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When does a random graph have constant cop number?

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Abstract

In this paper, we study the vertex pursuit game of *Cops and Robbers* where cops try to capture a robber loose on the vertices of a graph. The minimum number of cops required to win on a given graph G is the cop number of G. We present asymptotic results for the game of Cops and Robbers played on a random graph $G(n, p)$ focusing on the case when the cop number does not grow with the size of a graph. A few open problems are discussed.

1 Introduction

The game of *Cops and Robbers* was introduced independently by Nowakowski and Winkler [14], and Quilliot [15] over twenty years ago. The game is played on a fixed graph G and is our focus in this paper. We will always assume that G is undirected, simple, and finite. There are two players, a set of k *cops*, where $k \geq 1$ is a fixed integer, and the *robber*. The cops begin the game by occupying any set of k vertices (in fact, for a connected G, their initial position does not matter). The robber then chooses a vertex, and the cops and the robber move in alternate rounds. The players use edges to move from vertex to vertex. More than one cop is allowed to occupy a vertex, and the players may remain on their current positions. The players know each others current locations. The cops win and the game ends if at least one of the cops eventually occupies the same vertex as the robber; otherwise, that is, if the robber can avoid this indefinitely, he wins. As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c(G)$, which is the minimum number of cops needed to win on G. The cop number was introduced by Aigner and Fromme [1] who proved (among other things) that if G is planar, then $c(G) \leq 3$.

There are other well-studied versions of this game. Let us mention the one studied in [16], where at any point of the game the robber can run at great speed to any other vertex along a path of the graph. He is not permitted to run through a cop, however. Each cop either stands on a vertex or is in a helicopter (that is, is temporarily removed from the game). The robber can see the helicopter approaching its landing spot and may run to a new vertex before the helicopter actually lands. There are two two possible variants of this game: the robber is invisible (and so to capture him the cops must methodically search the whole graph) or visible. For more results on vertex pursuit games such as *Cops and Robbers*, the reader is directed to the survey [2].

Our main results refer to the probability space $G(n,p) = (\Omega, \mathcal{F}, \mathbb{P})$ of random graphs, where Ω is the set of all graphs with vertex set $[n] = \{1, 2, \dots, n\}$, $\mathcal{F}$ is the family of all subsets of Ω, and for every $G \in \Omega$

$$\mathbb{P}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2} - |E(G)|}.$$

It can be viewed as a result of $\binom{n}{2}$ independent coin flipping, one for each pair of vertices, where the probability of success (that is, drawing an edge) is equal to p ($p = p(n)$ can tend to zero with n). We say that an event holds *asymptotically almost surely* (a.a.s.) if it holds with probability tending to 1 as $n \to \infty$.

Bonato, Wang, and the author of this paper started investigating vertex pursuit games in dense random graphs and their generalizations used to model complex networks with a power-law degree distribution (see [8, 9]). From their results it follows that if $p = o(1)$ and $np = n^{\alpha + o(1)}$, where $1/2 < \alpha \leq 1$, then a.a.s.

$$c(G(n,p)) = \Theta(\log n / p) = n^{1-\alpha+o(1)},$$

a.a.s. $c(G(n,p)) = (1+o(1)) \log_{1/(1-p)} n$ for a constant $p < 1$, and $c(G(n, n^{-1/2+o(1)})) = n^{1/2+o(1)}$ a.a.s. On the other hand, Bollobás, Kun, and Leader [6] showed that the cop number of $G(n,p)$ is always bounded from above by $n^{1/2+o(1)}$ and this bound is achieved at the other end of the spectrum, that is, for sparse random graphs. More precisely, they showed that $c(G(n,p)) \leq 160000\sqrt{n} \log n$ for $p \geq 2.1 \log n / n$ and

$$c(G(n,p)) \geq \frac{1}{(np)^2} n^{\frac{1}{2} \frac{\log\log(np) - 9}{\log\log(np)}}$$

for $np \to \infty$. Since if either $np = n^{o(1)}$ or $np = n^{1/2+o(1)}$, then a.a.s. $c(G(n,p)) = n^{1/2+o(1)}$, it would be natural to assume that the cop number of $G(n,p)$ is close to $\sqrt{n}$ also for $np = n^{\alpha+o(1)}$, where $0 < \alpha < 1/2$. Łuczak and the author of this paper showed that the actual behaviour of $c(G(n,p))$ is slightly more complicated [13]. (See also [7] for a similar result for an extension of Cops and Robbers, distance k Cops and Robbers, where the cops win if they are distance at most k from the robber.) The function $f : (0,1) \to \mathbb{R}$ defined as

$$f(x) = \frac{\log \bar{c}(G(n, n^{x-1}))}{\log n},$$

where $\bar{c}(G(n,p))$ denotes the most likely value of the cop number for $G(n,p)$, has a characteristic zigzag shape (see Figure 1).

