AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **45** (2009), Pages 207–215

On regular $(1, q)$-extendable graphs

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Abstract

Let G be a graph with a maximum matching of size q, and let $p \leq q$ be a positive integer. Then G is called (p, q)*-extendable* if every set of p independent edges can be extended to a matching of size q. If G is a graph of even order n and $2q = n$, then (p, q)-extendable graphs are exactly the p-extendable graphs defined by Plummer [*Discrete Math.* 31 (1980), 201–210].

Let $d \geq 3$ be an integer, and let G be a d-regular graph of even order n with a maximum matching of size $q = \frac{n-2t}{2}$ for an integer $t \geq 0$. In this work we prove that if

(i) $n \leq (2t + 2)(d + 1)$ or

(ii) $n \leq (2t + 2)(d + 2) - 2$ when d is odd or

(iii) $n \leq (2t + 3)(d + 1) + 1$ when $d \leq 2t + 2$ is even and G is connected or

(iv) $n \leq (2t + 2)(d + 2)$ when $d \leq 2t + 1$ is odd,

then G is $(1, q)$-extendable. Examples show that the upper bounds (i) – (iv) on n are best possible. In addition, we present an analogue for regular odd order graphs.

We shall assume that the reader is familiar with standard terminology on graphs. In this paper, all graphs are finite and simple. The vertex set of a graph G is denoted by $V(G)$. If A is a subset of the vertex set of a graph G, then $G[A]$ is the subgraph induced by A. We denote by K_n the *complete graph* of order n and by $K_{r,s}$ the *complete bipartite graph* with partite sets A and B, where $|A| = r$ and $|B| = s$. If G is a graph and $A \subseteq V(G)$, then $o(G - A)$ is the number of odd components in the subgraph $G - A$.

A graph G is *p-extendable* if it contains a set of p independent edges and every set of p independent edges can be extended to a perfect matching. In 1980, Plummer [6] studied the properties of p-extendable graphs. As an extension of p-extendable graphs, Liu and Yu [5] defined (p, q)-extendable graphs as follows. Let G be a graph with a maximum matching of size q, and let $p \leq q$ be a positive integer. Then G is called (p, q)-*extendable* if every set of p independent edges can be extended to a matching of size q. If G is a graph of even order n and $2q = n$, then (p, q)-extendable graphs are exactly the p-extendable graphs defined by Plummer [6]. Examples of (p, q)-extendable graphs are complete bipartite graphs $K_{q,r}$ with $r \geq q$.

In 2001, Liu and Yu [5] have given a characterization of (p, q)-extendable graphs, which generalizes one by Little, Grant and Holton [3] and Yu [8] for p-extendable graphs. The proof is based on an extension of Tutte's famous 1-factor Theorem [7] by Berge [1]. For our proofs we mainly use the following special case of this characterization for $p = 1$.

Theorem 1 (Liu and Yu [5] 2001) *Let q and n be positive integers such that $1 < q \leq \frac{n}{2}$. A graph G of order n with a maximum matching of size q is $(1, q)$-extendable if and only if for any subset $A \subseteq V(G)$*

(1) $o(G - A) \leq |A| + n - 2q$ and

(2) $o(G - A) = |A| + n - 2q$ implies that $G[A]$ is the empty graph.

Theorem 2 *Let $d \geq 3$ be an integer, and let G be a d-regular graph of even order n with a maximum matching of size $q = \frac{n-2t}{2}$ for an integer $t \geq 0$. If*

(i) $n \leq (2t + 2)(d + 1)$ or

(ii) $n \leq (2t + 2)(d + 2) - 2$ when d is odd or

(iii) $n \leq (2t + 3)(d + 1) + 1$ when $d \leq 2t + 2$ is even and G is connected or

(iv) $n \leq (2t + 2)(d + 2)$ when $d \leq 2t + 1$ is odd,

then G is $(1, q)$-extendable.

Proof. Suppose to the contrary that G is not $(1, q)$-extendable. Then it follows from the hypothesis and Theorem 1 that there exists a non-empty set $A \subseteq V(G)$ such that $o(G - A) \geq |A| + 2t + 1$ or $o(G - A) = |A| + 2t$ and $G[A]$ contains an edge.

