AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **46** (2010), Pages 51–66

A class of group divisible designs with block size three and index λ

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Abstract

We investigate the existence of group divisible designs (GDDs) with block size three, group-type $g^t w^1$ and index λ. We prove that the elementary necessary conditions for the existence of such a GDD are also sufficient, which generalizes the result of Colbourn, Hoffman and Rees.

1 Introduction

Let λ be a positive integer. A *group divisible design* (GDD) of index λ is a triple $(X, \mathcal{G}, \mathcal{B})$ where

- X is a finite set of *points*,
- $\mathcal{G}$ is a partition of X into subsets called *groups*,
- $\mathcal{B}$ is a collection of subsets of X (called *blocks*) such that a group and a block contain at most one common point, and every pair of points from distinct groups occurs in exactly λ blocks.

The *group-type* (or *type*) of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. An "exponential" notation is usually to be used to describe group-type : a type $1^i 2^j 3^k \cdots$ denotes i occurrences of 1, j occurrences of 2, etc. We say that a GDD of index λ, $(X, \mathcal{G}, \mathcal{B})$, is a (K, λ)-GDD if $|B| \in K$ for every block B in $\mathcal{B}$, where K is a set of positive integers, each of which is at least 2. When $K = \{k\}$, we simply write k for K. Further, we denote $(K, 1)$-GDD as K-GDD and $(k, 1)$-GDD as k-GDD.

* The author was supported by the Natural Science Foundation of China under contract 10601035 and the Jiangsu Provincial Fund for Natural Sciences of China under contract BK2006538. E-mail: jmwang@suda.edu.cn.

A GDD is called *uniform* if all of its groups have the same size. A k-GDD of type n^k is also known as a *transversal design* and denoted by TD$(k,\ n)$. It is well known that the existence of a TD$(k,\ n)$ is equivalent to that of $k-2$ *mutually orthogonal Latin squares* (MOLS) of side n. So, GDDs can also be thought of as a natural generalization of MOLS.

Many researchers have contributed to the existence of group divisible designs for a variety of reasons. Important applications of group divisible designs include the construction of other types of combinatorial designs (see, for example, [1, 5]).

The existence of uniform GDDs with block size 3 or 4 and general index $\lambda \geq 1$ has been completely settled (see [8, 3]). Knowledge concerning the existence of uniform GDDs with block size 5 has also advanced quite a lot recently (see [7, 15]). Some important classes of non-uniform GDDs with block size 3 and index $\lambda = 1$ have been constructed in [4, 6]. For more results on the existence of non-uniform GDDs with index $\lambda = 1$, the interested reader may refer to [7].

In this paper, our purpose is to investigate the existence spectrum for $(3, \lambda)$-GDDs of type $g^t w^1$. We first give the elementary necessary conditions for the existence of such a GDD.

Lemma 1.1 *Let g, t, w and λ be nonnegative integers. The necessary conditions for the existence of a $(3, \lambda)$-GDD of type $g^t w^1$ are that*

(i) *if $g > 0$, then $t \geq 3$, or $t = 2$ and $w = g$, or $t = 1$ and $w = 0$, or $t = 0$;*

(ii) $w \leq g(t-1)$ *or* $gt = 0$*;*

(iii) $\lambda(g(t-1)+w) \equiv 0 \pmod 2$ *or* $gt = 0$*;*

(iv) $\lambda gt \equiv 0 \pmod 2$ *or* $w = 0$*; and*

(v) $\lambda(\frac{1}{2}g^2 t(t-1) + gtw) \equiv 0 \pmod 3$.

The necessity of the above conditions is quite straightforward. Condition (i) means that there cannot be exactly two groups, and if there are exactly three groups, they must be equal in size. For Condition (ii), fix a point x of a group of size g and observe that each point in the group of size w, say G, must appear in λ blocks containing x and some other point not in G. Since x appears in $\lambda g(t-1)$ such pairs, the inequality follows. For Condition (iii), observe that if there is a point in a group of size g, it must appear in blocks with pairs of the points not in its group, of which there are $\lambda(g(t-1)+w)$. Similarly, Condition (iv) follows from considering a point in the group size w. Finally, Condition (v) holds because the total number of the pairs must be divisible by 3 when each pair of points from distinct groups appears in exactly λ blocks.

Now we must deal with the sufficiency of these conditions. Some work has been done for the existence of $(3, \lambda)$-GDDs of type $g^t w^1$. Colbourn, Hoffman and Rees [6] proved the following essential result partly settling our problem in the case $\lambda = 1$.

Theorem 1.2 *Let g, t and w be nonnegative integers. There exists a 3-GDD of type $g^t w^1$ if and only if the following conditions are all satisfied:*

(i) *if* $g > 0$, *then* $t \geq 3$, *or* $t = 2$ *and* $w = g$, *or* $t = 1$ *and* $w = 0$, *or* $t = 0$;
(ii) $w \leq g(t-1)$ *or* $gt = 0$;
(iii) $g(t-1) + w \equiv 0 \pmod 2$ *or* $gt = 0$;
(iv) $gt \equiv 0 \pmod 2$ *or* $w = 0$; *and*
(v) $\frac{1}{2}g^2t(t-1) + gtw \equiv 0 \pmod 3$.

For the reader's convenience, we tabulate the necessary and sufficient conditions when g, t and w are all positive. In Table I, we list the corresponding congruence classes for $w \pmod 6$ for each combination of congruence classes of g and $t \pmod 6$.

TABLE I
The Corresponding Values of $w \pmod 6$ when $\lambda = 1$

	$g \pmod 6$					
$t \pmod 6$	0	1	2	3	4	5
0	0,2,4	1,3,5	0,2,4	1,3,5	0,2,4	1,3,5
1	0,2,4		0		0	
2	0,2,4	1	2	1,3,5	4	5
3	0,2,4		0,2,4		0,2,4	
4	0,2,4	3	0	1,3,5	0	3
5	0,2,4		2		4	

The spectrum for $(3, \lambda)$-GDDs of type g^t was completely determined in [8].

Theorem 1.3 *Let* g, t *and* λ *be positive integers. The necessary and sufficient conditions for the existence of a* $(3, \lambda)$*-GDD of type* g^t *are* $t \geq 3$, $\lambda g(t-1) \equiv 0 \pmod 2$ *and* $\lambda g^2 t(t-1) \equiv 0 \pmod 6$.

A $(3, \lambda)$-GDD of type $1^{v-w}w^1$ is also referred to as an *incomplete triple system*, denoted by ITS$(v, w; \lambda)$ (see [5]). Since the existence of an ITS$(v, w; \lambda)$ was determined in [11], we have the following theorem.

