the strong components of $D[A]$, where $q \geq 1$, and let $B_1, B_2, \ldots, B_r$ be an acyclic ordering of the strong components of $D[B]$, where $r \geq 1$. Then $N^+(A_q) - V(A_q) \subseteq \{y\}$ and thus, $|V(A_q)| \geq 2p - 1$ by Lemma 3.4(b). Analogously, $N^-(B_1) - V(B_1) \subseteq \{y\}$ and thus, $|V(B_1)| \geq 2p - 1$ by

Lemma 3.4(b). Let C_q be a Hamiltonian cycle of A_q. Then $V(D) - V(C_q) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C_q) \cap X| \geq 3$ by choice of C. It follows that

$$|X| \geq |V(C) \cap X| + |V(A_q)| + |V(B_1) - Y^-| \geq 4p.$$

Case 1.1.2: This case can be solved analogously to Case 1.1.1.

Case 1.1.3: Let $Y^+ = \{y_1\}$ and $Y^- = \{y_2\}$. Let $D_1, D_2, \ldots, D_q$ be an acyclic ordering of the strong components of $D[\hat{X}]$, where $q \geq 1$. Then $N^+(D_q) - V(D_q) \subseteq \{y_2\}$ and thus, $|V(D_q)| \geq 2p - 1$ by Lemma 3.4(b) and $|V(D_q)| \geq 2p$ if $D_q \not\to y_2$ by Lemma 3.4(c). Let C_q be a Hamiltonian cycle of D_q. Then $V(D) - V(C_q) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C_q) \cap X|$ by choice of C. If $|V(D_q)| \geq 2p$ or if $q \geq 3$ or if $q = 2$ and $|V(D_1)| \geq 3$, it follows that $|X| \geq 4p$. So it remains to check the cases $q = 2$, $|V(D_2)| = 2p - 1$ and $|V(D_1)| = 1$ (Subcase 1.1.3.1) and $q = 1$ and $|V(D_1)| = 2p - 1$ (Subcase 1.1.3.2). Note that $D_q \to y_2$, since $|V(D_q)| = 2p - 1$.

Subcase 1.1.3.1: Suppose that $q = 2$, $|V(D_2)| = 2p - 1$ and $|V(D_1)| = 1$. Let $V(D_1) = \{x\}$. Note that $N^+(\hat{X}, C) = N^+(\hat{X}, Y^+) = \emptyset$ by Claims 1 and 3. It follows that $N^+(x, D_2) \neq \emptyset$ and hence $x \to D_2$. Since $N^-(\hat{X}, C) = \emptyset$ by Claim 1 and $\delta(D) \geq 2$, it follows that $\{y_1, y_2\} \to x$. Let

$$C' = xC_2y_2x.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |V(C') \cap X| \geq 4p.$$

Subcase 1.1.3.2: Suppose that $q = 1$ and $|V(D_1)| = 2p-1$. Note that $N^-(\hat{X}, C) = \emptyset$ by Claim 1. The latter together with Lemma 3.4(c) implies that $y_1 \to D_1$.

Subcase 1.1.3.2.1: Suppose that $|N^-(y_1, C)| \geq 2$. We may assume, without loss of generality, that $\{v_i, v_k\} \to y_1$, where $1 \leq i \leq k - 1$. If $V(C[v_1, v_i]) - X \neq \emptyset$ and $V(C[v_{i+1}, v_k]) - X \neq \emptyset$, we may assume, without loss of generality, that $|V(C[v_1, v_i]) \cap X| \geq 2$. Let

$$C' = y_1C_1y_2C[v_1, v_i]y_1.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X|$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq |V(C') \cap X| + |V(D_1)| \geq 2|V(D_1)| + 2 = 4p.$$

So assume that $V(C[v_1, v_i]) \subseteq X$.

If $V(C) \not\subseteq X$, we consider

$$C^* = y_1C_1y_2C[v_1, v_i]y_1.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq |V(D_1)| + |V(C[v_1, v_i])|$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 2|V(D_1)| + |V(C[v_1, v_i])| = 4p - 2 + |V(C[v_1, v_i])|.$$

So it remains to consider the case that $p \geq 3$ and $i = 1$. If $v_k \in X$, let

$$\hat{C} = y_1 C_1 y_2 v_k v_1 y_1.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq |V(D_1)| + 2 = 2p + 1$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

So assume that $v_k \notin X$. Since $\delta(D) \geq 3$, there exists an arc $v_j y_1$ in D, where $j \neq k$. Then either $|V(C[v_1, v_j]) \cap X| \geq 2$ and $V(C[v_{j+1}, v_k]) - X \neq \emptyset$ or $|V(C[v_{j+1}, v_k]) \cap X| \geq 2$ and $V(C[v_1, v_j]) - X \neq \emptyset$. We may assume, without loss of generality, that the former holds. Let

$$\tilde{C} = y_1 C_1 y_2 C[v_1, v_j] y_1.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\tilde{C}) \cap X| \geq |V(D_1)| + 2 = 2p + 1$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

If $V(C) \subseteq X$, note that D_1 induces a $(p-1)$-regular tournament in D. Furthermore, note that $N^+(C) - V(C) \subseteq \{y_1\}$. So $|V(C)| \geq 2p - 1$ by Lemma 3.4(b) and $|V(C)| \geq 2p$ if $C \not\rightarrow y_1$ by Lemma 3.4(c). If $|V(C)| = 2p - 1$, it follows that $C \rightarrow y_1$. Since $\delta(D) \geq p$, the cycle C induces a $(p-1)$-regular tournament in D and thus, D is a member of $\mathcal{T}^*_{loc}$. If $|V(C)| = 2p$, the cycle C induces a tournament T in D such that $p - 1 \leq \delta(T), \Delta(T) \leq p$. So if $C \rightarrow y_1$, the digraph D is a local tournament and a member of $\mathcal{T}^{**}_{loc}$. If $C \not\rightarrow y_1$, the digraph D is not a local tournament and with the help of Lemma 3.5 we conclude that D is a member of $\mathcal{T}^*_{in}$.

