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On Ramsey unsaturated and saturated graphs

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Abstract

A graph is *Ramsey unsaturated* if there exists a proper supergraph of the same order with the same Ramsey number, and *Ramsey saturated* otherwise. This has been studied by Balister, Lehel and Schelp [*J. Graph Theory* **51** (2006), 22–32]. In this paper, we show that some circulant graphs, trees with diameter 3, and $K_{t,n} \cup mK_1$ for infinitely many t, n and m, are Ramsey unsaturated.

1 Introduction

Throughout this paper, $r(G, H)$ denotes the Ramsey number of a pair of graphs (G, H), i.e., the minimum n such that in any coloring of the edges of K_n with colors red and blue, we either obtain a red subgraph isomorphic to G, or a blue subgraph isomorphic to H. We write $r(G)$ for $r(G, G)$.

Definition. A graph G on n vertices is said to be *Ramsey unsaturated* if there exists an edge $e \in E(\overline{G})$ such that $r(G+e) = r(G)$. The graph G is *Ramsey saturated* if $r(G + e) > r(G)$ for all $e \in E(\overline{G})$, i.e. G is not Ramsey unsaturated.

This has been studied by Balister, Lehel and Schelp in [1]. In particular, they showed that the path P_k and the cycle C_k are Ramsey unsaturated for all $k \geq 5$. In this paper, we show that some circulant graphs, trees with diameter 3 and $K_{t,n} \cup mK_1$ for infinitely many t, n and m are Ramsey unsaturated.

In the following, for X and Y (not necessarily distinct) in the vertex set $V(G)$ of G, $E(X, Y)$ denotes the set of edges between X and Y. For $v \in V(G)$, $N_G(v)$ denotes the neighborhood of v in G, that is, $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$.

2 Results

2.1 Circulant graphs

The *circulant graph* $C(n; S)$ is the graph with the vertex set $V(C(n; S)) = \{i \mid 0 \leq i \leq n-1\}$ and the edge set $E(C(n; S)) = \{(i, j) \mid 0 \leq i \leq n-1, 0 \leq i \leq n-1, (i-j)$

mod $n \in S\}$, where $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. Given a circulant graph $C(n; S)$, if $x, y \in V(C(n; S))$ such that $|x - y| = k$, then we call (x, y) a k-chord. In [1], the following was shown.

Theorem 2.1. *Let n and k be integers with $n \geq 5$, $1 < k < n/2$, and $\gcd(k, n) = 1$. If there is an odd $i > 0$ such that $k^i \equiv \pm 1 \mod n$, then $r(C_n + k\text{-chord}) = r(C_n)$.*

By using the same technique in the proof of Theorem 2.1 in [1], we can show the following:

Theorem 2.2. *Let $n \geq 5$ such that $m^2 \equiv \pm 1 \mod n$ and $m \not\equiv \pm 1 \mod n$. If there exists k where $1 < k < n/2$ which is relatively prime to both n and m, and an odd $j > 0$ such that $k^j \equiv 1 \mod n$, then $r(C(n; \{1, m\}) + k\text{-chord}) = r(C(n; \{1, m\}))$.*

Proof. Assume that $r(C(n; \{1, m\}) + k\text{-chord}) > r(C(n; \{1, m\}))$. Then there exists a red-blue edge coloring on $K_{r(C(n;\{1,m\}))}$ such that it contains no monochromatic $C(n; \{1, m\}) + k$-chord. Without loss of generality, we assume that it contains a red $C(n; \{1, m\})$. Then each k-chord of this red $C(n; \{1, m\})$ must be blue, i.e. the edges $(ik, (i + 1)k)$ must be blue for $0 \leq i \leq n - 1$.

Consider the mapping f given by $i \longmapsto im$ where $0 \leq i \leq n - 1$ on the vertex set of $C(n; \{1, m\})$. Since $m^2 \equiv \pm 1 \mod n$, f is an automorphism of the circulant graph $C(n; \{1, m\})$. Then the k-chord of this circulant graph $f(C(n; \{1, m\}))$ must be blue, i.e. $(ikm, (i + 1)km)$ must be blue for all $0 \leq i \leq n - 1$.