Theorem 1.4 *A* $(3, \lambda)$*-GDD of type* $1^t w^1$ *exists if and only if*
(i) $t \geq 3$, *or* $t = 2$ *and* $w = 1$, *or* $t = 1$ *and* $w = 0$, *or* $t = 0$;
(ii) $w \leq (t-1)$ *or* $t = 0$;
(iii) $\lambda(t - 1 + w) \equiv 0 \pmod 2$ *or* $t = 0$;
(iv) $\lambda t \equiv 0 \pmod 2$ *or* $w = 0$; *and*
(v) $\lambda(\frac{1}{2}t(t-1) + tw) \equiv 0 \pmod 3$.

In what follows, we handle the remaining cases. We will show that the conditions in Lemma 1.1 are also sufficient for the existence of a $(3, \lambda)$-GDD of type $g^t w^1$. It is obvious that when $gt = 0$, the GDD is trivial, which has no blocks. Combining this with Theorems 1.2–1.4, we can restrict ourselves to the case where $\lambda \geq 2$, $g \geq 2$, $t \geq 3$ and $w > 0$ throughout the paper.

2 Preliminaries

In order to establish our result, we require several types of auxiliary designs and related results. We use [1] and [5] as our standard design-theoretic references.

A set of blocks of a GDD that partitions its point set is called a *parallel class*. A GDD is called *resolvable* if its block set admits a partition into parallel classes. We use the prefix "R" to indicate the resolvability of a GDD. Thus a (K,λ)-RGDD stands for a resolvable (K,λ)-GDD. The following known result can be found in [10].

Lemma 2.1 *A* $(3,\lambda)$*-RGDD of type* h^u *exists if and only if* $u \geq 3$, $\lambda h(u-1)$ *is even,* $hu \equiv 0 \pmod 3$*, and* $(\lambda,h,u) \notin \{(1,2,6),(1,6,3)\} \cup \{(2j+1,2,3),(4j+2,1,6) : j \geq 0\}$.

Let X be a gu-set (of *points*) and $\mathcal{H} = \{Y_1, Y_2, \cdots, Y_t\}$ be a set of disjoint subsets (called *holes*) of X. An *incomplete* (k,λ)-GDD of type g^u having hole set $\mathcal{H}$ is a structure $(X,\ \mathcal{H},\ \mathcal{G},\ \mathcal{B})$ where $\mathcal{G} = \{G_1, G_2, \cdots, G_u\}$ is a partition of $\mathcal{X}$ into u *groups* of g points each and $\mathcal{B}$ is a collection of k-subsets of X (called *blocks*) such that no block contains two distinct points of any group or any hole, but any other pair of points of $\mathcal{X}$ is contained in exactly λ block of $\mathcal{B}$. If $\mathcal{H} = \{Y_1, Y_2, \cdots, Y_t\}$ forms a partition of X , then the incomplete GDD $(X,\ \mathcal{H},\ \mathcal{G},\ \mathcal{B})$ is known as a *holey* GDD (HGDD). Furthermore, if each hole Y_i $(1 \leq i \leq t)$ is a set of uh_i points such that $|Y_i \cap G_j| = h_i$ for $1 \leq j \leq u$, we call the HGDD a (k,λ)-HGDD of type $(u,\ T)$, where T is the multiset $\{h_i : 1 \leq i \leq t\}$ and denoted by an "exponential" form.

The significance of HGDDs to our purpose is that we can fill in every hole of an existing HGDD with some GDD to obtain a new GDD, this construction is simple but useful.

Lemma 2.2 *If there exist a* $(3,\lambda)$*-HGDD of type* $(t,\ m_1^{n_1} \cdots m_r^{n_r})$ *and a* $(3,\lambda)$*-GDD of type* $m_i^t w^1$ *for each* $1 \leq i \leq r$*, then there exists a* $(3,\lambda)$*-GDD of type* $g^t w^1$*, where* $g = \sum_{i=1}^{r} m_i n_i$.

To apply Lemma 2.2, we need the following known results concerning HGDDs.

Lemma 2.3 ([13]) *There exists a* $(3,\lambda)$*-HGDD of type* $(u,\ h^t)$ *if and only if* $u \geq 3$, $t \geq 3$, $\lambda(u-1)(t-1)h \equiv 0 \pmod 2$ *and* $\lambda h^2 ut(u-1)(t-1) \equiv 0 \pmod 3$.

Lemma 2.4 ([12]) *Let* $u,\ t,\ g$ *and* w *be nonnegative integers. The necessary and sufficient conditions for the existence of a* 3*-HGDD of type* $(u,\ g^t w^1)$ *are that*

(1) $u \geq 3,\ t = 2$ *and* $g = w$*; or*

(2) $u \geq 3,\ t \geq 3,\ 0 \leq w \leq g(t-1),\ gt(u-1) \equiv 0 \pmod 2$, $(u-1)(w-g) \equiv 0 \pmod 2$ *and* $gtu(u-1)(g(t-1)-w) \equiv 0 \pmod 3$.

We will also make use of the following construction for HGDDs (see, for example, [12]), which is a modification of the PBD Construction for MOLS (see [2]).

Lemma 2.5 *Let $u \geq 3$. If there exists a $(3,\lambda)$-GDD of type $g_1^{t_1} \cdots g_r^{t_r}$, then there exists a $(3,\lambda)$-HGDD $(u,\, g_1^{t_1} \cdots g_r^{t_r})$.*

A *packing design* of pairs (or *packing*, in brief) of order v with block size k and index λ, denoted by $\mathrm{P}(k,\lambda;v)$, is a pair $(X,\mathcal{B})$ where X is a v-set (of *points*), and $\mathcal{B}$ is a collection of k-subsets (called *blocks*) of X such that every pair of distinct points from X occurs in at most λ blocks. In the particular case where every pair of distinct points from X occurs in exactly λ blocks of $\mathcal{B}$, the packing is exact, which is known as a *balanced incomplete block design*, $B(k,\lambda;v)$, and equivalent to a (k,λ)-GDD of type 1^v.

Consider a $\mathrm{P}(k,\lambda;v)$, $(X,\mathcal{B})$. Let π be a permutation on X. For any block $B=\{x_1,\ldots,x_k\}$, define $\pi(B)=\{\pi(x_1),\ldots,\pi(x_k)\}$. If $\{\pi(B) : B\in\mathcal{B}\}=\mathcal{B}$, then π is called an *automorphism* of the packing. Any automorphism π partitions $\mathcal{B}$ into equivalence classes called the *block orbits* under π. If a block orbit has v distinct blocks, then it is said to be *full*, otherwise *short*. A set of *base blocks* is an arbitrary set of representatives for these block orbits of $\mathcal{B}$.