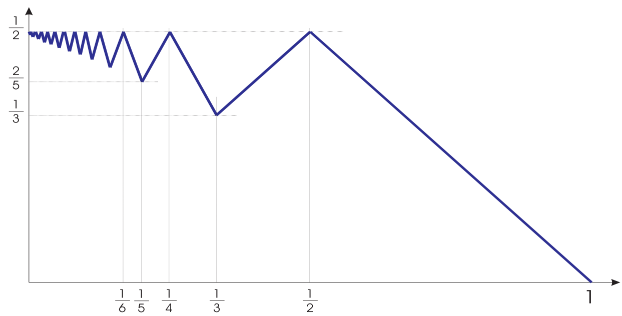

Figure 1: The 'zigzag' function f.

In this paper, we focus on a constant cop number that does not grow with the size of the graph. In particular, this answers a question when a random graph is *cop-win*. So-called cop-win graphs (that is, graphs G with $c(G) = 1$) were structurally characterized in [14, 15]. If x is a vertex, then define $N[x]$ to be x along with the vertices joined to x. The cop-win graphs are exactly those graphs which are *dismantlable*: there exists a linear ordering $(x_j : 1 \leq j \leq n)$ of the vertices so that for all $j \in \{2, 3, \ldots, n\}$, there is an $i < j$ such that

$$N[x_j] \cap \{x_1, x_2, \ldots, x_j\} \subseteq N[x_i] \cap \{x_1, x_2, \ldots, x_j\}.$$

This provides a useful tool to construct cop-win graphs. On the other hand, generating graphs with a constant cop number different than one is a challenging and nontrivial task.

It follows from the previous results that in order to get a constant cop number, $p(n)$ has to tend to 1 with n. Below we present the main theorem of this paper.

Theorem 1.1 *Let* $k \in \mathbb{Z}_+$ *and*

$$p = p(n) = 1 - \left(\frac{k \log n + a_n}{n} \right)^{\frac{1}{k}}.$$

Then the following holds:

- *if* $a_n \to -\infty$, *then a.a.s.* $c(G(n,p)) \leq k$,
- *if* $a_n \to a \in \mathbb{R}$, *then the probability that* $c(G(n,p)) = k$ *tends to* $1 - e^{-e^{-a}/k!}$; $c(G(n,p)) = k+1$ *otherwise,*
- *if* $a_n \to \infty$, *then a.a.s.* $c(G(n,p)) \geq k+1$.

(Let us note that if we add an additional condition that the sequence $(|a_n|)$ does not grow too fast with n (say, $|a_n| = n^{o(1)}$), then inequalities above can be replaced by equalities.)

On the other hand, it is possible that $p(n)$ tends to 1 and the cop number grows together with n. In this case clearly $p(n) = 1 - n^{o(1)}$ and $c(G(n,p)) = o(\log n)$.

Theorem 1.2 *Let* $1 \ll f(n) = o(\log n)$ *and* $p = p(n) = 1 - n^{-1/f(n)} = 1 - o(1)$. *Then a.a.s.* $c(G(n,p)) = (1+o(1))f(n)$.

The domination number of a dense random graph, used to show an upper bound on the cop number, is discussed in Section 2. The main result is shown in Section 3. We conclude the paper with a few open problems stated in Section 4.

2 Domination number

In order to prove the upper bound we require some background on the domination number of a graph. A set of vertices S is a *dominating set* in G if each vertex not in S is joined to some vertex of S. The *domination number* of G, written $\gamma(G)$, is the minimum cardinality of a dominating set in G. A straightforward observation is that $c(G) \le \gamma(G)$ (place a cop on each vertex of dominating set with minimum cardinality). However, if $n \ge 1$, then $c(P_n) = 1$ (where P_n is a path with n vertices) and $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ so the bound while useful, is far from tight in general. It is not surprising that the bound is of use when we deal with random graphs.