We call an odd component of $G - A$ large if it has more than d vertices and small otherwise. We denote by α and β the number of large and small components of $G - A$, respectively. Since G is a d-regular graph, it is easy to see that there are at least d edges in G joining each small component of $G - A$ with A. The d-regularity of G therefore implies

$$d\beta \leq d|A|. \tag{1}$$

Case 1. Assume that $o(G-A) \geq |A|+2t+1$. Since n is even, the numbers $o(G-A)$ and $|A|$ are of the same parity, and we deduce that

$$\alpha+\beta = o(G-A) \geq |A|+2t+2. \tag{2}$$

Inequality (1) shows that $\beta \leq |A|$ and thus (2) yields $\alpha \geq 2t+2$. Applying the hypothesis (i) that $n \leq (2t+2)(d+1)$ and using the fact that $A \neq \emptyset$, we arrive at the contradiction

$$\begin{aligned}(2t+2)(d+1) \geq n &\geq |A|+\alpha(d+1)+\beta \\ &\geq |A|+(2t+2)(d+1) \\ &> (2t+2)(d+1).\end{aligned}$$

If d is odd, then each large component contains at least $d+2$ vertices. Now the hypotheses (ii) or (iv) analogously lead to a contradiction.

If G is connected and d is even, then there are at least two edges in G joining each large component of $G-A$ with A, and hence we conclude that

$$2\alpha+d\beta \leq d|A|. \tag{3}$$

This inequality implies $\beta \leq |A|-1$, and thus it follows from (2) that $\alpha \geq 2t+3$. If $\alpha \geq 2t+4$, then the hypothesis (iii) leads to the contradiction

$$(2t+3)(d+1)+1 \geq |A|+(2t+4)(d+1).$$

In the remaining case $\alpha = 2t+3$, we deduce from (iii) that

$$(2t+3)(d+1)+1 \geq |A|+(2t+3)(d+1)$$

and thus $|A|=1$. Using (3), we obtain $2(2t+3) = 2\alpha \leq d$, a contradiction to the hypothesis $d \leq 2t+2$.

Case 2: Assume that $\alpha+\beta = o(G-A) = |A|+2t$ and $G[A]$ contains an edge. This implies that $|A| \geq 2$, and because of $\beta \leq |A|$, we obtain $\alpha \geq 2t$. Since G has a maximum matching of size $q = \frac{n-2t}{2}$, the graph G has at most $2t$ odd components, and we conclude that

$$\alpha - 2t + d\beta \leq d|A| - 2. \tag{4}$$

If G is connected and d is even, then we have the inequality

$$2\alpha+d\beta \leq d|A|-2. \tag{5}$$

Subcase 2.1: Assume that $\alpha \geq 2t+2$. Then (i) leads to the contradiction

$$(2t+2)(d+1) \geq n \geq |A|+(2t+2)(d+1)+\beta,$$

and (ii) or (iv) lead to the contradiction

$$(2t+2)(d+2) \geq n \geq |A|+(2t+2)(d+2)+\beta.$$

Now let G be connected and $d \leq 2t+2$ even. If $\alpha \geq 2t+3$, then (iii) yields the contradiction

$$\begin{aligned}(2t+3)(d+1)+1 \geq n &\geq |A|+(2t+3)(d+1)+\beta \\ &\geq (2t+3)(d+1)+2.\end{aligned}$$

If $\alpha = 2t+2$, then $\beta = |A|-2$. Using (5), we obtain $d \geq 2t+3$, a contradiction to the hypothesis $d \leq 2t+2$.

Subcase 2.2: Assume that $\alpha = 2t$. We deduce that $\beta = |A|$, a contradiction to (4).

Subcase 2.3: Assume that $\alpha = 2t+1$. We deduce that $\beta = |A|-1$.

If G is connected and $d \leq 2t+2$ is even, then (5) leads to a contradiction, and (iii) is proved.

If $d \leq 2t+1$ is odd, then there exists at least one edge joining every large component of $G-A$ with A and hence $\alpha + d\beta = \alpha + d(|A|-1) \leq d|A|-2$. This yields the contradiction $2t+1-d \leq -2$, and (iv) is proved.