A $\mathrm{P}(k,\lambda;v)$ is said to be *cyclic* if it admits an automorphism which is a permutation consisting of a single cycle of length v. It is important to note that a cyclic $\mathrm{P}(k,\lambda;v)$ may have short block orbits. Following [14], we use notation $\mathrm{CP}(k,\lambda;v)$ to denote a cyclic $\mathrm{P}(k,\lambda;v)$ without short block orbits. A cyclic $B(k,\lambda;v)$ will be denoted by $CB(k,\lambda;v)$. A straightforward fact is that a $\mathrm{CP}(k,\lambda;v)$ is uniquely determined by its base blocks. So, a $\mathrm{CP}(k,\lambda;v)$ can be defined equivalently as a family of k-subsets (called *base blocks*) of Z_v, $\mathcal{F}$, such that the difference list $\Delta(\mathcal{F})=\{a-b : a,b\in B, a\neq b \text{ and } B\in\mathcal{F}\}$ covers each nonzero element in Z_v at most λ times. One can obtain the packing by developing the base blocks in Z_v, that is, successively adding 1 to each base blocks modulo v. The *difference leave* of $\mathcal{F}$, $DL(\mathcal{F})$, is defined as the multiset $\lambda Z_v \setminus (\Delta(\mathcal{F})\cup\lambda\{0\})$, where the notation λS denotes the multiset containing each element of set S λ times.

Let H be an additive subgroup of Z_v having order g, which must be generated by integer v/g. We say that a $\mathrm{CP}(k,\lambda;v)$ with base block set $\mathcal{F}$ is *g-regular* if $\lambda G\setminus\Delta(\mathcal{F})=\lambda H$. It is easy to see that if a g-regular $\mathrm{CP}(k,\lambda;v)$ exists, then so does a (k,λ)-GDD of type $g^{v/g}$. A 1-regular $\mathrm{CP}(k,\lambda;v)$ is just a $CB(k,\lambda;v)$.

Lemma 2.6 ([9]) *For any positive integer $v\equiv 1 \pmod 6$, there exists a $CB(3,1;v)$.*

Lemma 2.7 *For any positive integer $v\equiv 5 \pmod 6$, there exists a $CP(3,2;v)$ having $(v-2)/3$ base blocks.*

Proof. For $v=12t+5$ and $t\geq 0$, consider the following base blocks on Z_v:

$$\{0,2i+2,3t+i+2\},\quad \{0,2i+1,5t+i+2\},\quad \{0,2i+2,3t+i+2\},$$
$$\{0,2i+1,5t+i+2\},\quad \{0,3t+1,6t+2\},$$

where $0\leq i\leq t-1$.

For $v = 12t + 11$ and $t \geq 0$, consider the following base blocks on Z_v:

$$\begin{array}{lll} \{0, 2i+1, 3t+i+4\}, & \{0, 2i+1, 3t+i+3\}, & \{0, 2i+2, 5t+i+6\}, \\ \{0, 2i+2, 5t+i+5\}, & \{0, 2t+1, 4t+3\}, & \{0, 2t+1, 6t+5\}, \\ \{0, 2t+2, 7t+6\}, & & \end{array}$$

where $0 \leq i \leq t-1$. □

Lemma 2.8 *If g is odd and $t \equiv 1 \pmod 6$, then there exists a g-regular $CP(3,1;gt)$.*

Proof. Let $\mathcal{F}$ be the base block set of a $CB(3,1;t)$ over Z_t, which exists by Lemma 2.6. It is easy to see that $|\mathcal{F}| = (v-1)/6$. The desired g-regular $CP(3,1;gt)$ is based on Z_{gt}, whose difference leave plus zero forms the subgroup $H = \{0, t, \ldots, (g-1)t\}$ of Z_{gt}. The required $g(t-1)/6$ base blocks can be constructed in the following way. For each base block $B = \{b_0, b_1, b_2\} \in \mathcal{F}$, we take g base blocks $B_j = \{b_0, b_1 + jt, b_2 + 2jt\}$, $j = 0, 1, \ldots, g-1$, where the additive operation is performed in Z_{gt}. □

Finally, we mention a construction for $(3,\lambda)$-GDDs.

Lemma 2.9 *Suppose that there exist a $(3,\lambda)$-GDD of type $m_1^{n_1} \cdots m_r^{n_r}$ and a $(3,\lambda)$-GDD of type $g^{m_i}w^1$ for $1 \leq i \leq r$. Then there exists a $(3,\lambda)$ -GDD of type $g^t w^1$, where $t = \sum_{i=1}^{r} m_i n_i$.*

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be the given $(3,\lambda)$-GDD of type $m_1^{n_1} \cdots m_r^{n_r}$. Let Y be a w-set such that $(X \times Z_g) \cap Y = \emptyset$. We construct the desired GDD on $(X \times Z_g) \cup Y$. For each $B \in \mathcal{B}$, construct a 3-GDD of type g^3 (from Lemma 1.3) with group set $\{\{x\} \times Z_g : x \in B\}$. For each $G \in \mathcal{G}$, construct a $(3,\lambda)$-GDD of type $g^{|G|}w^1$ with group set $\{\{x\} \times Z_g : x \in G\} \cup \{Y\}$. The resulting blocks then form a $(3,\lambda)$ -GDDs of type $g^t w^1$ with group set $\{\{x\} \times Z_g : x \in X\} \cup \{Y\}$. □

3 The case $\lambda = 2$

By Lemma 1.1, the necessary conditions for the existence of a $(3,2)$-GDD of type $g^t w^1$ with $g \geq 2$, $t \geq 3$ and $w > 0$ are that $w \leq g(t-1)$ and $\frac{1}{2}g^2 t(t-1) + gtw \equiv 0 \pmod 3$. In Table II (overleaf) we give the possible congruence classes for w (mod 6) for each combination of congruence classes of g and t (mod 6).

We first treat sufficiency for the cases when gt is not divisible by 2 or 3.

Lemma 3.1 *Let $g \equiv 1, 5 \pmod 6$, $t \equiv 1 \pmod 6$, $w \equiv 0 \pmod 3$ and $3 \leq w \leq g(t-1)$. Then there exists a $(3,2)$-GDD of type $g^t w^1$.*

TABLE II
Possible Values of w (mod 6) when $\lambda = 2$

t (mod 6)	g (mod 6) 0	1	2	3	4	5
0	all	all	all	all	all	all
1	all	0, 3	0, 3	all	0, 3	0, 3
2	all	1, 4	2, 5	all	1, 4	2, 5
3	all	all	all	all	all	all
4	all	0, 3	0, 3	all	0, 3	0, 3
5	all	1, 4	2, 5	all	1, 4	2, 5

Proof. We first repeat each base block of a g-regular $CP(3,1;gt)$ from Lemma 2.8 to obtain a g-regular $CP(3,2;gt)$ having $g(t-1)/3$ base blocks. Select arbitrarily $w/3$ base blocks from the packing. For each of these base blocks, say $B = \{x,y,z\}$, add three infinite points ∞_{B1}, ∞_{B2}, and ∞_{B3}, and form three new base blocks $\{x,y,\infty_{B1}\}$, $\{x,z,\infty_{B2}\}$ and $\{y,z,\infty_{B3}\}$. Combining the resulting w new base blocks with the other $g(t-1)/3 - w/3$ base blocks of the $CP(3,2;gt)$, we can construct a $(3,2)$-GDD of type $g^t w^1$. The required blocks of the GDD are obtained by developing these base blocks in Z_{gt}, where the infinite point, if it occurs in the base block, is always fixed. Note that all infinite points form a new group of size w. ∎

Lemma 3.2 *Let* $g \equiv 1 \pmod 6$, $t \equiv 5 \pmod 6$, $w \equiv 1 \pmod 3$ *and* $1 \le w \le g(t-1)$. *Then there exists a* $(3,2)$*-GDD of type* $g^t w^1$.

