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Complete k-ary trees and Hamming graphs

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Abstract

A Hamming graph $H(b,n)$ of base b and dimension n has vertex set $\{X = x_1x_2\dots x_n : x_i \in \{0,1,\dots,b-1\}, \text{ for } 1 \le i \le n\}$ and edge set $\{(X,Y) : X \text{ and } Y \text{ differ in exactly one bit position}\}$. In this paper, we are concerned with the following problem: Given positive integers b, k and h, what is the minimum integer $m = m(b,k,h)$ such that the complete k-ary tree, T_h^k, of height h, is a subgraph of $H(b,m)$? The value $m(b,k,h)$ is known for very few values of b, k and h. We show that
(i) $m(k,k,h) = h+1$, for every $k \ge 3$,
(ii) $\lceil(\log_3 2)(h+1)\rceil \le m(3,2,h) \le \lceil \frac{2}{3}(h+1)\rceil$,
(iii) $\left\lceil \frac{h+1}{\log_2 b} \right\rceil \le m(b,2,h) \le \left\lceil \frac{h+1}{\lfloor \log_2 b \rfloor} \right\rceil$, for every $b \ne 2^l$, and that
(iv) $\left\lceil \frac{h+1}{\log_2 b} \right\rceil \le m(b,2,h) \le \left\lceil \frac{h+2}{\log_2 b} \right\rceil$, for every $b = 2^l$.

1 Introduction

For standard graph theoretic notation and terminology, we refer to West [12]. The *Cartesian product* $G_1 \,\square\, G_2 \,\square \cdots \square\, G_n$ of n graphs $G_1, G_2, \dots, G_n$ has vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_n)$, where two vertices $(u_1,u_2,\dots,u_n)$ and $(v_1,v_2,\dots,v_n)$ are adjacent if and only if for exactly one i $(1 \le i \le n)$, $u_i \ne v_i$ and u_i, v_i are adjacent in G_i. The Cartesian product of complete graphs is called a *Hamming graph*. These graphs are widely studied in coding theory, and as mathematical models of interconnection networks in the topic of parallel computers; see [5] and [8]. The references [1], [6] and [7] contain extensive literature on graph theoretical properties of Hamming graphs.

In this paper, all our Hamming graphs are Cartesian products of complete graphs of the same order. So, given any two positive integers b and n, the *Hamming graph $H(b,n)$ of base b and dimension n* is defined as the Cartesian product of n complete graphs each of order b. It follows that $H(b,n) = H(b,p)\square H(b,n-p)$, where $1 \le p \le n-1$. Alternatively, $H(b,n)$ can be defined as a graph with vertex set $\{X = x_1x_2\ldots x_n : x_i \in \{0,1,\ldots,b-1\}$, for $1 \le i \le n\}$ and edge set $\{(X,Y) : X$ and Y differ in exactly one bit position$\}$. It has b^n vertices and $\frac{1}{2}n(b-1)b^n$ edges. Moreover, it is a $(b-1)n$-regular graph with diameter n. Clearly, $H(2,n)$ is the binary hypercube of dimension n. So $H(b,n)$ is also called a generalized hypercube or b-ary n-cube.

A graph is said to be *vertex symmetric* if given any pair of vertices x and y, there exists an automorphism α such that $\alpha(x) = y$. Analogously, a graph is said to be *edge symmetric* if given any pair of edges (a,b) and (u,v), there exists an automorphism α such that $\alpha(a) = u$ and $\alpha(b) = v$. It is shown in [9] that $H(b,n)$ is vertex symmetric and edge symmetric. We exploit these properties in proving our results.

Let $k \ge 1$, $h \ge 0$ be integers. The *complete k-ary tree of height h, T_h^k*, is a rooted tree in which every non-leaf vertex has exactly k children, and the distance from the root to each leaf is exactly h. By convention, $T_0^k = K_1$. The root of T_h^k is denoted by r_h^k.