Since k is relatively prime to both n and m, for each j, the edges $(ik^j, (i + 1)k^j)$ and $(ik^j m, (i + 1)k^j m)$ where $0 \leq i \leq n - 1$ form the circulant graph $C(n; \{1, m\})$. Denote the circulant graph $C(n; \{1, m\})$ formed by the edges $(ik^j, (i + 1)k^j)$ and $(ik^j m, (i + 1)k^j m)$ where $0 \leq i \leq n - 1$ by $C(j)$.

By our assumption $C(0)$ is red and we have shown that $C(1)$ is blue. Likewise, one can show that $C(j)$ is red for all even j and blue for all odd j. It is therefore apparent that $k^j \not\equiv 1 \mod n$ for all odd j. Otherwise, $C(j) = C(0)$ would need to be both red and blue. □

Let $\mathbb{Z}_n^\times$ denote the multiplicative group of integers mod n that are relatively prime to n. Then $|\mathbb{Z}_n^\times| = \phi(n)$, the Euler phi function. Then we have the following:

Corollary 2.1. *Let $n \geq 5$. If $\phi(n) = 4q$ where q is an odd number greater than 1, then $C(n; \{1, m\})$ is Ramsey unsaturated for some $m \not\equiv \pm 1$.*

Proof. Since $\phi(n)$ is divisible by an odd prime p, there exists a nontrivial element k in the p-Sylow subgroup of $\mathbb{Z}_n^\times$. Then k is relatively prime to n and $k^j \equiv 1 \mod n$ for some odd j.

Since $\phi(n) = 4q$ where q is odd, there exists a nontrivial element m in the 2-Sylow subgroup of $\mathbb{Z}_n^\times$. Then m is relatively prime to n and $m^2 \equiv \pm 1 \mod n$. Moreover, k and m must be relatively prime.

Applying Theorem 2.2 to either k or $(n - k)$-chords of $C(n, \{1, m\})$, we get Corollary 2.1. □

It is well-known that if $n = p_1^{k_1} \dots p_r^{k_r}$ where p_i are distinct primes and $k_i > 0$, then $\phi(n) = p_1^{k_1-1} \dots p_r^{k_r-1}(p_1-1)\dots(p_r-1)$. Let S be the set of all odd primes p such that $p-1 = 2d$ where d is odd; then for $p_1, p_2 \in S$, we have $\phi(p_1p_2) = \phi(2p_1p_2) = 4q$ where q is odd. Since S is an infinite set, by Corollary 2.1, there are infinitely many n such that $C(n, \{1, m\})$ is Ramsey unsaturated, namely, $n = p_1p_2$ or $2p_1p_2$ where $p_1, p_2 \in S$.

Note also that there exists some n which is not in the form of Corollary 2.1. For example, $13^2 \equiv 1 \mod 28$ and $9^3 \equiv 1 \mod 28$, by Theorem 2.3, $r(C(28, \{1, 13\}) +$ 9-chord$) = r(C(28, \{1, 13\}))$. In particular, $C(28, \{1, 13\})$ is Ramsey unsaturated. By finding n which is not in the form of Corollary 2.1 and which satisfies the condition in Theorem 2.2, one can find more circulant graphs which are Ramsey unsaturated.

Theorem 2.3. *Let $n \geq 5$ such that $m_1^2 \equiv \pm m_2 \mod n$ and $m_2^2 \equiv \pm m_1 \mod n$ for some $m_1, m_2 \not\equiv \pm 1 \mod n$. If there exists k where $1 < k < n/2$ which is relatively prime to both n, m_1, m_2, and an odd $j > 0$ such that $k^j \equiv 1 \mod n$, then $r(C(n; \{1, m_1, m_2\}) + k\text{-chord}) = r(C(n; \{1, m_1, m_2\}))$.*

Proof. Assume that $r(C(n; \{1, m_1, m_2\}) + k\text{-chord}) > r(C(n; \{1, m_1, m_2\}))$. Then there exists a red-blue edge coloring on $K_{r(C(n;\{1,m_1,m_2\})}$ containing no monochromatic $C(n; \{1, m_1, m_2\}) + k$-chord. Without loss of generality, we assume that it contains a red $C(n; \{1, m_1, m_2\})$. Then each k-chord of this red $C(n; \{1, m_1, m_2\})$ must be blue, i.e. the edges $(ik, (i + 1)k)$ must be blue for all $0 \leq i \leq n - 1$.