Proof. Let $r = (g-1)/3$, and write $w = 3a(t-1) + 3b + c$, where $0 \le a \le r-1$, $0 \le b \le t-1$, $1 \le c \le t-1$ and $c \equiv 1 \pmod 3$. Let $S_1, \ldots, S_r$ be the base blocks of a $CB(3,2;g)$ which exists by taking two copies of a $CB(3,1;g)$ from Lemma 2.6. We construct a $(3,2)$-GDD of type $g^t w^1$ on $Z_g \times Z_t$ plus w infinite points as follows.

Firstly, for every block $\{x,y,z\}$ in the orbit of S_i, $i = a+2, \ldots, r$, place a 3-HGDD of type $(t, 1^3)$ (from Lemma 2.3) on $\{x,y,z\} \times Z_t$.

Secondly, for every base block $S_i = \{x,y,z\}$, $i = 2, \ldots, a+1$, place a 3-HGDD of type $(t, 1^3)$ on $S_i \times Z_t$, which has $t-1$ parallel classes of triples. This design can be derived from a 3-RGDD of type t^3, which exists by Lemma 2.1. Now, for each parallel class $\mathcal{P}$, add three infinite points $\infty_{\mathcal{P}1}$, $\infty_{\mathcal{P}2}$, and $\infty_{\mathcal{P}3}$, and for each block $\{(x,\alpha),(y,\beta),(z,\gamma)\}$ in $\mathcal{P}$, we form three new base blocks $\{(x,\alpha),(y,\beta),\infty_{\mathcal{P}1}\}$, $\{(x,\alpha),(z,\gamma),\infty_{\mathcal{P}2}\}$ and $\{(y,\beta), (z,\gamma), \infty_{\mathcal{P}3}\}$. Develop all new base blocks mod $(g,-)$.

Thirdly, place a 3-HGDD of type $(t, 1^3)$ having $t-1$ parallel classes of triples on $S_1 \times Z_t$. Select arbitrarily b parallel classes from the HGDD. For each parallel class $\mathcal{P}$ of the selected parallel classes, add three infinite points $\infty_{\mathcal{P}1}$, $\infty_{\mathcal{P}2}$, and $\infty_{\mathcal{P}3}$. Then, for each block $\{(x,\alpha),(y,\beta),(z,\gamma)\}$ in $\mathcal{P}$, we form three new base blocks

$\{(x,\alpha),(y,\beta),\infty_{\mathcal{P}1}\}$, $\{(x,\alpha),(z,\gamma),\infty_{\mathcal{P}2}\}$ and $\{(y,\beta),(z,\gamma),\infty_{\mathcal{P}3}\}$. Develop all new base blocks and each block of the other $t-1-b$ parallel classes of the HGDD mod $(g,-)$.

Finally, we add c new infinite points, and for each $i \in Z_g$, place a (3,2)-GDD of type $1^t c^1$ from Theorem 1.4 on $\{i\} \times Z_t$ together with these c infinite points.

In this way, the above resulting blocks give a $(3,2)$-GDD of type $g^t w^1$, where all infinite points form a new group of size w. ∎

Lemma 3.3 *Let* $g \equiv 5 \pmod 6$, $t \equiv 5 \pmod 6$, $w \equiv 2 \pmod 3$ *and* $2 \le w \le g(t-1)$. *Then there exists a* $(3,2)$-*GDD of type* $g^t w^1$.

Proof. Let $r = (g-2)/3$ and $w = 3a(t-1)+3b+c$, where $0 \le a \le r-1$, $0 \le b \le t-1$, $2 \le c \le 2(t-1)$ and $c \equiv 2 \pmod 6$. We use a $CP(3,2;g)$ from Lemma 2.7, which has $(g-2)/3$ base blocks. Note that there is exactly one nonzero element $d \in Z_g$ such that $\pm d$ are not covered in the difference list of the base block set. The construction of the desired GDD is similar to that of the GDD in Lemma 3.2 except that at the last step, we add c new infinite points, and for each $i \in Z_g$, place a 3-GDD of type $2^t c^1$ from Theorem 1.2 on $\{i, i+d\} \times Z_t$ together with these c infinite points. ∎

Next, we handle the remaining cases for $gt \equiv 0 \pmod 3$.

Lemma 3.4 *Let* $g \ge 2$, $t \ge 3$, $gt \equiv 0 \pmod 3$ *and* $1 \le w \le g(t-1)$. *Then there exists a* $(3,2)$-*GDD of type* $g^t w^1$.

Proof. By Lemma 2.1 we have a $(3,2)$-RGDD type g^t, $(X,\mathcal{G},\mathcal{B})$. Its blocks admit a partition into $g(t-1)$ parallel classes. Take w parallel classes $\mathcal{P}_1,\dots,\mathcal{P}_w$ from the RGDD and add w infinite points $\infty_1,\dots,\infty_w$. For each block $B=\{x,y,z\} \in \mathcal{P}_i$, we construct three blocks: $\{x,y,\infty_i\}$, $\{x,z,\infty_i\}$ and $\{y,z,\infty_i\}$, $i=1,\dots,w$. For each i $(1 \le i \le w)$, we use $\mathcal{A}_i$ to denote the set of the gt blocks obtained from $\mathcal{P}_i$ in the above way. Now, let $X^* = X \cup \{\infty_1,\dots,\infty_w\}$, $\mathcal{G}^* = \mathcal{G} \cup \{\{\infty_1,\dots,\infty_w\}\}$, $\mathcal{B}^* = \mathcal{B} \setminus (\bigcup_{i=1}^{w} \mathcal{P}_i) \cup (\bigcup_{i=1}^{w} \mathcal{A}_i)$. Then $(X^*,\mathcal{G}^*,\mathcal{B}^*)$ is a $(3,2)$-GDD of type $g^t w^1$. ∎

Now we turn to the cases where gt is even and not divisible by 3. We start with the case $g=5$ and $w=2$.