Clearly, T_h^k has $h+1$ vertices if $k = 1$, and $\frac{k^{h+1}-1}{k-1}$ vertices if $k \ge 2$. It can be recursively constructed as follows. Consider k copies of T_{h-1}^k, and denote the i^{th} copy by iT_{h-1}^k and its root by ir_{h-1}^k, for $i = 1,2,\ldots,k$. Add a new vertex r_h^k and join it to the vertices ir_{h-1}^k, for $i = 1,2,\ldots,k$. The resultant tree is T_h^k with root r_h^k. Figure 1 shows the construction of T_h^3 from three copies of T_{h-1}^3.

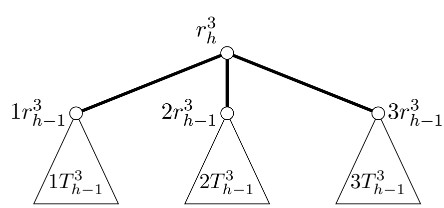


Figure 1: Construction of T_h^3 from three copies of T_{h-1}^3

Given two graphs G and H, if there exists an injection $f : V(G) \to V(H)$ such that $(f(u), f(v)) \in E(H)$ whenever $(u,v) \in E(G)$, we say that f is an embedding, and G is embeddable into H (and write $G \subseteq H$). Clearly, $G \subseteq H$ if and only if G is isomorphic with a subgraph of H.

The graph embeddings are useful (i) to establish equivalence between interconnection networks of two parallel computers, and (ii) to design efficient parallel algorithms and implement them on a given parallel computer with minimum communication delay. Extensive literature on embeddings and parallel processing is contained in [3] and [10].

In this paper, we are concerned with the following problem.

Given non-negative integers b, k and h, what is the minimum integer $m = m(b,k,h)$ such that $T_h^k \subseteq H(b,m)$?

For very few values of b, k and h, is $m(b,k,h)$ known:

1. (Obvious) $m(2, k, 1) = k$ and $m(2, 1, h) = \lceil \log_2 h \rceil$.

2. (Nebeský [11]) $m(2, 2, h) = h + 2$, $h \geq 2$.

3. (Havel and Liebl [4]) $m(2, k, 2) = \lceil \frac{3k+1}{2} \rceil$.

4. (Havel and Liebl [4]) $\frac{kh}{e} \leq m(2, k, h) \leq \frac{k(h+1)}{2} + h - 1$, where $e = 2.71\ldots$ is the base of the natural logarithm.

5. (Lakshmivarahan and Dhall [8]) $m(4, 2, h) = \lceil \frac{h}{2} \rceil$.

In the literature, the parameter $m(2, k, h)$ is called the *cubic dimension* of T_h^k and is unknown for $k \geq 3$ or $h \geq 3$; see [4, 6]. A few asymptotic results approximating $m(2, k, h)$ are proved in [2].

In this paper, we show that for every $h \geq 0$:

(i) $m(k, k, h) = h + 1$, for every $k \geq 3$;

(ii) $\lceil (\log_3 2)(h + 1) \rceil \leq m(3, 2, h) \leq \lceil \frac{2}{3}(h + 1) \rceil$;

(iii) $\left\lceil \frac{h+1}{\log_2 b} \right\rceil \leq m(b, 2, h) \leq \left\lceil \frac{h+1}{\lfloor \log_2 b \rfloor} \right\rceil$, for every $b \neq 2^l$; and

(iv) $\left\lceil \frac{h+1}{\log_2 b} \right\rceil \leq m(b, 2, h) \leq \left\lceil \frac{h+2}{\log_2 b} \right\rceil$, for every $b = 2^l$.

Given a graph G and a positive integer b, let n be the smallest integer such that $b^n \geq |V(G)|$. Then $H(b, n)$ is called the *optimal* Hamming graph of G, and $H(b, n + 1)$ is called the *next-to-optimal* Hamming graph of G.