It has been shown that the domination number of a random graph has a.a.s. two point concentration for a constant $p < 1$ or $p = p(n)$ tending to zero sufficiently slowly [17]. Here we focus on $p(n)$ tending to 1 with n.

Theorem 2.1 *Let* $k \in \mathbb{Z}_+$ *and*

$$p = p(n) = 1 - \left(\frac{k \log n + a_n}{n} \right)^{\frac{1}{k}}.$$

Then

$$\lim_{n\to\infty} \mathbb{P}(\gamma(G(n,p)) \le k) = \begin{cases} 0 & \text{if } a_n \to \infty \\ 1 - e^{-e^{-a}/k!} & \text{if } a_n \to a \in \mathbb{R} \\ 1 & \text{if } a_n \to -\infty. \end{cases}$$

(Again, let us note that inequalities above can be replaced by equalities provided that, say, $|a_n| = n^{o(1)}$.)

Before we move to the proof of this theorem let us mention that for $k = 1$ we get a well-known result since the threshold for having a vertex dominating a graph G corresponds to the threshold for isolated vertices in the complement of G. (See, for example, [12, 5] or any textbook on random graphs.)

We need the following lemma which is known as a *convergence of moments* method.

Lemma 2.2 *Consider a sequence of probability spaces* $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$*. Let* $A_1^n, A_2^n, \ldots, A_{r_n}^n \in \mathcal{F}_n$*,*

$$B_l^n = \sum_{1 \le j_1 < \ldots j_l \le r_n} \mathbb{P}\left(\bigcap_{i=1}^{l} A_{j_i}^n\right),$$

and let S_n *be the number of events among* $A_1^n, A_2^n, \ldots, A_{r_n}^n$ *that actually occur, that is,*

$$S_n = \sum_{i=1}^{r_n} \mathbb{I}_{A_i^n}.$$

Suppose that for some λ*,* $\lim_{n\to\infty} B_l^n = \frac{\lambda^l}{l!}$ *for all fixed positive integers* l*. Then* S_n *tends to the Poisson distribution with parameter* λ*, that is,*

$$\lim_{n\to\infty} \mathbb{P}(S_n = l) = e^{-\lambda}\frac{\lambda^l}{l!}.$$

Now we are ready to show the main result of this section.

Proof of Theorem 2.1. Let $k \in \mathbb{Z}_+$,

$$p = p(n) = 1 - \left(\frac{k \log n + a_n}{n}\right)^{\frac{1}{k}},$$

and $a_n \to a \in \mathbb{R}$ as $n \to \infty$. For $X \subseteq [n]$, $|X| = k$, and $v \in [n] \setminus X$, let $A(X, v)$ denote the event that vertex v is adjacent to at least one vertex in X and let $A(X)$ denote the event that X is a dominating set of a random graph $G(n, p)$, that is, $A(X) = \bigcap_{v\in[n]\setminus X} A(X, v)$. It is clear that

$$\begin{aligned} B_1^n &= \sum_{X\subseteq[n],|X|=k} \mathbb{P}\left(A(X)\right) \\ &= \binom{n}{k}\left(1-(1-p)^k\right)^{n-k} \\ &= (1+o(1))\frac{n^k}{k!}\left(1 - \frac{k\log n + a_n}{n}\right)^{n-O(1)} \\ &= (1+o(1))\frac{e^{-a}}{k!} = (1+o(1))\lambda, \end{aligned}$$

where $\lambda := \frac{e^{-a}}{k!}$.

Let $l \ge 2$, $X_i \subseteq [n]$, $|X_i| = k$ $(1 \le i \le l)$, $X_i \ne X_{i'}$ for $1 \le i < i' \le l$, and

$v \in [n] \setminus \bigcup_{i=1}^{l} X_i$. Then Bonferroni's inequality can be used to get that

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{l} A(X_i,v)\right) &= 1-\mathbb{P}\left(\bigcup_{i=1}^{l} A(X_i,v)^c\right)\\
&\le 1-\sum_{i=1}^{l}\mathbb{P}(A(X_i,v)^c)+\sum_{1\le i<i'\le l}\mathbb{P}(A(X_i,v)^c\cap A(X_{i'},v)^c)\\
&= 1-l(1-p)^k+O((1-p)^{k+1})\\
&= 1-l\left(\frac{k\log n+a_n}{n}\right)+O\left(\left(\frac{\log n}{n}\right)^{\frac{k+1}{k}}\right).
\end{aligned}$$