Lemma 3.5 *If* $t \equiv 2 \pmod 6$ *and* $t \ge 8$, *then there exists a* $(3,2)$-*GDD of type* $5^t 2^1$.

Proof. For $t=8$, we construct a $(3,2)$-GDD of type $5^t 2^1$ on $Z_{40} \cup \{\infty_1, \infty_2\}$, with group set $\{\{i,i+8,i+16,i+24,i+32\} : i=0,1,\dots,7\} \cup \{\{\infty_1,\infty_2\}\}$. The required blocks are obtained by developing the following base block in Z_{40}: {0, 1, 3}, {0, 1, 4}, {0, 2, 17}, {0, 5, 15}, {0, 5, 26}, {0, 6, 27}, {0, 6, 28}, {0, 7, 29}, {0, 9, 13}, {0, 9, 20}, {0, 7, 17}, $\{0, 14, \infty_1\}$, $\{0, 12, \infty_2\}$. For $t \ge 14$, we take $(3,2)$-GDDs of type $3^{(t-5)/3}5^1$, $5^3 2^1$ from Lemma 3.4 and $5^5 2^1$ from Lemma 3.3, then apply Lemma 2.9 to give the result. ∎

The following lemma constructs the other necessary GDDs.

Lemma 3.6 *If there exist* 3*-GDDs of type* $g^t(u+6)^1$ *and* g^tu^1*, then there exists a* $(3,2)$*-GDD of type* $g^t(u+3)^1$*.*

Proof. Let $(X,\mathcal{G},\mathcal{B})$ be the given 3-GDD of type $g^t(u+6)^1$, G be its group of size $u+6$, and $S=\{a_i : 1\le i\le 6\}\subset G$. For each block B containing point a_i $(4\le i\le 6)$, we replace a_i by a_{i-3}. The other blocks in $\mathcal{B}$ are preserved. The resulting block set is denoted by $\mathcal{B}_1$. Then we construct a 3-GDD of type g^tu^1 with point set $X\setminus S$, group set $(\mathcal{G}\setminus\{G\})\cup\{G\setminus S\}$ and block set $\mathcal{B}_2$. Now, let $X^*=X\setminus\{a_i : 4\le i\le 6\}$, $G^*=(\mathcal{G}\setminus\{G\})\cup\{G\setminus\{a_i : 4\le i\le 6\}\}$ and $\mathcal{B}^*=\mathcal{B}_1\cup\mathcal{B}_2$. It is easily checked that $(X^*,\mathcal{G}^*,\mathcal{B}^*)$ is a $(3,2)$-GDD of type $g^t(u+3)^1$. □

Lemma 3.7 *Let* g,t *and* w *satisfy the necessary conditions shown in Lemma 1.1, with* $\lambda=2$*,* $g\ge 2$*,* $t\ge 3$*,* $w>0$*,* $gt\equiv 0 \pmod 2$ *and* $gt\not\equiv 0\pmod 3$*. Then there exists a* $(3,2)$*-GDD of type* g^tw^1*.*

Proof. For the stated values of g,t and w such that there exists a 3-GDD of type g^tw^1 from Theorem 1.2, the required $(3,2)$-GDD can be obtained from this design by repeating every block twice. For the other stated values of g,t and $w\ge 3$, there exist 3-GDDs of type $g^t(w+3)^1$ and $g^t(w-3)^1$ from Theorem 1.2. Therefore, the result is obtained by applying Lemma 3.6. It remains only to consider the following cases:

(i) $g\equiv 4 \pmod 6$, $t\equiv 2,5 \pmod 6$ and $w=1$;

(ii) $g\equiv 5 \pmod 6$, $t\equiv 2 \pmod 6$ and $w=2$.

For the remaining case (i), we take a $(3,2)$-HGDD of type $(t,1^g)$ from Lemma 2.3, then apply Lemma 2.2 to get the desired $(3,2)$-GDD of type g^t1^1. The required $(3,2)$-GDD of type 1^{t+1} as ingredient follows from Theorem 1.3.

Now we handle the remaining case (ii). For $g=5$, the result has been given in Lemma 3.5. For $g=11$, take a $(3,2)$-GDD of type 3^32^1 from Lemma 3.4, and apply Lemma 2.5 to form a $(3,2)$-HGDDs of type $(t,3^32^1)$. Since there exist $(3,2)$-GDDs of type 3^t2^1 from Lemma 3.4 and 2^{t+1} from Theorem 1.3, we can apply Lemma 2.2 to obtain the desired GDD. For $g\ge 17$, we repeat each block of a 3-HGDD of type $(t,3^{(g-5)/3}5^1)$ from Lemma 2.4 to obtain a $(3,2)$-HGDD of type $(t,3^{(g-5)/3}5^1)$, then fill in each hole with a $(3,2)$-GDD of type of 3^t2^1 or 5^t2^1. The conclusion holds by Lemma 2.2. □

Summarizing the results in Lemmas 3.1–3.5 and 3.7, we have proved

Theorem 3.8 *For all positive integers* g,t *and* w *satisfying* $g\ge 2$*,* $t\ge 3$*,* $w\le g(t-1)$ *and* $\frac{1}{2}g^2t(t-1)+gtw\equiv 0 \pmod 3$*, there exists a* $(3,2)$*-GDD of type* g^tw^1*.*

4 The case $\lambda = 3$

By Lemma 1.1, the necessary conditions for the existence of a $(3,3)$-GDD of type $g^t w^1$ with $g \geq 2$, $t \geq 3$ and $w > 0$ are that $w \leq g(t-1)$, gt is even and $g \equiv w \pmod 2$. In Table III we give the possible congruence classes for $w \pmod 6$ for each combination of congruence classes of g and $t \pmod 6$.

TABLE III
Possible Values of $w \pmod 6$ when $\lambda = 3$

$t \pmod 6$	$g \pmod 6$					
	0	1	2	3	4	5
0	0,2,4	1,3,5	0,2,4	1,3,5	0,2,4	1,3,5
1	0,2,4		0,2,4		0,2,4	
2	0,2,4	1,3,5	0,2,4	1,3,5	0,2,4	1,3,5
3	0,2,4		0,2,4		0,2,4	
4	0,2,4	1,3,5	0,2,4	1,3,5	0,2,4	1,3,5
5	0,2,4		0,2,4		0,2,4	

We first treat the easier cases where $gt \equiv 0 \pmod 6$.

Lemma 4.1 *Let $g \geq 2$, $t \geq 3$, $0 < w \leq g(t-1)$, $gt \equiv 0 \pmod 6$ and $g \equiv w \pmod 2$. Then there exists a $(3,3)$-GDD of type $g^t w^1$.*

Proof. For each of the stated values of g, t and w, there exists a 3-GDD of type $g^t w^1$ from Theorem 1.2. Hence, we can obtain the result by repeating each block of the 3-GDD three times. ∎

To handle the other cases, we develop two general methods for $(3,3)$-GDDs of type $g^t w^1$ as follows.