2 k-ary trees in Hamming graphs $H(k, n)$

Since $k^h < |V(T_h^k)| < k^{h+1}$, it follows that $H(k, h + 1)$ is the optimal Hamming graph of T_h^k. In our first result we show that $T_h^k \subseteq H(k, h + 1)$, for any $k \geq 3$ and $h \geq 0$; consequently, $m(k, k, h) = h + 1$. However, it may be interesting to know that $T_h^2 \not\subseteq H(2, h + 1)$ and that $T_h^2 \subseteq H(2, h + 2)$, see [11]; so Theorem 2.1 is not true when $k = 2$. To exploit the recursive structure of T_h^k and $H(k, h+1)$, we find it convenient to embed a super tree $\mathbb{T}_h^k$ of T_h^k into $H(k, h + 1)$ instead of straightaway embedding T_h^k into $H(k, h+1)$. The tree $\mathbb{T}_h^k$ is obtained from T_h^k by adding two new vertices x and y and two new edges (r_h^k, x) and (x, y). The resultant path $\langle r_h^k, x, y \rangle$ is called the *auxiliary path* of $\mathbb{T}_h^k$. In any embedding f of a graph G into $H(b, n)$, the vertex $f(x)$ is denoted by X.

Theorem 2.1 *Let $k \geq 3$. Then for every $h \geq 0$, $T_h^k \subseteq H(k, h + 1)$.*

Proof: We prove that $\mathbb{T}_h^k \subseteq H(k, h+1)$ by induction on h. Since $T_h^k \subset \mathbb{T}_h^k$, the theorem follows. When $h = 0$ or 1 it can be easily verified that $\mathbb{T}_h^k \subseteq H(k, h+1)$. For the inductive step, we assume that $\mathbb{T}_{h-1}^k \subseteq H(k, h)$, where $h \geq 2$; by our notation, the auxiliary path $\langle r_{h-1}^k, x, y \rangle$ of $\mathbb{T}_{h-1}^k$ is mapped onto a path $\langle R_{h-1}^k, X, Y \rangle$ in $H(k, h)$. We embed $\mathbb{T}_h^k$ in $H(k, h+1)$ in three steps.

Step 1: First we decompose $H(k, h+1)$ into k copies of $H(k, h)$ denoted by $iH(k, h)$, $i = 0, 1, \ldots, k-1$; see Figure 2.

Step 2: By induction hypothesis, we have $\mathbb{T}_{h-1}^k \subseteq iH(k, h)$, $i = 0, 1, \ldots, k-1$, with T_{h-1}^k rooted at iR_{h-1}^k. For each i, $1 \leq i \leq k-1$, we apply an automorphism α to $V(iH(k, h))$ such that $\alpha(iR_{h-1}^k) = iX$ and $\alpha(iX) = iY$. Let α map the edge (iX, iY) onto the edge (iY, iZ), where clearly Z is some vertex adjacent to Y in $V(H(k, h))$. Thus, after applying the automorphism, we have $\mathbb{T}_{h-1}^k \subseteq iH(k, h)$ with its auxiliary path, mapped onto the path $\langle iX, iY, iZ \rangle$.

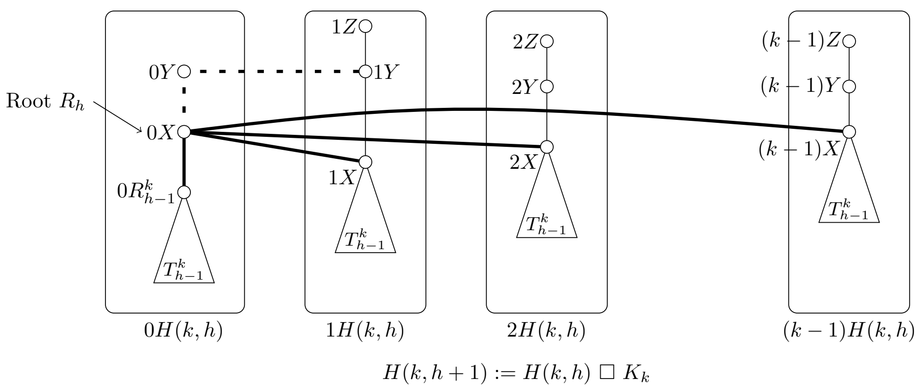


Figure 2: Embedding of T_h^k into $H(k, h+1)$.