For $t = 1, 2$, consider the mapping f_t given by $i \longmapsto im_t$ where $0 \leq i \leq n-1$ on the vertex set of $C(n; \{1, m_1, m_2\})$. Since $m_1^2 \equiv \pm m_2 \mod n$ and $m_2^2 \equiv \pm m_1 \mod n$ which implies $m_1m_2 \equiv \pm 1$, f_t is an automorphism of $C(n; \{1, m_1, m_2\})$. Then the k-chord of this circulant graph $f_t(C(n; \{1, m\}))$ must be blue, i.e. $(ikm_t, (i+1)km_t)$ must be blue for all $0 \leq i \leq n - 1$.

Since k is relatively prime to n, m_1, m_2, for each j, the edges $(ik^j, (i + 1)k^j)$, $(ik^jm_1, (i+1)k^jm_1)$ and $(ik^jm_2, (i+1)k^jm_2)$ where $0 \leq i \leq n-1$ form the circulant graph $C(n; \{1, m_1, m_2\})$. Denote the circulant graph $C(n; \{1, m_1, m_2\})$ formed by the edges $(ik^j, (i+1)k^j)$, $(ik^jm_1, (i+1)k^jm_1)$ and $(ik^jm_2, (i+1)k^jm_2)$ where $0 \leq i \leq n-1$ by $C(j)$.

By our assumption $C(0)$ is red and we have shown above that $C(1)$ is blue. Likewise, it can be shown that $C(j)$ is red for all even j and blue for all odd j. It is therefore apparent that $k^j \not\equiv 1 \mod n$ for all odd j. Otherwise, $C(j) = C(0)$ would need to be both red and blue. □

Corollary 2.2. *Let $n \geq 5$. If $\phi(n) = 3q$ where $q \not\equiv 0 \mod 3$ and $q \equiv 0 \mod p$ for some odd prime p, then $C(n; \{1, m_1, m_2\})$ is Ramsey unsaturated for some $m_1, m_2 \not\equiv \pm 1$.*

Proof. Since $\phi(n) = 3q$ and $q \not\equiv 0 \mod 3$, there exists a nontrivial element m_1 in the 3-Sylow subgroup of $\mathbb{Z}_n^\times$. This implies that $m_1 \equiv 1 \mod n$ and $m_1 \not\equiv \pm 1$. By taking $m_2 = m_1^2$, then we have $m_2 \not\equiv \pm 1$; $m_1^2 \equiv m_2 \mod n$ and $m_2^2 = m_1^4 \equiv m_1 \mod n$.

Moreover, since $\phi(n) = 3q$ where $q \equiv 0 \mod p$ for some odd prime p, there exists a nontrivial element k in the p-Sylow subgroup of $\mathbb{Z}_n^\times$. Then k is relatively prime to n, m_1, m_2 and $k^j \equiv 1 \mod n$ for some odd j.

Now considering either k or $(n-k)$-chords of $C(n, \{1, m_1, m_2\})$ and applying Theorem 2.3, we have Corollary 2.2. □

Note that there exists some n which is not in the form of Corollary 2.2. For example, we have $7^2 \equiv 11 \mod 19$; $11^2 \equiv -7 \mod 19$ and $5^9 \equiv 1 \mod 19$, by Theorem 2.3, $r(C(19, \{1, 7, 11\}) + 5\text{-chord}) = r(C(19, \{1, 7, 11\}))$. This shows that $C(19, \{1, 7, 11\})$ is Ramsey unsaturated. By finding n which is not in the form Corollary 2.2 and satisfies the condition in Theorem 2.3, one can find more circulant graphs which are Ramsey unsaturated.

2.2 Trees with diameter 3

In [1], the following conjecture is stated:

Conjecture 2.1. *If T_n is a non-star tree of order $n \geq 5$, then $r(T_n + e) = r(T_n)$ for each edge e such that $T_n + e$ is bipartite.*

Note that if T_n is a tree of order $n \geq 5$ such that $\text{diam}(T_n) = 3$, then T_n is a non-star tree. We will show that Conjecture 2.1 is true for all trees with diameter 3. Note also that if T_n is a tree of order $n \geq 5$ with $\text{diam}(T_n) = 3$, there exist two distinct vertices a, b in T_n such that $ab \in E(T_n)$; and for any $e \in E(T_n) - \{ab\}$, $e = xa$ or xb where $x \in V(T_n) - \{a, b\}$. See Figure 1.