Lemma 4.2 *If $w > 0$ and there exists a 3-GDD of type $g^t(w+2)^1$, then there exists a $(3,3)$-GDD of type $g^t w^1$.*

Proof. We consider the cases of $w \pmod 6$. For $w \equiv 0,1,2,3 \pmod 6$ the existence of a 3-GDD of type $g^t(w+2)^1$ implies the existence of a $(3,2)$-GDD of type $g^t(w-1)^1$ by Theorem 3.8, which is required for these cases. For $w \equiv 4,5 \pmod 6$ the existence of a 3-GDD of type $g^t(w+2)^1$ implies the existence of a 3-GDD of type $g^t(w-4)^1$ by Theorem 1.2, which is required for these cases. Let $(X, \mathcal{G}, \mathcal{B})$ be the given 3-GDD of type $g^t(w+2)^1$, where its group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$ and $|G| = w+2$.

Case I: $w \equiv 5 \pmod 6$

We may assume that the group G contains a subset of seven points, $A = \{a_i : 1 \leq i \leq 7\}$, and let $G_0 = G \setminus A$. Replace a_6 by a_1 and replace a_7 by a_2 in every block in $\mathcal{B}$ containing a_6 or a_7. The resulting block set is denoted by $\mathcal{B}_1$. Again, for another GDD $(X, \mathcal{G}, \mathcal{B})$, replace a_6 by a_3 and replace a_7 by a_4 in every block in $\mathcal{B}$ containing a_6 or a_7. The resulting block set is denoted by $\mathcal{B}_2$. Now, construct a 3-GDD of type $g^t(w-4)^1$ with group set $\mathcal{G}' \cup \{G_0 \cup \{a_5\}\}$ and block set $\mathcal{B}_3$. Thus, $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ gives the required block set of a $(3,3)$-GDD of type $g^t w^1$ with group set $\mathcal{G}' \cup \{G_0 \cup \{a_i : 1 \le i \le 5\}\}$.

Case II: $w \equiv 0 \pmod 6$

Assume that the group G contains a subset $A = \{a_i : 1 \le i \le 8\}$. First replace a_7 and a_8 by a_6 in $\mathcal{B}$. Then construct a $(3,2)$-GDD of type $g^t(w-1)^1$ with group set $\mathcal{G}' \cup \{G \setminus \{a_6, a_7, a_8\}\}$. This yields a $(3,3)$-GDD of type $g^t w^1$ with group set $\mathcal{G}' \cup \{G \setminus \{a_7, a_8\}\}$.

Case III: $w \equiv 1 \pmod 6$

In this case, we assume that the group G contains three points a_1, a_2 and a_3. Replace a_2 and a_3 by a_1 in $\mathcal{B}$. Then construct a $(3,2)$-GDD of type $g^t(w-1)^1$ with group set $\mathcal{G}' \cup \{G \setminus \{a_1, a_2, a_3\}\}$. This produces a $(3,3)$-GDD of type $g^t w^1$ with group set $\mathcal{G}' \cup \{G \setminus \{a_2, a_3\}\}$.

Case IV: $w \equiv 2 \pmod 6$

Assume that the group G contains four points a_1, a_2, a_3 and a_4. Replace a_3 and a_4 by a_2 in $\mathcal{B}$. Then construct a $(3,2)$-GDD of type $g^t(w-1)^1$ with group set $\mathcal{G}' \cup \{G \setminus \{a_2, a_3, a_4\}\}$. This produces a $(3,3)$-GDD of type $g^t w^1$ with group set $\mathcal{G}' \cup \{G \setminus \{a_3, a_4\}\}$.

Case V: $w \equiv 3 \pmod 6$

Assume that the group G contains five points a_1, a_2, a_3, a_4 and a_5. Replace a_4 and a_5 by a_3 in $\mathcal{B}$. Then construct a $(3,2)$-GDD of type $g^t(w-1)^1$ with group set $\mathcal{G}' \cup \{G \setminus \{a_3, a_4, a_5\}\}$. This gives a $(3,3)$-GDD of type $g^t w^1$ with group set $\mathcal{G}' \cup \{G \setminus \{a_4, a_5\}\}$.

Case VI: $w \equiv 4 \pmod 6$

Assume that the group G contains subset $A = \{a_i : 1 \le i \le 6\}$. Firstly, replace a_5 by a_1 and replace a_6 by a_2 in $\mathcal{B}$. Secondly, for another GDD $(X, \mathcal{G}, \mathcal{B})$, replace a_5 by a_3 and replace a_6 by a_4 in $\mathcal{B}$. Finally, construct a 3-GDD of type $g^t(w-4)^1$ with group set $\mathcal{G}' \cup \{G \setminus A\}$. Thus, we can obtain a $(3,3)$-GDD of type $g^t w^1$ with group set $\mathcal{G}' \cup \{G \setminus \{a_5, a_6\}\}$. ∎

Lemma 4.3 *If $w > 1$ and there exists a 3-GDD of type $g^t(w+4)^1$, then there exists a $(3,3)$-GDD of type $g^t w^1$.*

Proof. Note that the existence of a 3-GDD of type $g^t(w+4)^1$ implies the existence of a $(3,2)$-GDD of type $g^t(w-2)^1$. Let $(X, \mathcal{G}, \mathcal{B})$ be the given 3-GDD of type $g^t(w+4)^1$, where $\mathcal{G} = \mathcal{G}' \cup \{G\}$ and $|G| = w+4$. We may assume that the group G contains a subset $A = \{a_i : 1 \le i \le u\}$, where $6 \le u \le 11$ and $u - 4 \equiv w \pmod 6$. First replace a_1 and a_2 by a_5 in every block in $\mathcal{B}$ containing a_1 or a_2, and replace a_3 and a_4 by a_6 in every block in $\mathcal{B}$ containing a_3 or a_4. The resulting block set is denoted by $\mathcal{B}_1$. Then construct a $(3,2)$-GDD of type $g^t(w-2)^1$ with group set $\mathcal{G}' \cup \{G \setminus \{a_i : 1 \le i \le 6\}\}$ and block set $\mathcal{B}_2$. It is easily checked that $\mathcal{B}_1 \cup \mathcal{B}_2$ is the required block set of a $(3,3)$-GDD of type $g^t w^1$ with group set $\mathcal{G}' \cup \{G \setminus \{a_1, a_2, a_3, a_4\}\}$. ∎

Lemma 4.4 *If $g \ge 2$, $t \ge 3$, $0 < w \le g(t-1)$, gt is even, $gt \not\equiv 0 \pmod 6$ and $g \equiv w \pmod 2$, then there exists a $(3,3)$-GDD of type $g^t w^1$.*

Proof. Fix g, t and w satisfying the conditions of the lemma. For every $r \equiv g(t-1) \pmod 6$ the conditions of Theorem 1.2 are satisfied with $\lambda = 1$ and so a 3-GDD of type $g^t r^1$ exists for these values of g, t and r. Note that we must have $w \equiv g(t-1) \pmod 2$. If $w \equiv g(t-1) \pmod 6$, we are done since the corresponding $\lambda = 1$ design exists; if $w + 2 \equiv g(t-1) \pmod 6$, then the desired design exists by Lemma 4.2; if $w + 4 \equiv g(t-1) \pmod 6$, then the desired design exists by Lemma 4.3, except in the case when $w = 1$.