Step 3: We combine all these k embeddings by adding the edges $(0X, iX)$, $1 \leq i \leq k-1$. This operation results in an embedding of $\mathbb{T}_h^k$ into $H(k, h+1)$ with its auxiliary path mapped onto $\langle 0X, 0Y, 1Y \rangle$. $\square$

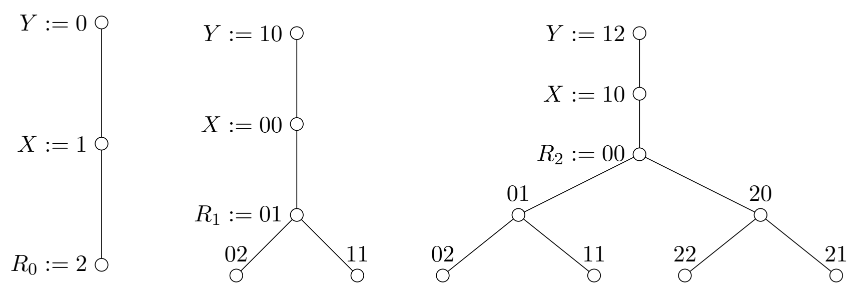
3 Complete binary trees in $H(3, n)$

While $m(2, 2, h)$ and $m(4, 2, h)$ are known (see the introduction), $m(3, 2, h)$ is not known. We fill this gap by showing that $\lceil (\log_3 2)(h+1) \rceil \leq m(3, 2, h) \leq \lceil \frac{2}{3}(h+1) \rceil$. The lower bound follows since the optimal ternary cube of T_h^2 has dimension $\lceil (\log_3 2)(h+1) \rceil$, for $h \geq 2$. The upper bound follows by Theorem 3.1. Note that the lower bound is approximately $\lceil (0.6309)(h+1) \rceil$ and the upper bound is approximately $\lceil (0.6667)(h+1) \rceil$. In the following, T_h^2 and its root r_h^2 are denoted by T_h and r_h, respectively.

Theorem 3.1 *For every integer $h \geq 0$, $T_h \subseteq H(3,n)$, where $n = n(3,2,h) = \lceil \frac{2}{3}(h+1) \rceil$.*

Proof: As above, we embed $\mathbb{T}_h$ into $H(3,n)$ by induction on $h \pmod 3$. That is, for the basic step, we construct the embeddings of $\mathbb{T}_0$ into $H(3,1)$, and $\mathbb{T}_1$ and $\mathbb{T}_2$ into $H(3,2)$. For the inductive step, given an embedding of $\mathbb{T}_h$ into $H(3,n)$, we describe an embedding of $\mathbb{T}_{h+3}$ into $H(3,n+2)$.

For the embeddings of $\mathbb{T}_0$ into $H(3,1)$, and $\mathbb{T}_1$ and $\mathbb{T}_2$ into $H(3,2)$ see Figure 3. In the figure, the embedding is shown by labeling the vertices of the tree using the vertex labels of the corresponding ternary cube. For the inductive step, we assume that $\mathbb{T}_h \subseteq H(3,n)$ with its auxiliary path $\langle r_h, x, y \rangle$ mapped onto a path $\langle R_h, X, Y \rangle$ in $H(3,n)$. We embed $\mathbb{T}_{h+3}$ in $H(3,n+2)$ in three steps. Since, $n(3,2,h+3) = \lceil \frac{2}{3}(h+4) \rceil = \lceil \frac{2}{3}(h+1) \rceil + 2 = n(3,2,h)+2$, it follows by induction that $\mathbb{T}_h \subseteq H(3,n)$, for all $h \geq 0$, where $n = \lceil \frac{2}{3}(h+1) \rceil$.