Subcase 1.1.3.2.2: Suppose that $|N^-(y_1, C)| = 1$. We conclude that $y_2 \rightarrow y_1$ and $p = 2$. If $|V(C) \cap X| \geq 2p = 4$, it follows that $|X| \geq 4p - 1 = 7$. So assume that $|V(C) \cap X| = |V(D_1)| = 2p - 1 = 3$. Let, without loss of generality, v_k be the negative neighbor of y_1 on C. If there exists a vertex $v_i \in V(C) \cap X$ such that $V(C[v_1, v_{i-1}]) - X \neq \emptyset$, we consider the cycle

$$C' = y_1 C_1 y_2 C[v_i, v_k] y_1.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq |V(D_1)| + 1 = 4$ by choice of C, a contradiction. So $V(C) \cap X = \{v_1, v_2, v_3\}$ and $v_i \notin X$ for every index $4 \leq i \leq k$. Note that $k \geq 4$. If D has an arc $v_i v_j$, where $1 \leq i \leq 3$ and $5 \leq j \leq k$, we consider the cycle

$$C^* = y_1 C_1 y_2 v_i C[v_j, v_k] y_1.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq |V(D_1)| + 1 = 4$ by choice of C, a contradiction. So $N^+(\{v_1, v_2, v_3\}) - \{v_1, v_2, v_3\} \subseteq \{v_4\}$ and thus, $\{v_1, v_2\} \rightarrow v_4$ and $v_3 \rightarrow v_1$. The latter implies that $v_k \rightarrow v_3$ and hence $k > 4$. Since $\{v_1, v_2, v_3\} \rightarrow v_4$, it follows that there is an arc $v_4 v_j$, where $5 < j \leq k$. Let

$$\hat{C} = y_1 C_1 y_2 C[v_1, v_4] C[v_j, v_k] y_1.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq |V(D_1)| + 3 = 6$ by choice of C, the final contradiction.

Case 1.1.4: Let $Y^+ = \{y\}$. Note that $N^-(X^- \cup \hat{X}) - (X^- \cup \hat{X}) \subseteq \{y\}$. Let $D_1, D_2, \ldots, D_q$ be an acyclic ordering of the strong components of $D[X^- \cup \hat{X}]$, where $q \geq 1$. Then $N^-(D_1) - V(D_1) \subseteq \{y\}$ and thus, $|V(D_1)| \geq 2p - 1$ by Lemma 3.4(b) and $|V(D_1)| \geq 2p$ if $y \not\rightarrow D_1$ by Lemma 3.4(c). Let C_1 be a Hamiltonian cycle of D_1. Then $V(D) - V(C_1) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C_1) \cap X|$ by choice of C. If $|V(D_1)| \geq 2p$ or if $q \geq 3$ or if $q = 2$ and $|V(D_1)| \geq 3$, it follows that $|X| \geq 4p$. So it remains to check the cases $q = 2$, $|V(D_1)| = 2p - 1$ and $|V(D_2)| = 1$ (Subcase 1.1.4.1) and $q = 1$ and $|V(D_1)| = 2p - 1$ (Subcase 1.1.4.2). Note that $y \rightarrow D_1$, since $|V(D_1)| = 2p - 1$.

Subcase 1.1.4.1: Suppose that $q = 2$, $|V(D_1)| = 2p - 1$ and $|V(D_2)| = 1$. Let $V(D_2) = \{x_2\}$. Since $d^+(x_2) \geq 2$, it follows that D has an arc leading from x_2 to C. So $x_2 \in X^-$ and hence, $x_2 \rightarrow C$. Since $d^-(x_2) \geq 2$, it follows that D has an arc leading from D_1 to x_2, say x_1x_2. If $x_2 \rightarrow y$, let

$$C' = yC_1[x_1^+, x_1]x_2y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| = 2p$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| + 1 \geq 4p.$$

If $x_2 \not\rightarrow y$, the vertex y has at least two negative neighbors on C. We may assume, without loss of generality, that $\{v_i, v_k\} \rightarrow y$, where $1 \leq i \leq k-1$. Since $|V(C) \cap X| \geq 3$ and $V(C) - X \neq \emptyset$, we may assume, without loss of generality, that $|V(C[v_{i+1}, v_k]) \cap X| \geq 1$ and $V(C[v_1, v_i]) - X \neq \emptyset$. Let

$$C^* = yC_1[x_1^+, x_1]x_2C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| + 1 \geq 4p + 1.$$

Subcase 1.1.4.2: Suppose that $q = 1$ and $|V(D_1)| = 2p - 1$. The latter together with Lemma 3.4(c) implies that every vertex of D_1 has at least one positive neighbor on C. So $D_1 \rightarrow C$. Recall that y has at least $p \geq 2$ negative neighbors on C.

Subcase 1.1.4.2.1: Suppose that $p = 2$. Let, without loss of generality, $\{v_i, v_k\} \rightarrow y$, where $1 \leq i \leq k - 1$. Since $V(C) - X \neq \emptyset$ and $|V(C) \cap X| \geq 2p - 1 = 3$, we may assume, without loss of generality, that $|V(C[v_{i+1}, v_k]) \cap X| \geq 1$ and $V(C[v_1, v_i]) - X \neq \emptyset$. Let

$$C' = yC_1C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p = 4$ by choice of C. Hence

$$|X| = |V(D_1)| + |V(C) \cap X| \geq 4p - 1 = 7.$$

Subcase 1.1.4.2.2: Suppose that $p \geq 3$. We may assume, without loss of generality, that $\{v_i, v_j, v_k\} \to y$, where $1 \leq i < j \leq k-1$. Since $|V(C) \cap X| \geq 3$ and $V(C) - X \neq \emptyset$, we may assume, without loss of generality, that $|V(C[v_{i+1}, v_k]) \cap X| \geq 2$ and $V(C[v_1, v_i]) - X \neq \emptyset$. Let

$$C^* = yC_1C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| = |V(D_1)| + |V(C) \cap X| \geq 4p.$$

Case 1.1.5: Let $Y^- = \{y\}$. Note that $N^+(X^+ \cup \hat{X}) - (X^+ \cup \hat{X}) \subseteq \{y\}$. Let $D_1, D_2, \ldots, D_q$ be an acyclic ordering of the strong components of $D[X^+ \cup \hat{X}]$, where $q \geq 1$. Then $N^+(D_q) - V(D_q) \subseteq \{y\}$ and thus, $|V(D_q)| \geq 2p-1$ by Lemma 3.4(b) and $|V(D_q)| \geq 2p$ if $D_q \not\to y$ by Lemma 3.4(c). Let C_q be a Hamiltonian cycle of D_q. Then $V(D) - V(C_q) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C_q) \cap X|$ by choice of C. If $|V(D_q)| \geq 2p$ or if $q \geq 3$ or if $q = 2$ and $|V(D_q)| \geq 3$, it follows that $|X| \geq 4p$. So it remains to check the cases $q = 2$, $|V(D_2)| = 2p-1$ and $|V(D_1)| = 1$ (Subcase 1.1.5.1) and $q = 1$ and $|V(D_1)| = 2p-1$ (Subcase 1.1.5.2). Note that $D_q \to y$, since $|V(D_q)| = 2p-1$.

Subcase 1.1.5.1: Suppose that $q = 2$, $|V(D_2)| = 2p-1$ and $|V(D_1)| = 1$. Let $V(D_1) = \{x\}$. Since $d^+(x) \geq 2$, it follows that D has an arc leading from x to D_2. Hence $x \to D_2$.