It remains to deal with the case when $w = 1$ and $g(t-1) \equiv 5 \pmod 6$, which implies $g \equiv 5 \pmod 6$ and $t \equiv 2 \pmod 6$. Since there exist a $(3,3)$-HGDD of type $(t, 1^g)$ from Lemma 2.3 and a $(3,3)$-GDD of type 1^{t+1} from Theorem 1.3, we can apply Lemma 2.2 to handle this remaining case. ∎

The foregoing can be summarized into the following theorem.

Theorem 4.5 *For all positive integers $g \ge 2$, $t \ge 3$ and w satisfying $gt \equiv 0 \pmod 2$, $g \equiv w \pmod 2$ and $1 \le w \le g(t-1)$, there exists a $(3,3)$-GDD of type $g^t w^1$.*

5 The case $\lambda = 6$

By Lemma 1.1, the necessary condition for the existence of a $(3,3)$-GDD of type $g^t w^1$ with $g \ge 2$, $t \ge 3$ and $w > 0$ is only $w \le g(t-1)$.

Lemma 5.1 *Let $g \ge 2$, $t \ge 3$, $gt \equiv 0 \pmod 3$ and $0 < w \le g(t-1)$. Then there exists a $(3,6)$-GDD of type $g^t w^1$.*

Proof. For each of the stated values of g, t and w, there exists a $(3,2)$-GDD of type $g^t w^1$ by Theorem 3.8. Thus, the conclusion holds by repeating every block of the GDD three times. ∎

Lemma 5.2 *Let $g \ge 2$, $t \ge 3$, $g \equiv 1, 2 \pmod 3$, $t \equiv 1 \pmod 3$ and $0 < w \le g(t-1)$. Then there exists a $(3,6)$-GDD of type $g^t w^1$.*

Proof. When $w \equiv 0, 3 \pmod 6$, we have a $(3,2)$-GDD of type $g^t w^1$ from Theorem 3.8. Hence, we can repeat every block of the GDD three times to obtain the desired result.

When $w \equiv 1 \pmod 6$, let $(X, \mathcal{G}, \mathcal{B})$ be a $(3,2)$-GDD of type $g^t(w+2)^1$, where $\mathcal{G} = \mathcal{G}' \cup \{G\}$ and $|G| = w+2$, and let $G = G_0 \cup \{a_1, a_2, a_3\}$. Replace a_2 and a_3 by a_1 in every block of $\mathcal{B}$ containing a_2 or a_3. Now, we construct two copies of a $(3,2)$-GDD of type $g^t(w-1)^1$ with group set $\mathcal{G}' \cup \{G_0\}$. Thus, all resulting blocks form a $(3,6)$-GDD of type $g^t w^1$ with group set $\mathcal{G}' \cup \{G_0 \cup \{a_1\}\}$.

When $w \equiv 2 \pmod 6$, take a $(3,2)$-GDD of type $g^t(w+1)^1$ with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $|G| = w+1$. Let $G = G_0 \cup \{a_1, a_2, a_3\}$. First, replace a_3 by a_1 in every block containing a_3. Next, for another copy of the $(3,2)$-GDD, replace a_3 by a_2 in every block containing a_3. Finally, construct a $(3,2)$-GDD of type $g^t(w-2)^1$ with group set $\mathcal{G}' \cup \{G_0\}$.

When $w \equiv 4 \pmod 6$, take a $(3,2)$-GDD of type $g^t(w+2)^1$ with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $|G| = w+2$. Let $G = G_0 \cup \{a_1, \ldots, a_6\}$. First, replace a_5 by a_1 and replace a_6 by a_2. Next, for another copy of the $(3,2)$-GDD, replace a_5 by a_3 and replace a_6 by a_4. Finally, construct a $(3,2)$-GDD of type $g^t(w-4)^1$ with group set $\mathcal{G}' \cup \{G_0\}$.

When $w \equiv 5 \pmod 6$, take a $(3,2)$-GDD of type $g^t(w+1)^1$ with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $|G| = w+1$. Let $G = G_0 \cup \{a_1, \ldots, a_6\}$. First, replace a_6 by a_5. Next, for another copy of the $(3,2)$-GDD, replace a_6 by a_4. Finally, construct a $(3,2)$-GDD of type $g^t(w-2)^1$ with group set $\mathcal{G}' \cup \{G_0 \cup \{a_1, a_2, a_3\}\}$. ∎

Lemma 5.3 *Let $g \geq 2$, $t \geq 3$, $g \equiv 1 \pmod 3$, $t \equiv 2 \pmod 3$ and $0 < w \leq g(t-1)$. Then there exists a $(3,6)$-GDD of type $g^t w^1$.*

Proof. When $w \equiv 1, 4 \pmod 6$, repeat each block of a $(3,2)$-GDD of type $g^t w^1$ from Theorem 3.8 three times.

When $w \equiv 0 \pmod 6$ and $w \geq 6$, take a $(3,2)$-GDD of type $g^t(w+1)^1$ with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $|G| = w+1$. Let $G = G_0 \cup \{a_1, \ldots, a_7\}$. First, replace a_7 by a_6. Next, for another copy of the $(3,2)$-GDD, replace a_7 by a_5. Finally, construct a $(3,2)$-GDD of type $g^t(w-2)^1$ with group set $\mathcal{G}' \cup \{G_0 \cup \{a_1, a_2, a_3, a_4\}\}$.

When $w \equiv 2 \pmod 6$, take a $(3,2)$-GDD of type $g^t(w+2)^1$ with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $|G| = w+2$. Let $G = G_0 \cup \{a_1, a_2, a_3, a_4\}$. Replace a_3 and a_4 by a_2. Then construct two copies of a $(3,2)$-GDD of type $g^t(w-1)^1$ with group set $\mathcal{G}' \cup \{G_0 \cup \{a_1\}\}$.