(a) $\mathbb{T}_0 \subseteq H(3,1)$ (b) $\mathbb{T}_1 \subseteq H(3,2)$ (c) $\mathbb{T}_2 \subseteq H(3,2)$

Figure 3: Embeddings of $\mathbb{T}_0$ into $H(3,1)$, and $\mathbb{T}_1, \mathbb{T}_2$ into $H(3,2)$.

Step 1: We first decompose $H(3,n+2)$ into nine copies of $H(3,n)$ denoted by $ijH(3,n)$, where $i,j \in \{0,1,2\}$; see Figure 4.

Step 2: By the induction hypothesis, we have an embedding of $\mathbb{T}_h$ into $ijH(3,n)$, for $i,j \in \{0,1,2\}$ with its auxiliary path mapped onto $\langle ijR_h, ijX, ijY \rangle$. For $ij \in \{01, 02, 10, 11\}$, we apply an automorphism α on $V(ijH(3,n))$ such that $\alpha(ijR_h) = ijX$ and $\alpha(ijX) = ijY$; Thus, after applying the automorphism, we have an embedding of $\mathbb{T}_h$ in $ijH(3,n)$ with its auxiliary path mapped onto $\langle ijX, ijY, ijZ \rangle$ in $ijH(3,n)$, where Z is clearly a vertex adjacent to Y in $V(H(3,n))$. For $ij \in \{21, 22\}$, we apply an automorphism β on $V(ijH(3,n))$, such that $\beta(ijR_h) = ijY$, to obtain an embedding of $\mathbb{T}_h$ in $ijH(3,n)$ with the root of T_h mapped onto ijY. These embeddings are shown in Figure 4.

Step 3: Combine all the above nine embeddings by adding the nine edges $(00X, 01X), (00Y, 02Y), (00Y, 10Y), (02Y, 22Y), (10Y, 12Y), (10Y, 20Y), (10X, 12X), (11Y, 12Y), (11Y, 21Y)$ of $H(3,n+2)$. This results in an embedding of $\mathbb{T}_{h+3}$ in $H(3,n+2)$ with its auxiliary path mapped onto $\langle 10Y, 20Y, 20X \rangle$; see Figure 5. $\square$