If $y \to x$, let

$$C' = yxC_2y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| = 2p$. Hence

$$|X| = |V(C) \cap X| + |V(D_2)| + 1 \geq 4p.$$

If $y \not\to x$, the vertex x has at least two negative neighbors on C. Let, without loss of generality, $\{v_i, v_k\} \to x$, where $1 \leq i \leq k-1$. Since $|V(C) \cap X| \geq 3$ and $V(C) - X \neq \emptyset$, we may assume, without loss of generality, that $|V(C[v_{i+1}, v_k]) \cap X| \geq 1$ and $V(C[v_1, v_i]) - X \neq \emptyset$. Let

$$C^* = yC[v_{i+1}, v_k]xC_2y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(D_2)| + 1 \geq 4p+1.$$

Subcase 1.1.5.2: Suppose that $q = 1$ and $|V(D_1)| = 2p-1$. The latter together with Lemma 3.4(c) yields that every vertex of D_1 has at least one negative neighbor on C. Note that $N^-(D_1, C) \to D_1$. Let, without loss of generality, $v_k \to D_1$.

Assume that there exists a vertex $v_i \in V(C) - X$ such that $|V(C[v_{i+1}, v_k]) \cap X| \geq 2$. Let

$$C' = yC[v_{i+1}, v_k]C_1y.$$

Then $V(D)-V(C')-X \neq \emptyset$ and thus, $|V(C)\cap X| \geq |V(C')\cap X| = 2p+1$ by choice of C. Hence

$$|X| = |V(C)\cap X| + |V(D_2)| \geq 4p.$$

So assume the contrary. That means that C has the following structure. Let $i = \min\{j \mid v_j \notin X\}$ be the minimal index such that v_i is not in X. Then there is at most one vertex on $C[v_{i+1}, v_k]$ that belongs to X, that is $|V(C[v_{i+1}, v_k]) \cap X| \leq 1$.

Subcase 1.1.5.2.1: Suppose that $i = k$. Then $V(C) - X = \{v_k\}$.

If there exists an index j such that $2 \leq j \leq k-1$ and $v_j \to D_1$, let

$$C' = yC[v_1, v_j]C_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C)\cap X| \geq |V(C')\cap X| \geq |V(D_1)| + 2 = 2p+1$ by choice of C. Hence

$$|X| = |V(C)\cap X| + |V(D_1)| \geq 4p.$$

If $v_1 \to D_1$, let

$$C^* = yv_1C_1y.$$

Then $V(D)-V(C^*)-X \neq \emptyset$ and thus, $|V(C)\cap X| \geq |V(C^*)\cap X| \geq |V(D_1)|+1 = 2p$ by choice of C. Hence

$$|X| = |V(C)\cap X| + |V(D_1)| \geq 4p-1$$

and it remains to consider the case that $p \geq 3$. In this case v_1 has a negative neighbor $v_r \neq v_k$ on C. Let

$$\hat{C} = yv_rv_1C_1y.$$

Then $V(D)-V(\hat{C})-X \neq \emptyset$ and thus, $|V(C)\cap X| \geq |V(\hat{C})\cap X| \geq |V(D_1)|+2 = 2p+1$ by choice of C. Hence

$$|X| = |V(C)\cap X| + |V(D_1)| \geq 4p.$$

So $v_j \not\to D_1$ for every index j with $1 \leq j \leq k-1$. Since $N^+(V(C)\cap X) - X \subseteq \{v_k\}$ and $v_k \to v_1 \in X$, it follows that $|V(C)\cap X| \geq 2p$ by Lemma 3.4(c). So if $|X| = 4p-1$ we conclude by Lemma 3.5 that D is a member of $\mathcal{T}_{in}^{**}$.

Subcase 1.1.5.2.2: Suppose that $i \neq k$ and $V(C[v_{i+1}, v_k]) \cap X = \emptyset$. We conclude that $|V(C) - X| \geq 2$.

If D has an arc v_jv_k, where $j \neq k-1$, let

$$C' = yC[v_1, v_j]v_kC_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C)\cap X| \geq |V(C')\cap X| \geq 2p+1$ by choice of C. Hence

$$|X| = |V(C)\cap X| + |V(D_1)| \geq 4p.$$

So $N^-(v_k, C) = \{v_{k-1}\}$ which implies that $p = 2$. It suffices to show that $|V(C)\cap X| \geq 4$. So assume to the contrary that $V(C)\cap X = \{v_1, v_2, v_3\}$ and $i = 4$. Note that $N^+(\{v_1, v_2, v_3\}, C) - \{v_1, v_2, v_3\} \subseteq \{v_4\}$.

If $v_j \to D_1$ for an index $j \in \{1,2,3\}$, let

$$C^* = yC[v_1, v_j]C_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 4$ by choice of C, a contradiction.

So $N^+(\{v_1,v_2,v_3\}) - \{v_1,v_2,v_3\} \subseteq \{v_4\}$. It follows that $v_3 \to v_1$ and $\{v_1,v_2,v_3\} \to v_4$. Then v_4 has a positive neighbor v_j on C, where $j \geq 6$. Let

$$\hat{C} = yC[v_1, v_4]C[v_j, v_k]C_1y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 6$, again a contradiction.

Subcase 1.1.5.2.3: Suppose that $i \neq k$ and $V(C[v_{i+1}, v_k]) \cap X = \{x\}$.

Assume that $x \neq v_k$. If $N^-(D_1, C) \neq \{v_k\}$, let $v_j \to D_1$, where $j \neq k$. If $1 \leq j \leq i-1$, let

$$C' = yC[x, v_j]C_1y$$

and if $i \leq j \leq k-1$, let

$$C' = yC[v_1, v_j]C_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

So $N^-(D_1, C) = \{v_k\}$. We show that $p = 2$. If $v_{k-1} = x$, assume that v_{k-1} has a negative neighbor $v_j \neq v_{k-2}$ on C. Let

$$C^* = yC[v_1, v_j]v_{k-1}v_kC_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

So $|N^-(v_{k-1}, C)| = 1$ and thus, $p = 2$. If $v_{k-1} \neq x$, assume that v_k has a negative neighbor $v_j \neq v_{k-1}$ on C. Let

$$\hat{C} = yC[v_1, v_j]v_kC_1y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