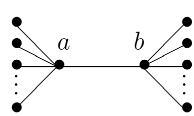

Figure 1. Tree with diameter 3.

Theorem 2.4. *If T_n is a tree of order $n \geq 5$ such that $\text{diam}(T_n) = 3$, then $r(T_n + e) = r(T_n)$ for each edge e such that $T_n + e$ is bipartite. In particular, T_n of order $n \geq 5$ with $\text{diam}(T_n) = 3$ is Ramsey unsaturated.*

Proof. Let $e = cd$ be an edge of T_n such that $T_n + e$ is bipartite. Then by renaming the vertices of c and d if necessary, we must have $c \in N_{T_n}(a) - \{b\}$ and $d \in N_{T_n}(b) - \{a\}$ where $N_G(x)$ denotes the neighborhood of x in G.

Suppose that $r(T_n + e) > r(T_n)$. Then we have a red-blue coloring on $K_{r(T_n)}$ such that it contains no monochromatic $T_n + e$. We can assume that it contains a red T_n. Then $E(N_{T_n}(a) - \{b\}, N_{T_n}(b) - \{a\})$ are blue.

Define $U = V(K_{r(T_n)}) - N_{T_n}(a) - N_{T_n}(b) + \{a, b\}$. If there exist two distinct vertices u, v in U such that ux' and vy' are blue for some $x' \in N_{T_n}(a) - \{b\}$ and $y' \in N_{T_n}(b) - \{a\}$, then $E(N_{T_n}(a) - \{b\}, N_{T_n}(b) - \{a\}) \cup ux' \cup vy'$ contains a blue $T_n + e$. Therefore, one of the following is true:

Case 1. $E(U, N_{T_n}(a) - \{b\})$ is red; or

Case 2. $E(U, N_{T_n}(b) - \{a\})$ is red.

Case 1. Note that $r(T_n) \geq \max\{n + |N_{T_n}(a)| - 1, n + |N_{T_n}(b)| - 1\}$ (see [4] for example). This implies $|U| \geq |N_{T_n}(b)| + 1$. If there exists $u \in U$ such that uy is red for some $y \in N_{T_n}(b) - \{a\}$, then $E(U, N_{T_n}(a) - \{b\}) \cup uy$ contains a red $T_n + e$.

Therefore we may assume that $E(U, N_{T_n}(b) - \{a\})$ is blue. However, $E(N_{T_n}(a) - \{b\}, N_{T_n}(b) - \{a\}) \cup E(U, N_{T_n}(b) - \{a\})$ contains a blue $T_n + e$.

Case 2. We can apply the same proof as in Case 1 by switching the labels a and b. □

2.3 $K_{t,n} \cup mK_1$

Theorem 2.5. *Let $K_{t,n}$ be the complete bipartite graph such that $1 \leq t \leq n$. Let $K_{t,n} \cup mK_1$ be the graph union of $K_{t,n}$ and m K_1's. If $r(K_{t,n} \cup mK_1) \geq n + 2t + 1$, then $r(K_{t,n} \cup mK_1) = r(K_{t,n} \cup mK_1 + e)$ where e is an edge connecting any one of the K_1 and a vertex in the n-side of $K_{t,n}$. In particular, $K_{t,n} \cup mK_1$ is Ramsey unsaturated.*

Proof. Suppose that we have a red-blue coloring on $K_{r(K_{t,n} \cup mK_1)}$. We can assume that there exists a red $K_{t,n}$ since $r(K_{t,n} \cup mK_1) \geq r(K_{t,n})$. Let $U = V(K_{r(K_{t,n} \cup mK_1)}) - V(K_{t,n})$ and V be the n-side of $K_{t,n}$.

If there exists $u \in U$, $v \in V$ such that uv is red, then we have a red $K_{t,n} \cup mK_1 + e$. Therefore, we may assume that $E(U, V)$ is blue. We claim that

$$|U| \geq t + 1. \tag{1}$$

To prove this, note that $|U| = r(K_{t,n} \cup mK_1) - (t + n)$. Combining this with the assumption that $r(K_{t,n} \cup mK_1) \geq n + 2t + 1$, we get (1).