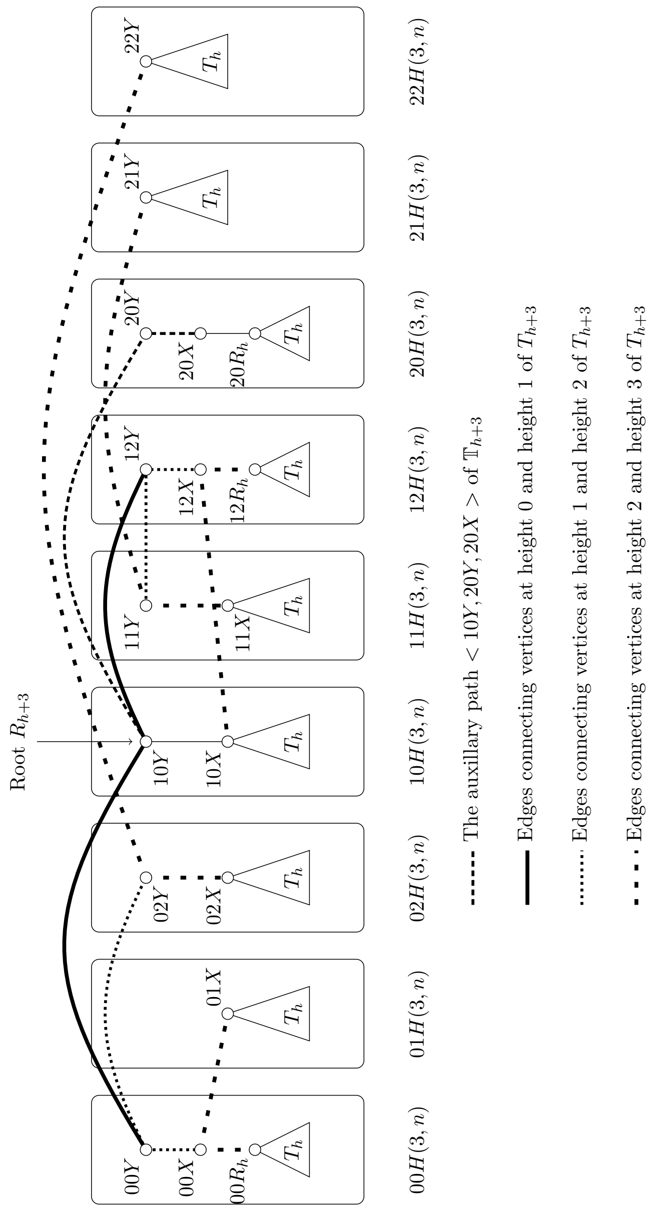


Figure 4: Embedding of T_{h+3} into $H(3, n+2) = H(3, n) \,\square\, H(3, 2)$

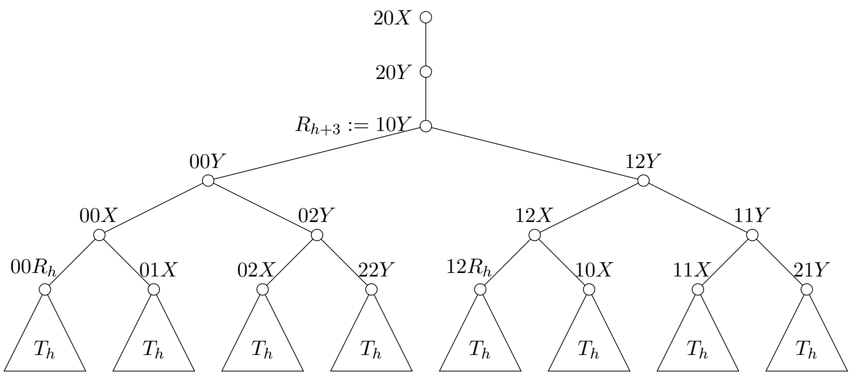


Figure 5: Embedding of T_{h+3} into $H(3, n+2)$ given an embedding of T_h into $H(3, n)$

4 Complete binary trees in $H(b, n)$

In this section, we show that (i) $n(b, 2, h) \leq \left\lceil \frac{h+1}{\lfloor \log_2 b \rfloor} \right\rceil$, when b is not a power of 2, and that (ii) $n(b, 2, h) \leq \left\lceil \frac{h+2}{\log_2 b} \right\rceil$, when b is a power of 2. The lower bound is given by $n(b, 2, h) \geq \left\lceil \frac{h+1}{\log_2 b} \right\rceil$, since the optimal Hamming graph $H(b, n)$ of T_h has dimension $\left\lceil \frac{h+1}{\log_2 b} \right\rceil$.

Theorem 4.1 *Let b be a positive integer. For every $n \geq 1$, $T_h \subseteq H(b, n)$, where*

$$h = \begin{cases} n\lfloor \log_2 b \rfloor - 1, & \text{if } b \text{ is not a power of 2,} \\ n \log_2 b - 2, & \text{if } b \text{ is a power of 2.} \end{cases}$$

Proof: Assume that b is not a power of 2; when b is a power of 2, the proof follows similarly. Let l be the largest integer such that $2^l \leq b$; so $l = \lfloor \log_2 b \rfloor$ and $h = nl - 1$. We embed $\mathbb{T}_{nl-1}$ in $H(b, n)$ by induction on n. For the basic case of $n = 1$, it can be easily verified that $\mathbb{T}_{l-1} \subseteq H(b, 1) = K_b$. For the inductive step, we assume that $\mathbb{T}_{nl-1} \subseteq H(b, n)$, with its auxiliary path mapped onto a path $\langle R_h, X, Y \rangle$ in $H(b, n)$. We embed $\mathbb{T}_{(n+1)l-1}$ in $H(b, n+1)$ in four steps.