When $w \equiv 3 \pmod 6$, take a $(3,2)$-GDD of type $g^t(w+1)^1$ with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $|G| = w+1$. Let $G = G_0 \cup \{a_1, a_2, a_3, a_4\}$. First, replace a_4 by a_3. Next, for another copy of the $(3,2)$-GDD, replace a_4 by a_2. Finally, construct a $(3,2)$-GDD of type $g^t(w-2)^1$ with group set $\mathcal{G}' \cup \{G_0 \cup \{a_1\}\}$.

When $w \equiv 5 \pmod 6$, take a $(3,2)$-GDD of type $g^t(w+2)^1$ with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $|G| = w+2$. Let $G = G_0 \cup \{a_1, \ldots, a_7\}$. Replace a_6 and a_7 by a_5. Then construct two copies of a $(3,2)$-GDD of type $g^t(w-1)^1$ with group set $(\mathcal{G}' \cup \{G_0 \cup \{a_1, a_2, a_3, a_4\}\}$. ∎

Lemma 5.4 *Let $g \geq 2$, $t \geq 3$, $g \equiv t \equiv 2$ (mod 3) and $2 \leq w \leq g(t-1)$. Then there exists a $(3,6)$-GDD of type $g^t w^1$.*

Proof. When $w \equiv 2, 5 \pmod 6$, repeat each block of a $(3,2)$-GDD of type $g^t w^1$ from Theorem 3.8 three times.

When $w \equiv 0 \pmod 6$ and $w \geq 6$, take a $(3,2)$-GDD of type $g^t(w+2)^1$ with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $|G| = w+2$. Let $G = G_0 \cup \{a_1, \ldots, a_8\}$. Replace a_7 and a_8 by a_6. Then, construct two copies of a $(3,2)$-GDD of type $g^t(w-1)^1$ with group set $\mathcal{G}' \cup \{G_0 \cup \{a_1, a_2, a_3, a_4, a_5\}\}$.

When $w \equiv 1 \pmod 6$ and $w \geq 7$, take a $(3,2)$-GDD of type $g^t(w+1)^1$ with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $|G| = w+1$. Let $G = G_0 \cup \{a_1, \ldots, a_8\}$. First, replace a_8 by a_7. Next, for another copy of the $(3,2)$-GDD, replace a_8 by a_6. Finally, construct a $(3,2)$-GDD of type $g^t(w-2)^1$ with group set $(\mathcal{G}' \cup \{G_0 \cup \{a_1, a_2, a_3, a_4, a_5\}\}$.

When $w \equiv 3 \pmod 6$, take a $(3,2)$-GDD of type $g^t(w+2)^1$ with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $|G| = w+2$. Let $G = G_0 \cup \{a_1, \ldots, a_5\}$. Replace a_4 and a_5 by a_3. Then construct two copies of a $(3,2)$-GDD of type $g^t(w-1)^1$ with group set $\mathcal{G}' \cup \{G_0 \cup \{a_1, a_2\}\}$.

When $w \equiv 4 \pmod 6$, take a $(3,2)$-GDD of type $g^t(w+1)^1$ with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $|G| = w+1$. Let $G = G_0 \cup \{a_1, \ldots, a_5\}$. First, replace a_5 by a_4. Next, for another copy of the $(3,2)$-GDD, replace a_5 by a_3. Finally, construct a $(3,2)$-GDD of type $g^t(w-2)^1$ with group set $\mathcal{G}' \cup \{G_0 \cup \{a_1, a_2\}\}$. ▯

Lemma 5.5 *Let $g \geq 2$, $t \geq 3$, $g \equiv t \equiv 2$ (mod 3). Then there exists a $(3,6)$-GDD of type $g^t 1^1$.*

Proof. For the case $g \geq 3$, take a $(3,6)$-HGDD of type $(t, 1^g)$ from Lemma 2.3 and a $(3,6)$-GDD of type 1^{t+1} from Theorem 1.3, then apply Lemma 2.2 to obtain the desired GDD. For the remaining case $g = 2$, take a $(3,3)$-GDD of type 2^{t+1} with group set $\mathcal{G} = \mathcal{G}' \cup \{G\}$, where $G = \{a_1, a_2\}$. Replace a_2 and a_1 in every block containing a_2. Then construct a $(3,3)$-GDD of type 2^t with group set $\mathcal{G}'$. The resulting blocks form a $(3,6)$-GDD of type $2^t 1^1$ with group set $\mathcal{G}' \cup \{a_1\}$. ▯

The results of Lemmas 5.1−5.5 can be summarized in the following theorem.

Theorem 5.6 *For any integers $g \geq 2$, $t \geq 3$ and $0 < w \leq g(t-1)$, there exists a $(3,6)$-GDD of type $g^t w^1$.*

6 Main theorem

We are now in a position to establish our main results concerning the existence of $(3,\lambda)$-GDDs of type $g^t w^1$.

Theorem 6.1 *Let g, t, w and λ be nonnegative integers. The necessary conditions for the existence of a $(3,\lambda)$-GDD of type $g^t w^1$, which are shown in Lemma 1.1, are also sufficient.*

Proof. By Theorems 1.2–1.4, we can restrict our attention to the case where $g \geq 2$, $t \geq 3$ and $w > 0$.

When $\lambda \equiv 1, 5 \pmod 6$, for any g, t and w satisfying the necessary conditions shown in Lemma 1.1, there exists a 3-GDD of type $g^t w^1$ by Theorem 1.2. Hence, the required $(3,\lambda)$-GDDs of type $g^t w^1$ can be obtained from this design by repeating every block λ times.

When $\lambda \equiv 2, 4 \pmod 6$, for any g, t and w satisfying the necessary conditions shown in Lemma 1.1, there exists a $(3,2)$-GDD of type $g^t w^1$ by Theorem 3.8. So, we can obtain the required $(3,\lambda)$-GDDs of type $g^t w^1$ from this design by repeating every block $\lambda/2$ times.

When $\lambda \equiv 3 \pmod 6$, for any g, t and w satisfying the necessary conditions shown in Lemma 1.1, there exists a $(3,3)$-GDD of type $g^t w^1$ by Theorem 4.5. Thus, we repeat every block of the design $\lambda/3$ times to obtain the required $(3,\lambda)$-GDDs of type $g^t w^1$.

When $\lambda \equiv 0 \pmod 6$, for any g, t and w satisfying the necessary conditions shown in Lemma 1.1, there exists a $(3,6)$-GDD of type $g^t w^1$ by Theorem 5.6. Therefore, the required $(3,\lambda)$-GDDs of type $g^t w^1$ can be obtained from this design by repeating every block $\lambda/6$ times. The proof is then completed. ∎

Acknowledgment

The authors would like to thank the anonymous referees for their careful reading of the manuscript and their helpful comments.

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(Received 23 Oct 2008; revised 15 Sep 2009)