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$\mathbb{Z}_k$-magic labelings of fans and wheels with magic-value zero

W.C. Shiu

Department of Mathematics
Hong Kong Baptist University
224 Waterloo Road, Kowloon Tong
Hong Kong
wcshiu@hkbu.edu.hk

Richard M. Low

Department of Mathematics
San Jose State University
San Jose, CA 95192
U.S.A.
low@math.sjsu.edu

Abstract

Let A be a non-trivial abelian group and $A^* = A - \{0\}$. A graph is A-magic if there exists an edge labeling using elements of A^* which induces a constant vertex labeling of the graph. For fan graphs F_n and wheel graphs W_n, we determine the sets of numbers $k \in \mathbb{N}$, where F_n and W_n have $\mathbb{Z}_k$-magic labelings with magic-value 0.

1 Introduction

Let $G = (V, E)$ be a connected simple graph. For any non-trivial abelian group A (written additively), let $A^* = A - \{0\}$. A function $f : E \to A^*$ is called a *labeling* of G. Any such labeling induces a map $f^+ : V \to A$, defined by $f^+(v) = \Sigma f(u, v)$, where the sum is over all $(u, v) \in E$. If there exists a labeling f whose induced map on V is a constant map, we say that f is an *A-magic labeling* of G and that G is an *A-magic* graph. The corresponding constant is called an *A-magic value*. The *integer-magic spectrum* of a graph G is the set $\mathrm{IM}(G) = \{k : G \text{ is } \mathbb{Z}_k\text{-magic and } k \geq 2\}$. Note that the integer-magic spectrum of a graph is not to be confused with the set of achievable magic values. $\mathbb{Z}$-magic (or $\mathbb{Z}_1$-magic) graphs were considered by Stanley [22, 23], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [1–3] and others [6, 8, 14, 15, 20] have studied A-magic graphs and $\mathbb{Z}_k$-magic graphs were investigated in [4, 5, 7, 9–13, 16–19, 21].

2 Null sets

With the purpose of constructing large classes of $\mathbb{Z}_k$-magic graphs, Salehi [17, 18] introduced the concept of a *null set* of a graph.

Definition. The **null set** of a graph G, denoted by $N(G)$, is the set of all numbers $k \in \mathbb{N}$, where G has a $\mathbb{Z}_k$-magic labeling with magic-value 0.

Among the various classes of graphs studied in [17], the null sets of two important classes of graphs were determined, namely $N(K_n)$ and $N(K_{m,n})$. Salehi concludes the paper with a few open problems for further consideration, one of which is to determine the null sets of wheel graphs and fan graphs. In this note, we provide an answer to this particular problem.

3 Preliminaries

In this section, definitions and notation are introduced. Known results which will be useful for determining the null sets of fans and wheels, will also be recalled.

Let $P_n = v_1 \cdots v_n$ be the path of order n. The *fan graph* $F_n = K_1 \vee P_n$ is the join of K_1 and P_n. The vertex of K_1 (in F_n) is called the *core* and will be denoted by c. The edges cv_i (where $1 \le i \le n$) in F_n are called *spokes*. Let C_n be the cycle of order n, for $n \ge 3$. The *wheel graph* $W_n = K_1 \vee C_n$ is the join of K_1 and C_n. Note that W_n can be obtained from F_{n+1} in the following way: in F_{n+1}, identify the vertex v_1 with v_{n+1} and identify the edge cv_1 with cv_{n+1}. In this case, the identified vertex in W_n will be denoted by v_1^*.

The following results from [15] will be very useful as we determine $N(F_n)$ and $N(W_n)$.

Theorem A. *A graph G is $\mathbb{Z}_2$-magic $\Longleftrightarrow$ every vertex of G is of the same parity.*

Theorem B. *Let $k \ge 2$ and suppose G has a $\mathbb{Z}_k$-magic labeling with magic value x. If $k|n$, then there exists a $\mathbb{Z}_n$-magic labeling of G with magic value $\frac{n}{k}x$.*

Theorem C. *Let $U(\mathbb{Z}_k)$ denote the multiplicative group of units of $\mathbb{Z}_k$. If $d \in U(\mathbb{Z}_k)$ and f is a $\mathbb{Z}_k$-magic labeling of graph G, then df is also a $\mathbb{Z}_k$-magic labeling of graph G.*

4 The null set of fan graphs

Theorem 1. *The null set of F_2 ($\cong C_3$) is $2\mathbb{N}$ and the null set of F_3 is $2\mathbb{N} \setminus \{2\}$.*

Proof. Let f be a $\mathbb{Z}_k$-magic labeling of $F_2 = C_3$, with magic value 0. Then, f labels two edges of F_2 with x and the third edge with $-x$, for some $x \in \mathbb{Z}_k^*$. Thus, $2x \equiv 0 \pmod{k}$. Since $x \not\equiv 0 \pmod{k}$, k must be even. Clearly, $2 \in N(F_2)$. Hence, $N(F_2) = 2\mathbb{N}$.