Since edges are chosen independently and a graph induced by the set $\bigcup_{i=1}^{l} X_i$ forms a clique a.a.s.,

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{l} A(X_i)\right) &= (1+o(1))\prod_{v\in[n]\setminus\bigcup X_i}\mathbb{P}\left(\bigcap_{i=1}^{l} A(X_i,v)\right)\\
&\le (1+o(1))\left(1-l\left(\frac{k\log n+a_n}{n}\right)+O\left(\left(\frac{\log n}{n}\right)^{\frac{k+1}{k}}\right)\right)^{n-O(1)}\\
&= (1+o(1))n^{-kl}e^{-al}=O(n^{-kl}). \qquad (1)
\end{aligned}$$

Now,

$$B_l^n=\sum_{*}\mathbb{P}\left(\bigcap_{i=1}^{l} A(X_i)\right)=\sum_{**}\mathbb{P}\left(\bigcap_{i=1}^{l} A(X_i)\right)+\sum_{***}\mathbb{P}\left(\bigcap_{i=1}^{l} A(X_i)\right), \qquad (2)$$

where $\sum_{*}$ is taken over all distinct sets X_i ($i=1,2,\dots,l$) such that $|X_i|=k$ and $X_i\subseteq[n]$; $\sum_{**}$ denotes the sum over all such sets that satisfy the property $|\bigcup_{i=1}^{l} X_i|=kl$ (that is, all $\binom{l}{2}$ intersections of two sets are empty); $\sum_{***}$ contains all remaining terms. From (2) and (1) it follows that

$$\begin{aligned}
B_l^n &= (1+o(1))\binom{n}{k}\binom{n-k}{k}\cdots\binom{n-(l-1)k}{k}\frac{1}{l!}\left(\left(1-(1-p)^k\right)^{n-kl}\right)^l\\
&\quad + O(n^{kl-1})\cdot O(n^{-kl})\\
&= (1+o(1))\left(\frac{n^k}{k!}\right)^l\frac{1}{l!}n^{-kl}e^{-al}+o(1)\\
&= (1+o(1))\frac{\lambda^l}{l!}.
\end{aligned}$$

Thus, Lemma 2.2 implies that S_n, the number of dominating sets of cardinality k, tends to the Poisson distribution with parameter λ and the assertion follows. □

3 Proof of the main result

In order to obtain the lower bound, we can use the following adjacency property. We say that G is $(1,k)$-*existentially closed* (or $(1,k)$-*e.c.*) if for each k-set S of vertices of G and vertex $u \notin S$, there is a vertex $z \notin S$ not joined to a vertex in S and joined to u. If G is $(1,k)$-*e.c.*, then $c(G) > k$ (when at most k cops are used, the robber may use the property to escape to a vertex not joined to any vertex occupied by a cop). We use this to show the following.

Lemma 3.1 *Fix $k \in \mathbb{Z}_+$. If*

$$p = p(n) < 1 - \left(\frac{(k+1)\log n + \log\log n}{n} \right)^{\frac{1}{k}},$$

then $c(G(n,p)) > k$ a.a.s.

Proof. Fix $k \in \mathbb{Z}_+$. As we already mentioned we can assume (without loss of generality) that $p(n)$ tends to 1 as n goes to infinity; otherwise the cop number grows with n.

Fix S, a k-subset of vertices and a vertex u not in S. For a vertex $x \in [n] \setminus (S \cup \{u\})$, the probability that the vertex x is joined to u and to no vertex of S is $p(1-p)^k$. Since edges are chosen independently, the probability that no suitable vertex can be found for this particular S and u is $(1-p(1-p)^k)^{n-k-1}$.

Let X be the random variable counting the number of S and u for which no suitable x can be found. We then have that

$$\begin{aligned} \mathbb{E}X &= \binom{n}{k}(n-k)\left(1-p(1-p)^k\right)^{n-k-1} \\ &= O(n^{k+1})\left(1 - \frac{(k+1)\log n + 0.5\log\log n}{n}\right)^{n-k-1} \\ &= O\left(\frac{1}{\sqrt{\log n}}\right) = o(1). \end{aligned}$$

The proof now follows by the Markov's inequality. □

Finally we are ready to prove the main theorem.

Proof of Theorem 1.1. Fix $k \in \mathbb{Z}_+$. Given Theorem 2.1 and Lemma 3.1, it is enough to show that the cop number is greater than k if

$$\left(\frac{k\log n - \log\log n}{n} \right)^{\frac{1}{k}} \le 1 - p \le \left(\frac{(k+1)\log n + \log\log n}{n} \right)^{\frac{1}{k}}$$

and there is no dominating set of cardinality k.