So $|N^-(v_k, C)| = 1$ and thus, $p = 2$. Therefore it remains to show that $|X| \geq 4p - 1 = 7$. Let $A = V(C[v_1, v_{i-1}])$. Note that $A \subseteq X$ and $|V(C) \cap X| = |A| + 1$. If there is a vertex $v_j \in A$ that has a positive neighbor $v_r \in V(C) - A$ besides v_i, let

$$\tilde{C} = yC[v_1, v_j]C[v_r, v_k]C_1y.$$

Then $V(D)-V(\tilde{C})-X \neq \emptyset$ and thus, $|V(C)\cap X| \geq |V(\tilde{C})\cap X| \geq 2p = 4$ by choice of C. Hence

$$|X| \geq |V(C)\cap X| + |V(D_1)| \geq 4p-1 = 7.$$

So $N^+(A) - A \subseteq \{v_i\}$ and thus, $|A| \geq 2p-1 = 3$ by Lemma 3.4(b). It follows that

$$|X| \geq |A| + 1 + |V(D_1)| \geq 4p-1 = 7.$$

Assume that $x = v_k$. Let $A = V(C[v_1, v_{i-1}])$. Note that $A \subseteq X$ and $|V(C)\cap X| = |A|+1$. If there is a vertex $v_j \in A$ that has a positive neighbor $v_r \in V(C) - A$ besides v_i, let

$$C' = yC[v_1, v_j]C[v_r, v_k]C_1y.$$

Then $V(D)-V(C')-X \neq \emptyset$ and thus, $|V(C)\cap X| \geq |V(C')\cap X| \geq 2p+1$ by choice of C. Hence

$$|X| \geq |V(C)\cap X| + |V(D_1)| \geq 4p.$$

If there is a vertex $v_j \in A$ that dominates D_1, let

$$C^* = yC[v_k, v_j]C_1y.$$

Then $V(D)-V(C^*)-X \neq \emptyset$ and thus, $|V(C)\cap X| \geq |V(C^*)\cap X| \geq 2p+1$ by choice of C. Hence

$$|X| \geq |V(C)\cap X| + |V(D_1)| \geq 4p.$$

So $N^+(A) - A \subseteq \{v_i\}$ and thus, $|A| \geq 2p-1$ by Lemma 3.4(b). Hence

$$|X| \geq |A| + 1 + |V(D_1)| \geq 4p-1.$$

It remains to check the case $p \geq 3$. In this case D has an arc v_jv_k, where $j \neq k-1$. Let

$$\hat{C} = yC[v_1, v_j]v_kC_1y.$$

Then $V(D)-V(\hat{C})-X \neq \emptyset$ and thus, $|V(C)\cap X| \geq |V(\hat{C})\cap X| \geq 2p+1$ by choice of C. Hence

$$|X| \geq |V(C)\cap X| + |V(D_1)| \geq 4p.$$

Case 1.2: Let $\hat{Y} = \{y\}$. Similar to the case $\hat{Y} = \emptyset$, we define

$A = \{v \in V(D) \mid$ there exists a path leading from $X^+ \cup Y^+$ to v in $D-y\}$ and
$B = \{v \in V(D) \mid$ there exists a path leading from v to $X^- \cup Y^-$ in $D-y\}$.

Note that $A \subseteq X^+ \cup Y^+ \cup \hat{X}$, $B \subseteq X^- \cup Y^- \cup \hat{X}$ and $A \cap B = \emptyset$ by Claim 6. Let $A_1, A_2, \ldots, A_q$ be an acyclic ordering of the strong components of $D[A]$, where $q \geq 1$. Then $N^+(A_q) - V(A_q) \subseteq \{y\}$ and thus, $|V(A_q)| \geq 2p-1$ by Lemma 3.4(b) and $|V(A_q)| \geq 2p$ if $A_q \not\to y$ by Lemma 3.4(c). Analogously, let $B_1, B_2, \ldots, B_r$ be an acyclic ordering of the strong components of $D[B]$, where $r \geq 1$. Then $N^-(B_1) - V(B_1) \subseteq \{y\}$ and thus, $|V(B_1)| \geq 2p-1$ by Lemma 3.4(b) and $|V(B_1)| \geq 2p$ if $y \not\to B_1$ by Lemma 3.4(c). Since $|V(A_q)\cap X| = |V(A_q) - Y^+| \geq 2p-2$ and

$|V(B_1) \cap X| = |V(B_1) - Y^-| \geq 2p - 2$, it follows that $|V(C) \cap X| \geq \max\{|V(A_q) \cap X|, |V(B_1) \cap X|\} \geq 2p - 2$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(A_q) \cap X| + |V(B_1) \cap X| \geq 6p - 6.$$

Since $6p - 6 \geq 4p$ if and only if $p \geq 3$, it remains to check the case $p = 2$. In this case we have to show that $|X| \geq 4p - 1 = 7$.

So assume to the contrary that $|X| \leq 4p - 2 = 6$. Then $|V(A_q) \cap X| = |V(B_1) \cap X| = |V(C) \cap X| = 2$, $V(A_q) = A$, $V(B_1) = B$, $X - A - B - V(C) = \emptyset$ and $A \to y \to B$. Furthermore, $V(C) - X \neq \emptyset$. Let $C_A = x_1x_2y_1x_1$ be a Hamiltonian cycle of A, where $\{y_1\} = Y^+$, and let $C_B = x_3x_4y_2x_3$ be a Hamiltonian cycle of B, where $\{y_2\} = Y^-$. If D has an arc uv from B to A, let

$$C' = yu^+u^-uvv^+v^-y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| = 4$ by choice of C, a contradiction. So D has no arc leading from B to A. Since $\delta(D) \geq 2$, it follows that $N^+(u, C) \neq \emptyset$ and $N^-(v, C) \neq \emptyset$ for every vertex $u \in B$ and $v \in A$. Hence $B \to C$ by Claim 1. Let $w \in V(C)$ be a negative neighbor of x_1 and let

$$C^* = yx_3x_4y_2wx_1x_2y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 4$ by choice of C, again a contradiction.

Case 1.3: Note that $N^+(\hat{Y}, C) = N^-(\hat{Y}, C) = \emptyset$ by Claim 2. Let $P = z_0z_1 \dots z_r$ be a shortest $(X^+ \cup Y^+)$-$(X^- \cup Y^-)$-path in D. Then $z_0 \in X^+ \cup Y^+$, $z_r \in X^- \cup Y^-$ and $\{z_1, z_2, \dots, z_{r-1}\} \subseteq \hat{X} \cup \hat{Y}$. Furthermore, $\hat{Y} \subseteq V(P)$ by Claim 6 and thus, $r \geq 3$. Let $i = \min\{s \mid z_s \in \hat{Y}\}$ be the smallest integer such that $z_i \in \hat{Y}$ and let $j = \max\{s \mid z_s \in \hat{Y}\}$ be the greatest integer such that $z_j \in \hat{Y}$. We consider the positive and negative neighborhood of $z_s \in V(P)$, where $1 \leq s \leq r - 1$. In view of Claims 1 and 2, we have $N^-(z_s, C) = \emptyset$ and in view of Claims 1-4 we have $N^+(z_s) \subseteq X^- \cup Y^- \cup \hat{X} \cup \hat{Y}$. Because of the choice of P, it follows that $N^-(z_s) \subseteq \hat{X} \cup \hat{Y} \cup X^- \cup Y^-$ for $s \geq 2$ and $N^+(z_s) \subseteq \hat{X} \cup \hat{Y}$ for $s \leq r - 2$.