Let g be a $\mathbb{Z}_k$-magic labeling of F_3 with magic value 0. Then, F_3 must be labeled as shown in Figure 1, for some $x, z \in \mathbb{Z}_k^*$. Thus, we have $2z \equiv 0 \pmod{k}$. Since

$z \not\equiv 0 \pmod{k}$, k must be even. By Theorem A, $2 \notin N(F_3)$. If $k \geq 3$, then it is possible to choose z not equal to x. Hence, $N(F_3) = 2\mathbb{N} \setminus \{2\}$.

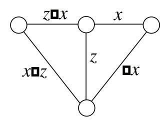

Figure 1: Potential $\mathbb{Z}_k$-magic labeling with magic value 0 of F_3.

□

Let A be an abelian group. Suppose $f : E(F_n) \to A^* = A - \{0\}$ is an edge labeling of F_n. Let $f(v_i v_{i+1}) = -a_i$ for $1 \leq i \leq n-1$ for some $a_i \in A^*$. See Figure 2. In order for f to be an A-magic labeling of F_n with magic value 0, the following necessary and sufficient condition must hold:

$$a_i \neq 0 \text{ for all } i,\ a_{i-1} + a_i \neq 0 \text{ for } 2 \leq i \leq n-1, \text{ and } 2\sum_{i=1}^{n-1} a_i = 0. \tag{4.1}$$

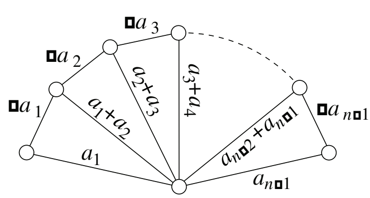

Figure 2: A-magic labeling of F_n.

Theorem 2. *Let $n \geq 4$. Then, $1 \in N(F_n)$.*

Proof. Consider the edge labeling f of F_n defined by $a_i = 1$ for $1 \leq i \leq n-3$ or $i = n-1$; and $a_{n-2} = 2 - n$, along with

$$a_{i-1} + a_i = \begin{cases} 3 - n, & i = n-2, n-1; \\ 2, & \text{otherwise (if any).} \end{cases}$$

Note that $2\sum_{i=1}^{n-1} a_i = 2[(n-3-1+1+1) + (2-n)] = 0$. So, all a_i's satisfy condition (4.1). Hence, f is a $\mathbb{Z}$-magic labeling with magic value 0. □

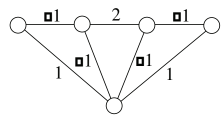

Figure 3: A $\mathbb{Z}$-magic labeling of F_4 with magic value 0.

When $n \geq 3$, $2 \notin N(F_n)$. For $k \geq 3$, $\mathbb{Z}_k$ contains a subgroup isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_p$, where p is an odd prime. Thus by Theorem B, we need to only consider $\mathbb{Z}_k$-magic labelings (with magic value 0) where $k = 4$ or k is an odd prime.

Theorem 3. *Let $n \geq 4$. Then, $3 \in N(F_n) \iff n \equiv 1 \pmod 3$.*

Proof. Let f be a $\mathbb{Z}_3$-magic labeling of F_n with magic value 0. By Theorem C, we may assume $f(cv_1) = -1$. Then $f(v_1v_2) = 1$. Since the magic value is 0, $f(cv_2) = f(v_2v_3) = 1$. By the same argument, we have $f(cv_i) = f(v_iv_{i+1}) = 1$ for $2 \leq i \leq n-1$. Hence $f(cv_n) = -1$. Then $0 \equiv f^+(c) = 2(-1)+(n-2) = n-4 \equiv n-1 \pmod 3$. Clearly, when $n \equiv 1 \pmod 3$, then above labeling is a $\mathbb{Z}_3$-magic labeling with magic value 0. □

Theorem 4. *Let $n \geq 4$ and $k = 4$ or $k \geq 5$ be an odd prime. Then, $k \in N(F_n)$.*

Proof. The proof is divided into several cases.

CASE 1: $n \not\equiv 2, 3 \pmod k$. In this case, the labeling f (as defined in the proof of Theorem 2) provides a $\mathbb{Z}_k$-magic labeling of F_n with magic value 0.

CASE 2: $n \geq 8$ and either $n \equiv 2 \pmod k$ or $n \equiv 3 \pmod k$.

In particular, $n \not\equiv 0 \pmod k$. Consider the edge labeling g of F_n defined by $a_i = 1$ for $i = 1, 3, n-1$ or $5 \leq i \leq n-3$; $a_2 = a_4 = 2$; and $a_{n-2} \equiv -n \pmod k$, along with

$$a_{i-1} + a_i = \begin{cases} 3, & 2 \leq i \leq 5; \\ 1-n, & i = n-2, n-1; \\ 2, & \text{otherwise (if any).} \end{cases}$$

Note that $2\sum_{i=1}^{n-1} a_i \equiv 2[(n-3-5+1)+3+2(2)+(-n)] = 0 \pmod k$. So, all a_i's satisfy condition (4.1). Hence, g is a $\mathbb{Z}_k$-magic labeling of F_n with magic value 0.