Since $\beta = |A|-1$, there exists at least one small component in $G-A$. If U is a small component of minimum order in $G-A$, then we observe that

$$|V(U)| \geq d-|A|+1. \qquad (6)$$

Subcase 2.3.1: Assume that $|A| \geq d$. Then the hypothesis (i) leads to

$$\begin{aligned}(2t+2)(d+1) &\geq n \geq |A|+(2t+1)(d+1)+\beta \\ &= 2|A|-1+(2t+1)(d+1) \\ &\geq 2d-1+(2t+1)(d+1),\end{aligned}$$

a contradiction to $d \geq 3$. If $d \geq 3$ is odd, then (ii) yields the contradiction

$$\begin{aligned}(2t+2)(d+2)-2 &\geq n \geq |A|+(2t+1)(d+2)+\beta \\ &= 2|A|-1+(2t+1)(d+2) \\ &\geq 2d-1+(2t+1)(d+2).\end{aligned}$$

Subcase 2.3.2: Assume that $3 \leq |A| \leq d-1$. Applying (i) and (6), we arrive at the contradiction

$$\begin{aligned}(2t+2)(d+1) \geq n &\geq |A|+(2t+1)(d+1)+(|A|-1)|V(U)| \\ &\geq |A|+(2t+1)(d+1)+2(d-|A|+1) \\ &\geq |A|+(2t+1)(d+1)+(d-|A|+1)+2 \\ &= (2t+2)(d+1)+2.\end{aligned}$$

If $d \geq 3$ is odd, then the hypothesis (ii) and (6) lead to the contradiction

$$\begin{aligned}(2t+2)(d+2)-2 \geq n &\geq |A|+(2t+1)(d+2)+(|A|-1)|V(U)| \\ &\geq |A|+(2t+1)(d+2)+2(d-|A|+1) \\ &\geq |A|+(2t+1)(d+2)+(d-|A|+1)+2 \\ &= (2t+2)(d+2)+1.\end{aligned}$$

Subcase 2.3.3: Assume that $|A| = 2$. If $d \geq 3$ is odd, then the hypothesis (i) or (ii) and (6) yield the contradiction

$$\begin{aligned}(2t+2)(d+2)-2 \geq n &\geq |A| + (2t+1)(d+2) + |V(U)| \\ &\geq |A| + (2t+1)(d+2) + (d - |A| + 1) \\ &= (2t+2)(d+2) - 1.\end{aligned}$$

If d is even, then it follows from (6) that $|V(U)| \geq d - |A| + 1 = d - 1$ and thus $|V(U)| = d - 1$. Hence there are at least $2(d-1)$ edges in G joining U with A and at least one edge in G joining one large component of $G - A$ with A. Since the subgraph $G[A]$ contains also an edge, there exists at least one vertex in A of deree greater than d, a contradiction to the d-regularity of G. This completes the proof of Theorem 2. □

The following examples will demonstrate that the bounds in Theorem 2 are best possible.

Example 3 Let $d \geq 4$ be an even integer. Let H be a complete graph K_2 with vertex set $\{u, v\}$, let H_1 be a complete graph K_{d+1} with vertex set $\{x_1, x_2, \ldots, x_{d+1}\}$ without the matching $\{x_1x_2, x_3x_4, \ldots, x_{d-1}x_d\}$, and let H_2 be a complete graph K_{d+1} with vertex set $\{y_1, y_2, \ldots, y_{d+1}\}$ without the matching $\{y_1y_2, y_3y_4, \ldots, y_{d-3}y_{d-2}\}$. In addition, let $H_3, H_4, \ldots, H_{2t+2}$ be complete graphs K_{d+1}. We define the graph G of order $n = (2t+2)(d+1) + 2$ as the disjoint union of $H, H_1, H_2, \ldots, H_{2t+2}$ together with the edges ux_i for $1 \leq i \leq d-1$, vx_d and vy_j for $1 \leq j \leq d-2$. The resulting graph G is d-regular, and its maximum matching is of size $q = \frac{n-2t}{2}$. However, the edge uv is not contained in any matching of size q. This example shows that Theorem 2 (i) is best possible when $d \geq 4$ is even.