If z_s has a positive neighbor $z_t \in V(P)$ besides z_{s+1}, we conclude $t \leq s - 2$ because of the choice of P. Using the in-tournament property of D and the choice of P, it follows that $z_s \to z_0 \in X^+ \cup Y^+$, a contradiction to Claim 3 or 4. So $N^+(z_s, P) = \{z_{s+1}\}$.

If z_s has a negative neighbor $z_t \in V(P)$ besides z_{s-1}, we conclude $t \geq s + 2$ because of the choice of P. Due to the observations above, it follows that $t = r$. So $N^-(z_s, P) \subseteq \{z_{s-1}, z_r\}$.

Now we consider $D - z_i$ and $D - z_j$ and define the sets

$$A = \{v \in V(D) \mid \text{there exists a path leading from } z_{i-1} \text{ to } v \text{ in } D - z_i\} \text{ and}$$
$$B = \{v \in V(D) \mid \text{there exists a path leading from } v \text{ to } z_{j+1} \text{ in } D - z_j\}.$$

Note that $A \subseteq X^+ \cup Y^+ \cup V(P[z_0, z_{i-1}]) \cup (\hat{X} - V(P))$, $B \subseteq X^- \cup Y^- \cup V(P[z_{j+1}, z_r]) \cup (\hat{X} - V(P))$ and $A \cap B = \emptyset$ by Claim 6. Let $A_1, A_2, \ldots, A_q$ be an acyclic ordering of the strong components of $D[A]$, where $q \geq 1$, and let $B_1, B_2, \ldots, B_s$ be an acyclic ordering of the strong components of $D[B]$, where $s \geq 1$. Then $N^+(A_q) - V(A_q) \subseteq \{z_i\}$ and thus, $|V(A_q)| \geq 2p - 1$ by Lemma 3.4(b) and $|V(A_q)| \geq 2p$ if $A_q \not\to z_i$ by Lemma 3.4(c). Analogously, $N^-(B_1) - V(B_1) \subseteq \{z_j\}$ and thus, $|V(B_1)| \geq 2p - 1$ by Lemma 3.4(b) and $|V(B_1)| \geq 2p$ if $z_j \not\to B_1$ by Lemma 3.4(c). Since $|V(A_q) \cap X| = |V(A_q) - Y^+| \geq 2p - 2$ and $|V(B_1) \cap X| = |V(B_1) - Y^-| \geq 2p - 2$, it follows that $|V(C) \cap X| \geq \max\{|V(A_q) \cap X|, |V(B_1) \cap X|\} \geq 2p - 2$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(A_q) \cap X| + |V(B_1) \cap X| \geq 6p - 6.$$

Since

$$6p - 6 \geq 4p \Leftrightarrow p \geq 3,$$

it remains to check the case $p = 2$. In this case we have to show that $|X| \geq 4p - 1 = 7$.

So assume to the contrary that $|X| \leq 4p - 2 = 6$. Then $|V(A_q) \cap X| = |V(B_1) \cap X| = |V(C) \cap X| = 2$, $V(A_q) = A$, $V(B_1) = B$, $X - A - B - V(C) = \emptyset$, $A \to z_i$ and $z_j \to B$. Furthermore, $V(C) - X \neq \emptyset$. Hence $Y^+, Y^- \neq \emptyset$, $X^+ \cup Y^+ \subseteq A$ and $X^- \cup Y^- \subseteq B$. Since $z_j \to B$ and P has minimal length, it follows that $j = r - 1$. Analogously, since $A \to z_i$ and P has minimal length, it follows that $i = 1$. Let $A = \{x_1, x_2, y\}$, where $\{x_1, x_2\} \in X$ and $Y^+ = \{y\}$. Since A induces a 3-cycle in D, we may assume, without loss of generality, that $D[A] = x_1 x_2 y x_1$. So $x_2 \notin \hat{X}$ by Claim 3. Hence $x_2 \in X^+$. Let v be a negative neighbor of x_2 on C and let

$$C' = x_2 P[z_i, z_r] C[v^+, v] x_2.$$

Then $V(D) - V(C') - X \neq \emptyset$ and $|V(C') \cap X| \geq |V(C) \cap X| + 1$, a contradiction to the choice of C.

Case 2: Suppose that $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- = \emptyset$. In this case $N^-(C) - V(C) \subseteq \hat{Y}$, since $N^+(\hat{X}, C) = \emptyset$ by Claim 1. Since D is strong, it follows that D has an arc from $\hat{Y}$ to C and thus, $|\hat{Y}| = 1$ and $Y^+ = \emptyset$ by Claim 2. Let $\hat{Y} = \{y\}$. Since $y \notin Y^-$, it follows that $N^-(y, C) \neq \emptyset$.

If $\hat{X} \neq \emptyset$, note that $N^+(\hat{X}, C) = \emptyset$ by Claim 1 and $N^+(\hat{X}, X^+) = \emptyset$ by Claim 3. Since $N^-(y, C) \neq \emptyset$, it follows that $N^+(\hat{X}, y) = \emptyset$ and thus, D is not strong, a contradiction. So $\hat{X} = \emptyset$.

Let $D_1, D_2, \ldots, D_q$ be an acyclic ordering of the strong components of $D[X^+]$, where $q \geq 1$. Then $N^+(D_q) - V(D_q) \subseteq \{y\}$ and thus, $|V(D_q)| \geq 2p - 1$ by Lemma 3.4(b) and $|V(D_q)| \geq 2p$ if $D_q \not\to y$ by Lemma 3.4(c). Let C_q be a Hamiltonian cycle of D_q. Then $V(D) - V(C_q) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C_q)|$ by choice of C. So if $|V(D_q)| \geq 2p$ or if $q \geq 3$ or if $|V(D_{q-1})| \geq 3$, it follows that $|X| \geq 4p$. It remains to check the case that $q = 2$, $|V(D_2)| = 2p - 1$ and $|V(D_1)| = 1$ (Subcase 2.1) and the case that $q = 1$ and $|V(D_1)| = 2p - 1$ (Subcase 2.2). Note that in both cases $D_q \to y$, since $|V(D_q)| = 2p - 1$.

Subcase 2.1. Suppose that $q = 2$, $|V(D_2)| = 2p - 1$ and $|V(D_1)| = 1$. Let $V(D_1) = \{x\}$. Since $d^+(x) \geq 2$ and $N^+(x, C) = \emptyset$, it follows that x has a positive neighbor in D_2. Hence $x \to D_2$. Since $d^-(x) \geq 2$, either $y \to x$ or y has at least two positive neighbors on C.

Subcase 2.1.1. Suppose that $y \to x$. Let

$$C' = yxC_2y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| = 2p$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

Subcase 2.1.2. Suppose that $y \not\to x$. Then $|N^+(y, C)| \geq 2$ and $|N^-(x, C)| \geq 2$. We may assume, without loss of generality, that $v_k \to y \to v_1$. Since D is an in-tournament, it follows subsequently that $v_k \to D_2$ and $v_k \to x$. Let v_i be a second negative neighbor of x on C, where $1 \leq i \leq k-1$.