We say that a vector $(r, c_1, c_1, \ldots, c_k)$ is *dangerous* if $\bigcap_{i=1}^k N^c(c_i) \subseteq N^c(r)$. Vertex r is *dangerous* if there is a vector $(c_1, c_2, \ldots, c_k)$ such that $(r, c_1, c_1, \ldots, c_k)$ is dangerous. It is clear that a graph with the cop number at most k must have at least one dangerous vector. If the game can be won by the cops, then a position at the end of the game yields such a vector. It does not matter where the robber goes, at least one cop can capture him and the game is finished (in other words, the robber should escape to $\bigcap_{i=1}^k N^c(c_i)$ but those vertices are not reachable).

The expected number of dangerous vectors with $|\bigcap_{i=1}^k N^c(c_i)| = l$ is at most

$$\begin{aligned} & O(n^{k+l+1})(1-p)^{(k+1)l}\left(1-(1-p)^k\right)^{n-k-l-1} \\ &= O(n^{k+l+1})\left(\frac{\log n}{n}\right)^{\frac{(k+1)l}{k}}\left(1-\frac{k\log n - \log\log n}{n}\right)^{n-O(1)} \\ &\leq n^{1-\frac{l}{k}+o(1)}, \end{aligned}$$

provided l is a constant. For $l = l(n)$ tending to infinity with n, we get that the expectation is at most

$$\begin{aligned} O(n^{k+l+1})(1-p)^{(k+1)l} &= O(n^{k+l+1})\left(\frac{(1+o(1))(k+1)\log n}{n}\right)^{\frac{(k+1)l}{k}} \\ &= n^{-l/k(1+o(1))}\Big((1+o(1))(k+1)\log n\Big)^{l(1+1/k)} \\ &= n^{-l/k(1+o(1))}. \end{aligned}$$

Thus, a.a.s. there is no dangerous vector with $l > k$ by the Markov's inequality. Moreover, $l \geq 1$ since it is assumed that there is no dominating set of cardinality k. Thus, the Markov's inequality can be used again to show that a.a.s. the number of dangerous vertices (which can be bounded from above by the number of dangerous vectors) is at most $n^{1-1/k+o(1)}$.

We will show now that the robber can avoid dangerous configurations. Note that, since a.a.s. $G(n,p)$ is connected, without loss of generality one can assume that at the beginning of the game all cops are placed at the same vertex w of degree at most $n - \Omega(\log n)$ (there must be a vertex of degree at most the average one). Indeed, if there is a winning strategy for cops that starts with an initial configuration C, then cops can always move from w to C and continue from there. The robber goes to any vertex in $N^c(w)$ and the corresponding vector is not dangerous a.a.s. (since there is no dangerous vector with $|\bigcap_{i=1}^k N^c(c_i)| = l = \Omega(\log n)$ a.a.s.). In order to force the robber occupying vertex r (that is not dangerous) to go to a dangerous vertex cops must occupy vertices $c_1, c_2, \dots, c_k$ and $(\bigcap_{i=1}^k N^c(c_i)) \setminus N^c(r)$ contains s dangerous vertices only, that is, the robber has to escape to $\bigcap_{i=1}^k N^c(c_i)$ to be safe but the only reachable vertices are dangerous. (Note that $s \geq 1$ since $(r, c_1, c_2, \dots, c_k)$ is not dangerous.) Using exactly the same argument as before, we get that

$$\left|\left(\bigcap_{i=1}^k N^c(c_i)\right) \cap N^c(r)\right| = t \leq k.$$

The expected number of vertices $c_1, c_2, \dots, c_k$ with the property that $\bigcap_{i=1}^{k} N^c(c_i)$ contains s dangerous vertices ($s \geq 1$) and $|\bigcap_{i=1}^{k} N^c(c_i)| = s + t$ ($0 \leq t \leq k$) is at most

$$\sum_{t=0}^{k} \sum_{s \geq 1} n^{k+t} \left(n^{1-\frac{1}{k}+o(1)}\right)^s (1-p)^{k(s+t)} (1-(1-p)^k)^{n-k-s-t} = \sum_{t=0}^{k} n^{-1/k+o(1)} = o(1).$$

Therefore, a.a.s. there is no way to force the robber to move to a dangerous vertex and the proof is complete. □

In order to prove the upper bound in Theorem 1.2, one can show that any fixed set of

$$K = f(n)\left(1 + \frac{1}{\log\log n}\right)$$

vertices is a dominating set a.a.s. The lower bound follows from the fact that a.a.s. $G(n,p)$ is $(1,k)$-*e.c.* with

$$k = f(n)\left(1 - \frac{1}{\log\log n}\right).$$

The argument is exactly the same as what we had before and therefore is omitted.