Example 4 Let $t \geq 0$ an integer, and let $d \geq 2t+3$ be an odd integer. Let H be a complete graph K_2 with vertex set $\{u, v\}$, let H_1 be a complete graph K_d with vertex set $\{x_1, x_2, \ldots, x_d\}$, let H_2 be graph of order $d+2$ with $d - (2t+2)$ vertices $y_1, y_2, \ldots, y_{d-(2t+2)}$ of degree $d-1$ and the remaining vertices of degree d, and let $H_3, H_4, \ldots, H_{2t+2}$ be graphs of order $d+2$ with exactly one vertex w_i of degree $d-1$ for $3 \leq i \leq 2t+2$ and the remaining vertices of degree d. We define the graph G of order $n = (2t+2)(d+2)$ as the disjoint union of $H, H_1, H_2, \ldots, H_{2t+2}$ together with the edges ux_i for $1 \leq i \leq d-1$, vx_d, vy_j for $1 \leq j \leq d-(2t+2)$ and vw_k for $3 \leq k \leq 2t+2$. The resulting graph G is d-regular and its maximum matching is of size $q = \frac{n-2t}{2}$. However, the edge uv is not contained in any matching of size q. This example shows that Theorem 2 (ii) is best possible when $d \geq 2t+3$ is odd.

Example 5 Let $t \geq 1$ be an integer, and let $d = 2t+2$. Let H consist of the vertex set $\{u, v, w\}$ and the edge uv. Let $H_1, H_2, \ldots, H_{t-1}$ be complete graphs K_{d+1} without the edges a_ib_i and x_iy_i for $1 \leq i \leq t-1$ and $t \geq 2$, where a_i, b_i, x_i and y_i are distinct vertices contained in H_i. In addition, let $H_t, H_{t+1}, \ldots, H_{2t+3}$ be complete graphs K_{d+1} without the edges a_ib_i for $t \leq i \leq 2t+3$, where a_i and b_i are distinct vertices contained in H_i. We define the graph G of order $n = (2t+3)(d+1)+3$ as the disjoint union of $H, H_1, H_2, \ldots, H_{2t+3}$ together with the edges

ua_i for $1 \leq i \leq t+1$ and ub_i for $1 \leq i \leq t$ and

vx_i for $1 \leq i \leq t-1$, vy_i for $1 \leq i \leq t-1$, va_{t+2}, vb_{t+2} and va_{t+3} and

wb_{t+1}, wb_i for $t+3 \leq i \leq 2t+3$ and wa_i for $t+4 \leq i \leq 2t+3$.

The resulting graph G is connected, d-regular, and its maximum matching is of size $q = \frac{n-2t}{2}$. However, the edge uv is not contained in any matching of size q. This example shows that Theorem 2 (iii) is best possible when G is connected and $d = 2t+2$.

Example 6 Let $t \geq 1$ be an integer, and let $d = 2t+1$. Let H be a complete graph K_2 with vertex set $\{u, v\}$, let H_1 be a graph of order $d+2$ with $d-2$ vertices $x_1, x_2, \ldots, x_{d-2}$ of degree $d-1$ and four vertices of degree d, and let $H_2, H_3, \ldots, H_{2t+2}$ be graphs of order $d+2$ with exactly one vertex y_i of degree $d-1$ for $2 \leq i \leq 2t+2$ and the remaining vertices of degree d. We define the graph G of order $n = (2t+2)(d+2)+2$ as the disjoint union of $H, H_1, H_2, \ldots, H_{2t+2}$ together with the edges ux_i for $1 \leq i \leq d-2$, uy_2 and vy_j for $3 \leq j \leq 2t+2$. The resulting graph G is d-regular and its maximum matching is of size $q = \frac{n-2t}{2}$. However, the edge uv is not contained in any matching of size q. This example shows that Theorem 2 (iv) is best possible when $d = 2t+1$.

Theorem 7 *Let $d \geq 4$ be an even integer, and let G be a d-regular graph of odd order n with a maximum matching of size $q = \frac{n-(2t+1)}{2}$ for an integer $t \geq 0$. If*

(a) $n \leq (2t+3)(d+1)$ or

(b) $n \leq (2t+4)(d+1)+1$ when $d \leq 2t+2$ and G is connected,

then G is $(1, q)$-extendable.