If $V(C[v_{i+1}, v_k]) - X \neq \emptyset$, let

$$C' = yC[v_1, v_i]xC_2y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $V(C[v_{i+1}, v_k]) \subseteq X$ and thus, $V(C) - X \subseteq V(C[v_1, v_i])$. Let v_j be a second positive neighbor of y on C, where $2 \leq j \leq k-1$.

If $i+1 \leq j \leq k-1$, let

$$C^* = yC[v_j, v_k]xC_2y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $2 \leq j \leq i$.

If $V(C[v_1, v_{j-1}]) - X \neq \emptyset$, let

$$\hat{C} = yC[v_j, v_k]xC_2y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 2p + 1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $V(C[v_1, v_{j-1}]) \subseteq X$. Let $r = \min\{s \mid v_s \notin X\}$. Then $r \leq i$. Let $A = \{v_1, v_2, \ldots, v_{r-1}\}$.

If there is a vertex $v_s \in A$ that has a positive neighbor v_t on C outside of $A \cup \{v_r\}$, let

$$\tilde{C} = yC[v_1, v_s]C[v_t, v_k]xC_2y,$$

if there is a vertex $v_s \in A$ that dominates x, let

$$\tilde{C} = yC[v_1, v_s]xC_2y$$

and if there is a vertex $v_s \in A$ that has a positive neighbor in D_2, let

$$\tilde{C} = yC[v_1, v_s]C_2y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\tilde{C}) \cap X| \geq 2p$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^+(A) - A = \{v_r\}$ and thus, $|A| \geq 2p - 1$ by Lemma 3.4(b). Since $v_k \notin A$ and $v_k \in X$, it follows that $|V(C) \cap X| \geq 2p$ and thus,

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

Subcase 2.2. Suppose that $q = 1$ and $|V(D_1)| = 2p-1$. In this case $|N^+(y, C)| \geq 2$. We consider two subcases depending on the structure of the negative neighborhood of y on C.

Subcase 2.2.1. Suppose that there exist two vertices $v \neq w$ on C such that $\{v, w\} \to y \to \{v^+, w^+\}$. Let, without loss of generality, $v = v_k$ and $w = v_i$, where $i \neq k$. Then $\{v_i, v_k\} \to D_1$. We may assume, without loss of generality, that $V(C[v_1, v_i]) \cap X \neq \emptyset$ and $V(C[v_{i+1}, v_k]) - X \neq \emptyset$.

If $|V(C[v_1, v_i]) \cap X| \geq 2$, let

$$C' = yC[v_1, v_i]C_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $|V(C[v_1, v_i]) \cap X| = 1$. Since $|V(C) \cap X| \geq 3$, it follows that $|V(C[v_{i+1}, v_k]) \cap X| \geq 2$. If $V(C[v_1, v_i]) - X \neq \emptyset$, let

$$C^* = yC[v_{i+1}, v_k]C_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $V(C[v_1, v_i]) \subseteq X$ and thus, $v_1 = v_i \in X$, again a contradiction.

Subcase 2.2.2. Suppose that there do not exist two vertices $v \neq w$ on C such that $\{v, w\} \to y \to \{v^+, w^+\}$. Then we may assume, without loss of generality, that $v_k \to y \to \{v_1, v_2\}$. It follows that $v_k \to D_1$. If $v_1 \notin X$, let

$$C' = yC[v_2, v_k]C_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p+2$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p + 1.$$

So $v_1 \in X$. Analogously we can show that $v_2 \in X$ if $p \geq 3$.

Subcase 2.2.2.1. Suppose that $N^-(D_1, C) \neq \{v_k\}$. Let i be the minimal index such that $v_i \to D_1$.

Subcase 2.2.2.1.1: Suppose that $p = 2$. Then it suffices to show that $|X| \geq 4p - 1 = 7$.

If $V(C[v_{i+1}, v_k]) - X \neq \emptyset$, let

$$C^* = yC[v_1, v_i]C_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p = 4$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p - 1 = 7.$$

So assume that $V(C[v_{i+1}, v_k]) \subseteq X$. Let j be the minimal index such that $v_j \notin X$. Then $j \leq i$. Let $A = \{v_1, v_2, \ldots, v_{j-1}\}$. If there is a vertex $v_r \in A$ that has a positive neighbor v_s on C outside of $A \cup \{v_j\}$, let

$$\tilde{C} = yC[v_1, v_r]C[v_s, v_k]C_1y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\tilde{C}) \cap X| \geq 2p = 4$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p - 1 = 7.$$

So $N^+(A) - A = \{v_j\}$ and thus, $|A| \geq 2p - 1 = 3$ by Lemma 3.4(b). Since $v_k \notin A$ and $v_k \in X$, it follows that $|V(C) \cap X| \geq 2p = 4$ and thus,

$$|X| = |V(C) \cap X| + |X^+| \geq 4p - 1 = 7.$$

Subcase 2.2.2.1.2: Suppose that $p \geq 3$. We have to show that $|X| \geq 4p$. Due to our assumption we conclude that $y \to \{v_1, v_2, v_3\}$ and $\{v_1, v_2\} \subseteq X$.

Assume that $i \neq 1$. If $V(C[v_{i+1}, v_k]) - X \neq \emptyset$, let

$$C^* = yC[v_1, v_i]C_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $V(C[v_{i+1}, v_k]) \subseteq X$. Let j be the minimal integer such that $v_j \notin X$. Then $j \leq i$. Let $A = \{v_1, v_2, \ldots, v_{j-1}\}$. If there is a vertex $v_r \in A$ that has a positive neighbor v_s on C outside of $A \cup \{v_j\}$, let

$$\tilde{C} = yC[v_1, v_r]C[v_s, v_k]C_1y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\tilde{C}) \cap X| \geq 2p + 1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^+(A) - A = \{v_j\}$ and thus, $|A| \geq 2p-1$ by Lemma 3.4(b). If $i \leq k-2$, we conclude that

$$|X| \geq |X^+| + |A| + |\{v_{k-1}, v_k\}| \geq 4p.$$

So $i = k-1$. If $V(C[v_j, v_i]) \cap X \neq \emptyset$, it follows that

$$|X| \geq |X^+| + |A| + |\{v_k\}| + 1 \geq 4p.$$

So assume that $V(C[v_j, v_i]) \cap X = \emptyset$. Since $d^-(v_k) \geq p \geq 3$, the vertex v_k has a negative neighbor $v_t \neq v_{k-1}$ on C. Let

$$\hat{C} = yC[v_1, v_t]v_kC_1y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