4 Open problems

It follows from the result of Bollobás, Kun, and Leader [6] (see also [13]) that the cop number of $G(n,p)$ is $O(\sqrt{n}\log n)$ for $np \geq 2.1 \log n$. This supports then the conjecture of Meyniel that $c(G) = O(\sqrt{|V|})$ for any connected graph G. In [11] one can read "that Meyniel conjectures $c(G) = O(\sqrt{|V|})$, which would be best possible" so it seems that Meyniel was able to construct a family of graphs with large cop numbers but unfortunately we do not know his construction. The Zigzag Theorem [13] implies only that a.a.s. $c(G) = n^{1/2+o(1)}$ for $G \in G(n, n^{-1/(2k)})$ ($k = 1, 2, \dots$), but it suggests that one should search for a candidate among graphs with the average degree around $\sqrt{|V|}$. We show that this is indeed the case. Before we move to the main result of this section, we need to introduce a few more definitions.

According to the combinatorial definition, a *finite projective plane* consists of a set of *lines* and a set of *points* with the following properties:

- given any two distinct points, there is exactly one line incident with both of them,
- given any two distinct lines, there is exactly one point incident with both of them,
- there are four points such that no line is incident with more than two of them.

One can show that a projective plane has the same number of lines as it has points. A finite projective plane has q^2+q+1 points, where q is an integer called the *order* of the projective plane. A projective plane of order q has $q+1$ points on every line, and $q+1$ lines passing through every point. It has been shown that there exists a finite projective plane of order q, if q is a prime power (that is, $q=p^a$ for a prime number p and $a \geq 1$), and for all known finite projective planes, the order q is a prime power. The existence of finite projective planes of other orders is an open question.

Finally, for a fixed prime power q, let $G_q=(P,L,E)$ be a bipartite graph with bipartition P,L where P and L denote the set of points and, respectively, lines in the projective plane. A point is joined to a line if it is contained in it. Then G_q has $2(q^2+q+1)$ many vertices and is $(q+1)$-regular.

Theorem 4.1 $c(G_q)=q+1$.

Proof. In order to prove that $c(G_q) \geq q+1$, we show that G_q is $(1,q)$-e.c. Fix $S=S_L \cup S_P$ a q-subset of vertices ($S_L \subset L$, $S_P \subset P$) and a vertex u not in S. Without loss of generality (by the duality of projective geometry), we can assume that $u \in L$. By the construction, each vertex in S_L controls exactly one neighbour of u. Since the graph is bipartite, each vertex in S_P can control (that is, occupy) at most one neighbour of u. Since G_q is $q+1$ regular, the assertion follows.

In order to show that $c(G_q) \leq q+1$, we present the following winning strategy for $(q+1)$ cops. The cops begin the game by occupying any set of points $c_1, c_2, \dots, c_{q+1}$. If the robber chooses a line v to start with, then each neighbour of v (points $b_1, b_2, \dots, b_{q+1}$) can be assigned to the unique cop and the cop occupying c_i can be moved to line l_i incident with both c_i and b_i to control all neighbours of v. The robber should stay in v but he will be captured in the next round. Otherwise, that is, if the robber chooses a point to start with, then one cop can be moved to attack the robber so that the robber has to escape to a line u. In the next round, the cop occupying a line is chasing the robber by moving to v; the other cops can be moved to lines to control remaining neighbours of u. □

Corollary 4.2 *There is an infinite family of graphs* $\{\hat{G}_n=([n],\hat{E}_n)\}$ *with* $c(\hat{G}_n) > \sqrt{n/8}$ *and* $c(\hat{G}_n) > \sqrt{n/2} - n^{0.2625}$ *for* n *sufficiently large.*

Proof. Fix any $n \in \mathbb{N}$ and let q be the largest prime power such that $2(q^2+q+1) \leq n$. (It is straightforward to show that $q \geq \sqrt{n/2}-1$ for $n \geq 2$.) We construct $\hat{G}_n$ by contracting into a single vertex a leaf of a path of length $n-2(q^2+q+1)$ and any vertex of the graph G_q based on the projective plane of order q. It is easy to show that $c(\hat{G}_n)=c(G_q)=q+1$ (the last equality follows from Theorem 4.1). In fact, Berarducci and Intrigila [4] proved that if G retracts to an induced subgraph H (that is, there is a homomorphism $f: V(G) \to V(H)$ which is the identity on $V(H)$), then $c(G)=c(H)$.