Proof. Suppose to the contrary that G is not $(1, q)$-extendable. Then it follows from the hypothesis and Theorem 1 that there exists a non-empty set $A \subseteq V(G)$ such that $o(G-A) \geq |A|+2t+2$ or $o(G-A) = |A|+2t+1$ and $G[A]$ contains an edge.

We call an odd component of $G-A$ large if it has more than d vertices and small otherwise. We denote by α and β the number of large and small components of $G-A$, respectively. As in the proof of Theorem 2, this implies

$$d\beta \leq d|A|. \tag{7}$$

Case 1. Assume that $o(G-A) \geq |A|+2t+2$. Since n is odd, the numbers $o(G-A)$ and $|A|$ are of different parity, and we deduce that

$$\alpha + \beta = o(G-A) \geq |A|+2t+3. \tag{8}$$

Inequality (7) implies $\beta \leq |A|$ and thus (8) yields $\alpha \geq 2t{+}3$. Applying the hypothesis (a) that $n \leq (2t{+}3)(d{+}1)$, and using the fact that $A \neq \emptyset$, we obtain the contradiction

$$(2t+3)(d+1) \geq n \geq |A| + \alpha(d+1) + \beta > (2t+3)(d+1).$$

If G is connected, then there are at least two edges in G joining each large component of $G - A$ with A, and hence we conclude that

$$2\alpha + d\beta \leq d|A|. \tag{9}$$

Because of this inequality, we observe that $\beta \leq |A| - 1$, and hence it follows from (8) that $\alpha \geq 2t + 4$. Now the hypothesis (b) leads to

$$(2t+4)(d+1) + 1 \geq n \geq |A| + (2t+4)(d+1)$$

and thus $|A| = 1$. Using the bound (9), we arrive at a contradiction to the condition $d \leq 2t + 2$.

Case 2: Assume that $\alpha + \beta = o(G - A) = |A| + 2t + 1$ and $G[A]$ contains an edge. This implies that $|A| \geq 2$, and because of $\beta \leq |A|$, we obtain $\alpha \geq 2t{+}1$. Since G has a maximum matching of size $q = \frac{n-(2t+1)}{2}$, the graph G has at most $2t{+}1$ odd components, and we conclude that

$$\alpha - 2t - 1 + d\beta \leq d|A| - 2. \tag{10}$$

If G is connected, then we have the inequality

$$2\alpha + d\beta \leq d|A| - 2. \tag{11}$$

Subcase 2.1: Assume that $\alpha \geq 2t + 3$. Then (a) leads to the contradiction

$$(2t+3)(d+1) \geq n \geq |A| + \alpha(d+1) + \beta \geq 2 + (2t+3)(d+1).$$

Now let G be connected and $d \leq 2t + 2$. If $\alpha \geq 2t + 4$, then (b) implies the contradiction

$$(2t+4)(d+1) + 1 \geq n \geq |A| + (2t+4)(d+1) + \beta \geq (2t+4)(d+1) + 2.$$

If $\alpha = 2t + 3$, then $\beta = |A| - 2$. Applying (11), we deduce that $d \geq 2t + 4$, a contradiction to $d \leq 2t + 2$.

Subcase 2.2: Assume that $\alpha = 2t + 1$. This yields $\beta = |A|$, a contradiction to (10).

Subcase 2.3: Assume that $\alpha = 2t + 2$. It follows that $\beta = |A| - 1$.

If G is connected and $d \leq 2t + 2$, then (11) leads to a contradiction, and (b) is proved.