Hence, $E(U, V)$ contains a blue $K_{t+1,n}$ in $K_{r(K_{t,n} \cup K_1)}$. By considering this blue $K_{t+1,n}$ union with the vertices which are not in this blue $K_{t+1,n}$, one can show that there exists a blue $K_{t,n} \cup mK_1 + e$ in $K_{r(K_{t,n} \cup K_1)}$. □

From Theorem 2.5, we have the following:

Corollary 2.3. *If $m \geq t + 1$, then $K_{t,n} \cup mK_1$ is Ramsey unsaturated.*

Proof. It is obvious that $r(K_{t,n} \cup mK_1) \geq m + n + t$. Hence, by assumption $m \geq t + 1$, we have $r(K_{t,n} \cup mK_1) \geq n + 2t + 1$. Now apply Theorem 2.5. □

Corollary 2.4. *If $1 \leq t \leq n/2 - 1$, then $K_{t,n} \cup mK_1$ is Ramsey unsaturated.*

Proof. By Theorem 2.5 and $r(K_{t,n} \cup mK_1) \geq r(K_{t,n})$, it is sufficient to show that $r(K_{t,n}) \geq n + 2t + 1$ if $1 \leq t \leq n/2 - 1$. From [3], we know that $r(K_{1,n}) \geq 2n - 1$. Therefore, if $1 \leq t \leq n/2 - 1$, we have $r(K_{t,n}) \geq r(K_{1,n}) \geq 2n - 1 \geq n + 2t + 1$. □

A (v, k, λ, μ) *strongly regular graph* is a graph with v vertices that is regular of degree k in which any two distinct vertices have λ common neighbors if they are adjacent and μ common neighbors if they are nonadjacent.

Corollary 2.5. *If a* $(4n-3, 2n-2, n-2, n-1)$ *strongly regular graph exists, then* $K_{t,n} \cup mK_1$ *is Ramsey unsaturated for* $1 \leq t \leq n$ *and* $n \geq 3$.

Proof. By [2], $r(K_{2,n}) \geq 4n-2$ if there exists a $(4n-3, 2n-2, n-2, n-1)$ strongly regular graph. Therefore, we have $r(K_{t,n}) \geq r(K_{2,n}) \geq 4n-2 \geq n+2t+1$ since $1 \leq t \leq n$ and $n \geq 3$. By Theorem 2.5, $K_{t,n} \cup mK_1$ is Ramsey unsaturated. □

Corollary 2.6. *If* $4n-3$ *is a prime power, or* $n = 12$*, then* $K_{t,n} \cup mK_1$ *is Ramsey unsaturated for* $1 \leq t \leq n$.

Proof. From [7], we know that a $(4n-3, 2n-2, n-2, n-1)$ strongly regular graph exists for n if $4n-3$ is a prime power. From [5], we know that a $(4n-3, 2n-2, n-2, n-1)$ strongly regular graph exists for $n = 12$. Hence, Corollary 2.6 follows from Corollary 2.5. □

Corollary 2.7. *If* n *is odd and there exists a symmetric Hadamard matrix of order* $2n-2$*, then* $K_{t,n} \cup mK_1$ *is Ramsey unsaturated if* $1 \leq t \leq n$ *and* $n \geq 4$*. If there exists a symmetric Hadamard matrix of order* $4n-4$*, then* $K_{t,n} \cup mK_1$ *is Ramsey unsaturated if* $1 \leq t \leq n$ *and* $n \geq 5$.

Proof. From [2], we know that $r(K_{2,n}) \geq 4n-3$ if n is odd and there exists a symmetric Hadamard matrix of order $2n-2$. Therefore, $r(K_{t,n}) \geq r(K_{2,n}) \geq 4n-3 \geq n+2t+1$ since $1 \leq t \leq n$ and $n \geq 4$. By Theorem 2.5, $K_{t,n} \cup K_1$ is Ramsey unsaturated.

From [2], we also know that $r(K_{2,n}) \geq 4n-4$ if there exists a symmetric Hadamard matrix of order $4n-4$. Therefore, $r(K_{t,n}) \geq r(K_{2,n}) \geq 4n-4 \geq n+2t+1$ since $1 \leq t \leq n$ and $n \geq 5$. By Theorem 2.5, $K_{t,n} \cup mK_1$ is Ramsey unsaturated. □

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