Step 1: We first decompose $H(b, n+1)$, into b copies of $H(b, n)$ denoted by $iH(b, n)$, for $i = 0, 1, \ldots, b-1$.

Step 2: By induction hypothesis, we have an embedding of $\mathbb{T}_h$ into $iH(b, n)$, $i = 0, 1, \ldots, b-1$, with the auxiliary path mapped onto $\langle iR_h, iX, iY \rangle$ in $iH(b, n)$. For $p = 4i+1, 4i+2$, $0 \leq i \leq 2^{l-2} - 1$, we apply an automorphism α on $pH(b, n)$ such that $\alpha(pR_h) = pX$ and $\alpha(pX) = pY$; we denote $\alpha(pY)$ by pZ, where Z is clearly some vertex adjacent with Y in $H(b,n)$. After applying α, we have an embedding of $\mathbb{T}_h$ in $pH(b,n)$ with its auxiliary path mapped onto $\langle pX, pY, pZ\rangle$. For $r = 4i+3$, $0 \le i \le 2^{l-2}-1$, we apply another automorphism β such that $\beta(rR_h) = rY$; β embeds T_h in $rH(b,n)$ with its root mapped onto rY.

Step 3: For $0 \le i < j \le b-1$, let $W(i,j) = [V(iH(b,n)) \cup V((i+1)H(b,n)) \cup \cdots \cup V(jH(b,n))]$ be the induced subgraph of $H(b,n+1)$. If $l = 1$, we obtain an embedding of $\mathbb{T}_{h+1}$ in $W(0,1)$ by adding the edges $(0X,1X)$ and $(0Y,1Y)$. Its auxiliary path is $\langle 0X, 0Y, 1Y\rangle$; see the first two boxes in Figure 6. If $l = 2$, we obtain an embedding of $\mathbb{T}_{h+2}$ in $W(0,3)$ by adding the edges $(0X,1X)$, $(0Y,1Y)$, $(0Y,2Y)$ and $(2Y,3Y)$ as shown in Figure 6. In this embedding, the auxiliary path of $\mathbb{T}_{h+2}$ is mapped onto $\langle 0Y, 1Y, 1Z\rangle$.

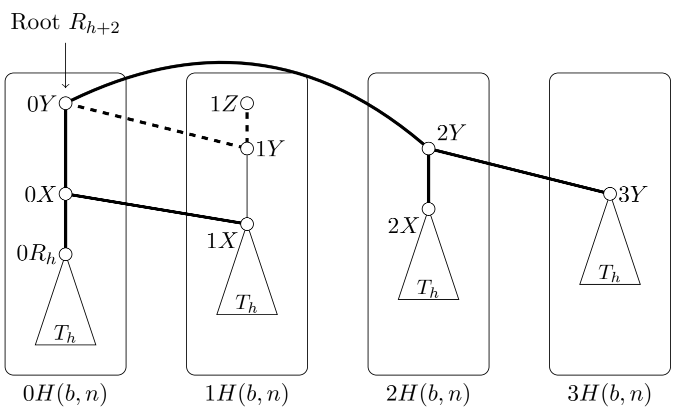


Figure 6: Construction of $\mathbb{T}_{h+2}$ from $\mathbb{T}_h$ using four copies of $H(b,n)$

Step 4: If $l \ge 3$, we embed $\mathbb{T}_{h+3}$ in $W(0,7)$ as follows. Given the embedding of $\mathbb{T}_{h+2}$ in $W(0,3)$ (and $W(4,7)$) with its auxiliary path mapped onto the path $\langle 0Y, 1Y, 1Z\rangle$ (respectively $\langle 4Y, 5Y, 5Z\rangle$ in $W(4,7)$), we obtain an embedding of $\mathbb{T}_{h+3}$ in $W(0,7)$ by adding the edges $(0Y,5Y)$ and $(1Y,5Y)$. Its auxiliary path is mapped onto the path $\langle 5Y, 1Y, 1Z\rangle$; see Figure 7.