Since $\beta = |A| - 1$, there exists at least one small component in $G - A$. If U is a small component of minimum order in $G - A$, then

$$|V(U)| \geq d - |A| + 1. \tag{12}$$

Subcase 2.3.1: Assume that $|A| \geq d+1$. Then the hypothesis (a) leads to

$$\begin{aligned}(2t+3)(d+1) \geq n &\geq |A| + (2t+2)(d+1) + \beta \\ &\geq d+1+(2t+2)(d+1)+1 \\ &= (2t+3)(d+1)+1.\end{aligned}$$

Subcase 2.3.2: Assume that $3 \leq |A| \leq d$. Using (a) and (12), we obtain the contradiction

$$\begin{aligned}(2t+3)(d+1) \geq n &\geq |A| + (2t+2)(d+1) + (|A|-1)|V(U)| \\ &\geq |A| + (2t+2)(d+1) + 2(d-|A|+1) \\ &\geq |A| + (2t+2)(d+1) + (d-|A|+1)+1 \\ &= (2t+3)(d+1)+1.\end{aligned}$$

Subcase 2.3.3: Assume that $|A| = 2$. It follows from (12) that $|V(U)| \geq d - |A| + 1 = d-1$ and thus $|V(U)| = d-1$. Hence there are at least $2(d-1)$ edges in G joining U with A and at least one edge in G joining one large component of $G-A$ with A. Since the subgraph $G[A]$ contains also an edge, there exists at least one vertex in A of deree greater than d, a contradiction to the d-regularity of G. This completes the proof of Theorem 7. □

The next two examples will show that the bounds in Theorem 7 are best possible.

Example 8 Let $d \geq 4$ be an even integer. Let H be a complete graph K_2 with vertex set $\{u, v\}$, let H_1 be a complete graph K_{d+1} with vertex set $\{x_1, x_2, \ldots, x_{d+1}\}$ without the matching $\{x_1x_2, x_3x_4, \ldots, x_{d-1}x_d\}$, and let H_2 be a complete graph K_{d+1} with vertex set $\{y_1, y_2, \ldots, y_{d+1}\}$ without the matching $\{y_1y_2, y_3y_4, \ldots, y_{d-3}y_{d-2}\}$. In addition, let $H_3, H_4, \ldots, H_{2t+3}$ be complete graphs K_{d+1}. We define the graph G of order $n = (2t+3)(d+1)+2$ as the disjoint union of $H, H_1, H_2, \ldots, H_{2t+3}$ together with the edges ux_i for $1 \leq i \leq d-1$, vx_d and vy_j for $1 \leq j \leq d-2$. The resulting graph G is d-regular, and its maximum matching is of size $q = \frac{n-(2t+1)}{2}$. However, the edge uv is not contained in any matching of size q. This example shows that Theorem 7 (a) is best possible.

Example 9 Let $t \geq 0$ be an integer, and let $d = 2t+4$. Let H be a complete graph K_2 with vertex set $\{u, v\}$. In addition, let $H_1, H_2, \ldots, H_{2t+3}$ be complete graphs K_{d+1} without the edges x_iy_i for $1 \leq i \leq 2t+3$, where x_i and y_i are distinct vertices contained in H_i. We define the graph G of order

$$n = (2t+3)(d+1)+2 \leq (2t+4)(d+1)+1$$

as the disjoint union of $H, H_1, H_2, \ldots, H_{2t+3}$ together with the edges ux_i and vy_i for $1 \leq i \leq 2t+3$. The resulting graph G is connected, d-regular, and its maximum matching is of size $q = \frac{n-(2t+1)}{2}$. However, the edge uv is not contained in any matching of size q. This example shows that Theorem 7 (b) is not valid when G is connected and $d = 2t+4$.

Remark 10 If G is a 2-regular graph of order n with a maximum matching of size $q = \frac{n-s}{2}$ for an integer $s \geq 0$ of the same parity as n, then G contains exactly s odd cycles, and the remaining components of G are even cycles. Now it is evident that such a 2-regular graph is always $(1, q)$-extendable.

Remark 11 If q is the size of a maximum matching in a d-regular graph of order n with $d \geq 3$, then Henning and Yeo [2] have recently proved that

$$q \geq \min\left\{\left(\frac{d^2+4}{d^2+d+2}\right) \times \frac{n}{2}, \frac{n-1}{2}\right\} \text{ when } d \text{ is even}$$

and

$$q \geq \frac{(d^3-d^2-2)n-2d+2}{2(d^3-3d)} \text{ when } d \text{ is odd}$$

In the papers by Yu [8] and Liu and Yu [4] one can find other extensions of p-extendability, which are stronger and which are only defined for graphs with a perfect or almost perfect matching.

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(Received 4 Sep 2008; revised 10 Mar 2009)