Assume that $i = 1$. If $v_k \notin X$, let $v_t \neq v_k$ be a negative neighbor of v_1 on C. Let

$$C^* = yC[v_2, v_t]v_1C_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $v_k \in X$. Let j be the minimal integer such that $v_j \notin X$ and let $A = \{v_1, v_2, \ldots, v_{j-1}\}$. If there is a vertex $v_r \in A$ that has a positive neighbor v_s on C outside of $A \cup \{v_j\}$, let

$$\tilde{C} = yC[v_1, v_r]C[v_s, v_k]C_1y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(\tilde{C}) \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^+(A) - A = \{v_j\}$ and thus, $|A| \geq 2p-1$ by Lemma 3.4(b). If $V(C[v_j, v_{k-2}]) \cap X \neq \emptyset$, we conclude that

$$|X| \geq |X^+| + |A| + |\{v_k\}| + 1 \geq 4p.$$

So $V(C[v_j, v_{k-2}]) \cap X = \emptyset$. Since $d^-(v_k) \geq p \geq 3$, the vertex v_k has a negative neighbor $v_t \neq v_{k-1}$ on C. Let

$$\hat{C} = yC[v_1, v_t]v_kC_1y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

Subcase 2.2.2.2: Suppose that $N^-(D_1, C) = \{v_k\}$. Let j be the smallest index such that $v_j \notin X$.

Subcase 2.2.2.2.1: Suppose that $v_k \in X$. Let $A = \{v_1, v_2, \dots, v_{j-1}\}$. If there is a vertex $v_r \in A$ that has a positive neighbor v_s on C outside of $A \cup \{v_j\}$, let

$$C^* = yC[v_1, v_r]C[v_s, v_k].$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^+(A) - A = \{v_j\}$ and thus, $|A| \geq 2p - 1$ by Lemma 3.4(b). If $v_{k-1} \in X$, we conclude that

$$|X| \geq |A| + |X^+| + |\{v_{k-1}, v_k\}| \geq 4p.$$

So $v_{k-1} \notin X$. Since $d^-(v_k) \geq p \geq 3$, there is a negative neighbor $v_t \neq v_{k-1}$ of v_k on C. Let

$$\tilde{C} = yC[v_1, v_t]v_k C_1 y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\tilde{C}) \cap X| \geq 2p + 1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

Subcase 2.2.2.2.2: Suppose that $v_k \notin X$.

If $j < k$, we consider v_{k-1}. If $v_{k-1} \notin X$, let $v_t \neq v_{k-1}$ be a negative neighbor of v_k on C. Let

$$C^* = yC[v_1, v_t]v_k y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $v_{k-1} \in X$ and $j \leq k - 2$. Let $A = \{v_1, v_2, \dots, v_{j-1}\}$. If there is a vertex $v_r \in A$ that has a positive neighbor v_s on C outside of $A \cup \{v_j\}$, let

$$\tilde{C} = yC[v_1, v_r]C[v_s, v_k]C_1 y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(\tilde{C}) \cap X| \geq 2p + 1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^+(A) - A = \{v_j\}$ and thus, $|A| \geq 2p - 1$ by Lemma 3.4(b). If $|V(C[v_j, v_k]) \cap X| \geq 2$, we conclude that

$$|X| \geq |A| + |X^+| + 2 = 4p.$$

So assume that $V(C[v_j, v_k]) \cap X = \{v_{k-1}\}$. If v_{k-1} has a negative neighbor $v_t \neq v_{k-2}$ on C, let

$$\hat{C} = yC[v_1, v_t]v_{k-1} v_k C_1 y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 2p + 1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^-(v_{k-1}, C) = \{v_{k-2}\}$ and thus, $p = 2$. Since

$$|X| = |X^+| + |A| + 1 = 4p - 1 = 7,$$

there is nothing left to show.

If $j = k$, it follows that $|A| = |V(C) \cap X| \geq 2p$ by Lemma 3.4(c). So if $|X| = 4p-1$ we conclude by Lemma 3.5 that D is a member of $\mathcal{T}_{in}^{**}$.

Case 3: Recall that $N^+(C, \hat{X}) = \emptyset$ by Claim 1. Since D is strong, it follows that $\hat{Y} \neq \emptyset$ and $N^+(C, \hat{Y}) \neq \emptyset$. By Claim 2 we conclude that $|\hat{Y}| = 1$ and $Y^- = \emptyset$. The latter implies that $X^- \neq \emptyset$. Let $\hat{Y} = \{y\}$. Then $N^+(y, C) \neq \emptyset$ and $N^-(y, C) \neq \emptyset$. Since D is an in-tournament, the vertex y is adjacent to every vertex of X^-. Let $D_1, D_2, \dots, D_q$ be an acyclic ordering of the strong components of $D[X^- \cup \hat{X}]$, where $q \geq 1$. Then $N^-(D_1) - V(D_1) \subseteq \{y\}$ and thus, $|V(D_1)| \geq 2p - 1$ by Lemma 3.4(b) and $|V(D_1)| \geq 2p$ if $y \not\rightarrow D_1$ by Lemma 3.4(c). Let C_1 be a Hamiltonian cycle of D_1. Since $V(D) - V(C_1) - X \neq \emptyset$, it follows that $|V(C) \cap X| \geq |V(C_1) \cap X| = |V(D_1)|$ by choice of C. If $|V(D_1)| \geq 2p$ or if $q \geq 3$ or if $q = 2$ and $|V(D_2)| \geq 3$, we conclude that $|X| \geq 4p$. It remains to check the case $q = 2$, $|V(D_1)| = 2p-1$ and $|V(D_2)| = 1$ (Subcase 3.1) and the case $q = 1$ and $|V(D_1)| = 2p - 1$ (Subcase 3.2). Note that in both cases $y \rightarrow D_1$, since $|V(D_1)| = 2p - 1$.

Subcase 3.1. Suppose that $q = 2$, $|V(D_1)| = 2p - 1$ and $|V(D_2)| = 1$. Let $V(D_2) = \{x_2\}$. Since $x_2 \in X^- \cup \hat{X}$, it follows that $N^-(x_2, C) = \emptyset$. The latter together with $d^-(x_2) \geq p \geq 2$ yields that there is a vertex $x_1 \in V(D_1)$ that dominates x_2.

Subcase 3.1.1. Suppose that $x_2 \rightarrow y$. Let

$$C' = yC_1[x_1^+, x_1]x_2y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| = 2p$ by choice of C. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| + 1 \geq 4p.$$

Subcase 3.1.2. Suppose that $x_2 \not\rightarrow y$. Then $|N^-(y, C)| \geq 2$. In addition, note that $N^+(x_2, C) \neq \emptyset$ and thus, $x_2 \rightarrow C$. Let, without loss of generality, $\{v_i, v_k\} \rightarrow y$, where $i \leq k-1$. We may assume, without loss of generality, that $V(C[v_1, v_i]) - X \neq \emptyset$ and $V(C[v_{i+1}, v_k]) \cap X \neq \emptyset$. Let

$$C' = yC_1[x_1^+, x_1]x_2C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |V(D_1)| + 1 \geq 4p + 1.$$

Subcase 3.2: Suppose that $q = 1$ and $|V(D_1)| = 2p - 1$. Then $D_1 \rightarrow C$ and $|N^-(y, C)| \geq p$.