CASE 3: $4 \leq n \leq 7$.

Here, only two possible cases can occur, namely $n = 6$, $k = 4$; and $n = 7$, $k = 4$. Figures 4 and 5 give the required $\mathbb{Z}$-magic labelings with magic value 0.

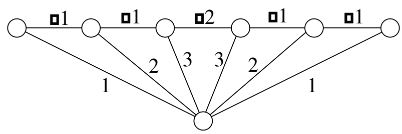

Figure 4: $\mathbb{Z}_4$-magic labeling of F_6 with magic value 0.

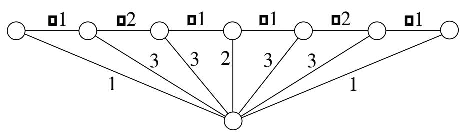

Figure 5: $\mathbb{Z}_4$-magic labeling of F_7 with magic value 0.

□

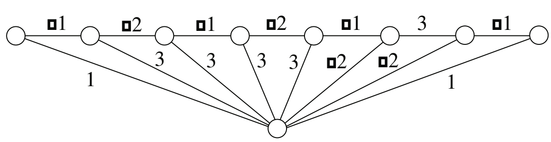

Figure 6: $\mathbb{Z}_5$-magic labeling of F_8 with magic value 0, illustrating CASE 2 of Theorem 4.

To summarize, we have the following corollary.

Corollary 1. *Let $n \geq 2$. Then,*

$$N(F_n) = \begin{cases} 2\mathbb{N}, & n = 2; \\ 2\mathbb{N} \setminus \{2\}, & n = 3; \\ \mathbb{N} \setminus \{2\}, & n \geq 4 \text{ and } n \equiv 1 \pmod 3; \\ \mathbb{N} \setminus \{2, 3\}, & n \geq 4 \text{ and } n \not\equiv 1 \pmod 3. \end{cases}$$

5 The null set of wheel graphs

As we mentioned in Section 3, W_n can be obtained from F_{n+1} in the following way: in F_{n+1}, identify the vertex v_1 with v_{n+1} and identify the edge cv_1 with cv_{n+1}. In this case, the identified vertex in W_n will be denoted by v_1^*.

Suppose there is an A-magic labeling f of F_{n+1}. Let W_n be obtained from F_{n+1} by the above procedure. We label all the edges of W_n with the corresponding edge labels in F_{n+1}, except the edge cv_1^*. We label the edge cv_1^* with $f(cv_1) + f(cv_{n+1})$. Thus we obtain an A-magic labeling of W_n with magic value 0, if $f(cv_1) + f(cv_{n+1}) \neq 0$.

Lemma 1. *Let $n \geq 3$ and $n \not\equiv 0 \pmod 3$. Then, $3 \notin N(W_n)$.*

Proof. (Indirect). Assume that W_n has a $\mathbb{Z}_3$-magic labeling f with magic value 0. Without loss of generality (because of Theorem C), f assigns 1 to each edge of W_n which is incident to a vertex of degree 3. Thus, every edge in W_n is labeled 1 by f. In particular, the central vertex has value n, which must be congruent to $0 \pmod 3$. This gives a desired contradiction and the result is established. □

Theorem 5. *Let $n \geq 3$. Then,*

$$N(W_n) = \begin{cases} \mathbb{N} \setminus \{2\}, & n \equiv 0 \pmod 3; \\ \mathbb{N} \setminus \{2,3\}, & n \not\equiv 0 \pmod 3. \end{cases}$$

Proof. If $n \equiv 0 \pmod 3$, we use the labelings of F_{n+1} found in Theorems 2, 3, and 4 and identify vertex v_1 with v_{n+1} and edge cv_1 with cv_{n+1}. In all instances, $f(cv_1) + f(cv_{n+1}) \neq 0$. Thus, $\{1, 3, 4, 5, \dots\} \subseteq N(W_n)$. Clearly, $2 \notin N(W_n)$. Hence, $N(W_n) = \mathbb{N} \setminus \{2\}$.

If $n \not\equiv 0 \pmod 3$, we use the labelings of F_{n+1} found in Theorems 2 and 4 and identify vertex v_1 with v_{n+1} and edge cv_1 with cv_{n+1}. In all instances, $f(cv_1) + f(cv_{n+1}) \neq 0$. Thus, $\{1, 4, 5, 6, \dots\} \subseteq N(W_n)$. Clearly, $2 \notin N(W_n)$. By Lemma 1, $3 \notin N(W_n)$. Hence, $N(W_n) = \mathbb{N} \setminus \{2, 3\}$. □

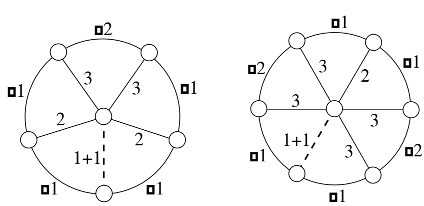

Figure 7: $\mathbb{Z}_4$-magic labelings of W_5 and W_6 with magic value 0, converted from $\mathbb{Z}_4$-magic labelings of F_6 and F_7 respectively.

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