For $3 \le i < l$, given an embedding of $\mathbb{T}_{h+i}$ in $W(0, 2^i - 1)$ with its auxiliary path mapped onto the path $\langle (2^{i-1}+1)Y, 1Y, 1Z\rangle$, we obtain the embedding $\mathbb{T}_{h+i+1}$ in $W(0, 2^{i+1}-1)$ by adding the edges $((2^{i-1}+1)Y, (2^i+1)Y)$ and $(1Y, (2^i+1)Y)$. Its auxiliary path is $\langle (2^i+1)Y, 1Y, 1Z\rangle$; see Figure 8.

It now follows by recursion that $\mathbb{T}_{h+l} \subseteq W(0, 2^l - 1) \subseteq H(b, n+1)$, with its auxiliary path mapped onto the path $\langle (2^{l-1}+1)Y, 1Y, 1Z\rangle$. □

Corollary 1 *Let b be a positive integer. For every integer $h \ge 0$, $T_h \subseteq H(b,n)$ where*

$$n = \begin{cases} \left\lceil \frac{h+1}{\lfloor \log_2 b \rfloor} \right\rceil, & \text{if } b \text{ is not a power of } 2, \\ \left\lceil \frac{h+2}{\log_2 b} \right\rceil, & \text{if } b \text{ is a power of } 2. \end{cases}$$

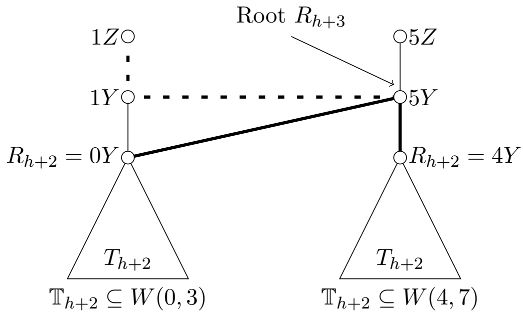


Figure 7: Embedding of $\mathbb{T}_{h+3}$ in $W(0,7)$ from two copies of $\mathbb{T}_{h+2}$

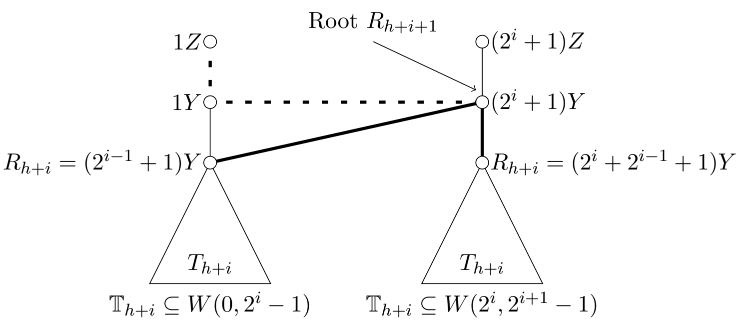


Figure 8: Embedding of $\mathbb{T}_{h+i+1}$ in $W(0,2^{i+1}-1)$ from two copies of $\mathbb{T}_{h+i}$

Proof: When b is not a power of 2, we have:

$$\begin{aligned} T_h &\subseteq T_{n\lfloor \log_2 b\rfloor -1} \quad &&(\text{since } h \le n\lfloor \log_2 b\rfloor - 1)\\ &\subseteq H(b,n) &&(\text{by Theorem 4.1}). \end{aligned}$$

When b is a power of 2, the proof follows similarly. □

Remark: When b is a power of 2, $n(b,2,h) = \left\lceil \frac{h+2}{\log_2 b} \right\rceil$ is the dimension of the optimal or the next-to-optimal Hamming graph $H(b,n)$ of T_h. In particular, $n(b,2,h)$ is the dimension of the next-to-optimal Hamming graph $H(b,n)$ of T_h, when $h = n\log_2 b - 1$, for all $n \ge 1$.

Acknowledgements

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