Now, the theorem follows from a well-known Chebyshev's Theorem that says that there exists a prime number between x and $2x$ for any integer $x \geq 2$. For sufficiently large x, one can use a stronger result which says that there is always a prime in the interval $[x - x^{0.525}, x]$ (see [3] for more details). $\square$

Now we are ready to formulate two open questions. Let $c(n)$ denote the maximum of $c(G)$ over all connected graphs with n vertices. It follows from Corollary 4.2 that $c(n) > \sqrt{n/2} - n^{0.2625}$ for n sufficiently large.

Question 1: Is it possible to improve a lower bound of $c(n)$?

On the other hand Frankl showed that $c(n) < (1 + o(1))n\frac{\log\log n}{\log n}$ [11]. A very little improvement has been made up to date. It took more than twenty years to improve an upper bound to $O(n/\log n)$. This result is due to Chiniforooshan [10] and is the best known upper bound of $c(n)$ (See also [7] for a similar result for an extension of Cops and Robbers, distance k Cops and Robbers, where the cops win if they are distance at most k from the robber.) Recall that Meyniel conjectured that $c(n) = O(\sqrt{n})$ so we are far away from what we believe is true. Therefore, an innocent looking upper bound of $n^{1-\varepsilon}$ for any fixed $\varepsilon > 0$ would be very significant progress.

Question 2: Is it possible to improve an upper bound of $c(n)$?

References

[1] M. Aigner and M. Fromme, A game of cops and robbers, *Discrete Appl. Math.* **8** (1984), 1–12.

[2] B. Alspach, Sweeping and searching in graphs: a brief survey, *Matematiche (Catania)* **59** (2006), 5–37.

[3] R.C. Baker, G. Harman and J. Pintz, The difference between consecutive primes. II. *Proc. London Math. Soc.* (3)**83** (2001), no. 3, 532–562.

[4] A. Berarducci and B. Intrigila, On the Cop Number of a Graph, *Adv. Appl. Math.* **14** (1993), 389–403.

[5] B. Bollobás, *Random Graphs*, Cambridge University Press, Cambridge, 2001.

[6] B. Bollobás and G. Kun, I. Leader, Cops and robbers in a random graph, preprint.

[7] A. Bonato, E. Chiniforooshan and P. Prałat, Cops and Robbers from a Distance, preprint.

[8] A. Bonato, P. Prałat and C. Wang, Pursuit-evasion in models of complex networks, *Internet Mathematics* **4** (2009), 419–436.

[9] A. Bonato, P. Prałat and C. Wang, Vertex pursuit games in stochastic network models, *Proc. 4th Workshop Combin. Algorithmic Aspects of Networking*, Lec. Notes Comp. Sci., Springer, 2007, 46–56.

[10] E. Chiniforooshan, A better bound for the cop number of general graphs, *J. Graph Theory* **58** (2008), 45–48.

[11] P. Frankl, Cops and robbers in graphs with large girth and Cayley graphs, *Discrete Appl. Math.* **17** (1987), 301–305.

[12] S. Janson, T. Łuczak and A. Ruciński, *Random Graphs*, Wiley, New York, 2000.

[13] T. Łuczak and P. Prałat, Chasing robbers on random graphs: zigzag theorem, *Random Structures and Algorithms* (to appear); 9pp.

[14] R. Nowakowski and P. Winkler, Vertex to vertex pursuit in a graph, *Discrete Math.* **43** (1983), 230–239.

[15] A. Quilliot, Jeux et pointes fixes sur les graphes, Ph.D. Dissertation, Université de Paris VI, 1978.

[16] P. Seymour and R. Thomas, Graph searching and a min-max theorem for tree-width, *J. Combin. Theory Ser. B* **58** (1993), no. 1, 22–33.

[17] B. Wieland and A.P. Godbole, On the domination number of a random graph, *Elec. J. Combin.* **8** (2001), #R37.

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