Subcase 3.2.1: Suppose that $|N^-(y, C)| \geq 3$. Let $\{v_i, v_j, v_k\} \rightarrow y$, where $1 \leq i < j \leq k-1$. We may assume, without loss of generality, that $V(C[v_1, v_i]) - X \neq \emptyset$ and $|V(C[v_{i+1}, v_k]) \cap X| \geq 2$. Let

$$C' = yC_1C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p+1$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |V(D_1)| \geq 4p.$$

Subcase 3.2.2: Suppose that $|N^-(y, C)| = p = 2$. We have to show that $|X| \geq 4p - 1 = 7$. Let, without loss of generality, $\{v_i, v_k\} \rightarrow y$, where $i \leq k - 1$. We may assume, without loss of generality, that $V(C[v_1, v_i]) - X \neq \emptyset$ and $V(C[v_{i+1}, v_k]) \cap X \neq \emptyset$. Let

$$C' = yC_1C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p = 4$ by choice of C. Hence

$$|X| = |V(C) \cap X| + |V(D_1)| \geq 4p - 1$$

which completes the proof of this lemma. □

6 Generalizations of Kotani's Theorems

In this section we use the results of Section 5 to generalize Theorems 1.8, 1.9 and 1.10 to in-tournaments. The following three results summarize our present achievements.

Theorem 6.1. *Let $p \geq 2$ be an integer and let D be a strong in-tournament with $\delta(D) \geq p$. If $X \subseteq V(D)$ with $|X| \leq 4p - 3$ and $X \neq V(D)$, then there exists a cycle C in D such that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$.*

Proof. Let X be a subset of $V(D)$ such that $|X| \leq 4p - 3$ and $X \neq V(D)$. Let C, X^+, X^-, $\hat{X}$, X, Y^+, Y^-, $\hat{Y}$ and Y be defined as above.

Assume that $X - V(C) \neq \emptyset$. If $|X| = |V(D)| - 1$, then we get a contradiction by Lemma 5.1. If $|X| \leq |V(D)| - 2$, then we get a contradiction by Lemma 5.2.

Hence, we obtain $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$ by Claim 10. □

Theorem 6.2. *Let $p \geq 2$ be an integer and let D be a strong in-tournament with $|V(D)| \geq 4p$ and $\delta(D) \geq p$. If $X \subseteq V(D)$ with $|X| \leq 4p - 2$, then either*

(a) there exists a cycle C in D such that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$ or

(b) $D \in \mathcal{T}^*_{loc}$.

Proof. Let X be a subset of $V(D)$ such that $|X| \leq 4p - 2$ and let C, X^+, X^-, $\hat{X}$, X, Y^+, Y^-, $\hat{Y}$ and Y be defined as above. In addition, let $D \notin \mathcal{T}^*_{loc}$.

Assume that $X - V(C) \neq \emptyset$. Since $D \notin \mathcal{T}^*_{loc}$, Lemma 5.2(a) implies that $|X| \geq 4p - 1$, a contradiction to our assumption. Hence, by Claim 10, it follows that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$. □

Theorem 6.3. *Let $p \geq 3$ be an integer. Let D be a strong in-tournament with $|V(D)| \geq 4p + 1$ and $\delta(D) \geq p$. If $X \subseteq V(D)$ with $|X| \leq 4p - 1$, then either*

(a) there exists a cycle C in D such that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$ or

(b) $D \in \mathcal{T}^*_{in} \cup \mathcal{T}^{**}_{in} \cup \mathcal{T}^{**}_{loc}$.

Proof. Let X be a subset of $V(D)$ such that $|X| \leq 4p-1$ and let C, X^+, X^-, $\hat{X}$, X, Y^+, Y^-, $\hat{Y}$ and Y be defined as above. In addition, let $D \notin \mathcal{T}^*_{in} \cup \mathcal{T}^{**}_{in} \cup \mathcal{T}^{**}_{loc}$.

Assume that $X - V(C) \neq \emptyset$. Note that $D \notin \mathcal{T}^*_{loc}$, since $|V(D)| \geq 4p+1$. Additionally $D \notin \mathcal{T}^*_{in} \cup \mathcal{T}^{**}_{in} \cup \mathcal{T}^{**}_{loc}$ and thus, Lemma 5.2(b) implies that $|X| \geq 4p$, a contradiction to our assumption. Hence, by Claim 10, it follows that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$. □

The combination of Lemma 3.3 and the theorems above yields the following results.

Theorem 6.4. *Let D be a strong in-tournament and let $p \geq 2$ be an integer. If $\delta(D) \geq p$, then D has at least $k = \min\{|V(D)|, 4p-2\}$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Theorem 6.5. *Let D be a strong in-tournament such that $D \notin \mathcal{T}^*_{loc}$ and let $p \geq 2$ be an integer. If $\delta(D) \geq p$ and $|V(D)| \geq 4p$, then D has at least $k = 4p-1$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Theorem 6.6. *Let D be a strong in-tournament such that $D \notin \mathcal{T}^*_{in} \cup \mathcal{T}^{**}_{in} \cup \mathcal{T}^{**}_{loc}$ and let $p \geq 3$ be an integer. If $\delta(D) \geq p$ and $|V(D)| \geq 4p+1$, then D has at least $k = 4p$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

The corresponding results for the class of local tournaments can be formulated as follows.

Corollary 6.7 (Meierling & Volkmann [8] 2007)**.** *Let D be a strong local tournament and let $p \geq 2$ be an integer. If $\delta(D) \geq p$, then D has at least $k = \min\{|V(D)|, 4p-2\}$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Corollary 6.8 (Meierling & Volkmann [8] 2007)**.** *Let D be a strong local tournament such that $D \notin \mathcal{T}^*_{loc}$ and let $p \geq 2$ be an integer. If $\delta(D) \geq p$ and $|V(D)| \geq 4p$, then D has at least $k = 4p-1$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Corollary 6.9 (Meierling & Volkmann [8] 2007)**.** *Let D be a strong local tournament such that $D \notin \mathcal{T}^{**}_{loc}$ and let $p \geq 3$ be an integer. If $\delta(D) \geq p$ and $|V(D)| \geq 4p+1$, then D has at least $k = 4p$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Since the exceptional classes of in-tournaments and local tournaments do not contain any tournaments, Theorems 1.8, 1.9 and 1.10 by Kotani [5] are direct consequences of our results.

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(Received 4 Mar 2008; revised 26 Aug 2008)