AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **48** (2010), Pages 111–115

On restricted edge connectivity of regular Cartesian product graphs*

Jingyu Wang Jianping Ou Tiedan Zhu

Department of Mathematics
Wuyi University
Jiangmen 529020
China
wjingyu_wyu@163.com jpou@263.net td_zhu@126.com

Abstract

Explicit expressions of the restricted edge connectivity of the Cartesian product of regular graphs are presented; some sufficient conditions for regular Cartesian product graphs to be maximally or super restricted edge connected are obtained as a result.

1 Introduction

All graphs appearing in this work are simple and connected with order at least four. A restricted edge cut is an edge cut S of a connected graph G such that every component of $G - S$ has order at least two. The minimum cardinality $\lambda'(G)$ over all restricted edge cuts of a graph G is its restricted edge connectivity. As is pointed out by Esfahanian in [4], restricted edge connectivity is a more accurate tool for analyzing the reliability of communication networks than the traditional edge connectivity of graphs. It is known that under some reasonable conditions, among networks that have maximum edge connectivity and least minimum edge cuts, those that have greater restricted edge connectivity are more reliable [8]. And so, much attention is paid to the optimization of restricted edge connectivity of graphs.

Let $\xi(G)$ denote the minimum edge degree of a graph G. It is know that if graph G is not isomorphic to any star and has order at least four, then $\lambda' \le \xi(G)$ [4]. The graph G is called maximally restricted edge connected if the equality holds in the previous inequality, and super restricted edge connected if every minimum restricted edge cut of G separates an isolated edge from G. It is not difficult to see that super restricted edge connected graphs are maximally restricted edge connected,

* Supported by the National Natural Science Foundation of China (Grant No. 10801091); Natural Science Foundation of Guangdong Province (Grant No. 9151051501000072)

but the converse is not true. The authors obtain various conditions for graphs to be maximally or super restricted edge connected in [1, 2, 7, 9, 10, 12–14], and Hellwig surveys the recent advances of this field in [6]. We remark here that the concepts of restricted edge cut and restricted edge connectivity are generalized in [5], [11] and elsewhere.

The Cartesian product $G_1\Box G_2$ of two graphs $G_1{=}(V_1, E_1)$ and $G_2{=}(V_2, E_2)$ has vertex-set $V(G_1) \times V(G_2)$, where two vertices (u_1, u_2) and (v_1, v_2) are adjacent to each other if and only if $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. Cartesian product graphs are the topology of a lot of networks, whose edge connectivities are determined in [12]. This work studies the restricted edge connectivity properties of Cartesian product graphs. Our main result is:

Theorem 2.5 *Let G_i be connected k_i-regular graphs with $k_i \geq 2, i = 1, 2$. If $\lambda^{'}(G_1), \lambda^{'}(G_2) \geq 2$, then $\lambda'(G_1\Box G_2) = \min\{|G_1|\lambda(G_2), |G_2|\lambda(G_1), \xi(G_1\Box G_2)\}$.*

Here λ is the edge connectivity of the corresponding graphs. This observation yields some sufficient conditions for Cartesian graphs to be maximally or super restricted edge connected.

Before proceeding, let us introduce some more symbols and terminology. For any subgraph X of G or subset of its vertex-set $V(G)$, let $G \setminus X$ denote the subgraph obtained by deleting all vertices of X from G. For any vertex $x \in V(G_1)$, let G_2^x represent the subgraph of $G_1\Box G_2$ induced by $V_2^x = \{(x, y)|y \in V(G_2)\}$, namely it has edge set $E_2^x = \{(x, y)(x, z)|yz \in E(G_2)\}$. For any vertex $y \in V(G_2)$, G_1^y is defined similarly. If S is a minimum restricted edge cut of a graph G, then $G - S$ contains exactly two components, which are called restricted fragments, or simply fragments, corresponding to S. Fragments of order not more than half of the order of the graph are called its normal fragments. A subgraph G_2^x is said to be separated by an edge cut S if $G_2^x \cap F \neq \emptyset \neq G_2^x \cap \overline{F}$. For other symbols and terminology not specified, we follow that of [3].

2 Restricted edge connectivity

For convenience, it is assumed that all graphs discussed in this section are regular with order at least three. Since $\lambda^{'}(G) \leq \xi(G)$ whenever the graph G has order at least four and is not isomorphic to any star, it follows that:

Lemma 2.1 *If G_i are connected k_i-regular graphs with $k_i \geq 2, i = 1, 2$, then $\lambda(G_1\Box G_2) \leq \lambda^{'}(G_1\Box G_2) \leq \xi(G_1\Box G_2) = 2(k_1 + k_2 - 1)$.*

Lemma 2.2 *Let F be a fragment of $G_1\Box G_2$. If $|F| \geq 3$, then $|F| \geq k_1 + k_2 - 1$.*

Proof Let S be the minimum restricted edge cut corresponding to the fragment F. Since G_i is a regular graph of order at least three, it has degree at least 2. By Lemma 2.1, we have

$$|F|(k_1 + k_2) - |F|(|F| - 1) \leq |S| \leq 2(k_1 + k_2) - 2;$$

$$[|F| - (k_1 + k_2 - 1)][|F| - 2] \geq 0.$$

Recalling that $|F| \geq 3$, we have $|F| \geq k_1 + k_2 - 1$. □

Lemma 2.3 *If F is a fragment of $G_1 \Box G_2$ with $|F| \geq 3$, then $F \not\subset G_2^i$ holds for any vertex $i \in V_1$ and $F \not\subset G_1^j$ holds for any vertex $j \in V_2$.*

Proof If $F \subset G_2^i$ for some vertex $i \in V(G_1)$, by Lemma 2.2 we have $|S| \geq (k_1+k_2-1)k_1+\lambda_2 > 2(k_1+k_2-1)$. This contradiction confirms the first statement: the second one follows similarly. □

Lemma 2.4 *If $G_1 \Box G_2$ contains restricted edge cuts, then it contains a minimum restricted edge cut $S' = [F, \overline{F}]$ such that $F = X \Box Y$ for some connected subgraphs $X \subseteq G_1$ and $Y \subseteq G_2$.*

Proof Let S be a minimum restricted edge cut of $G_1 \Box G_2$. If G_1^y is separated by S, then $G_1^y - S_y$ consists of two components, where $S_y = S \cap E(G_1^y)$. This observation is also true for any subgraph G_2^x. Let $|S_u| = \min\{|S_x| : G_2^x \text{ is separated by } S\}$, $|S_v|= \min\{|S_y| : G_1^y \text{ is separated by } S\}$, $r = |\{x \in V(G_1) : G_2^x \text{ is separated by } S\}|$ and $s = |\{y \in V(G_2) : G_1^y \text{ is separated by } S\}|$. Consider at first the case when $r \geq 1$ and $s \geq 1$. In this case, $S_u = [\{u\}\Box Y, \{u\}\Box\overline{Y}]$ and $S_v = [X\Box\{v\}, \overline{X}\Box\{v\}]$ for some connected subgraphs $X \subseteq G_1$ and $Y \subseteq G_2$. Assume without loss of generality that $|X| \leq |\overline{X}|$ and $|Y| \leq |\overline{Y}|$.

Let $F = X \Box Y$. If $\max\{|X|, |Y|\} \geq 2$, then $S' = [F, \overline{F}]$ is a restricted edge cut such that

$$|S| \geq \sum_{x \in X} |S_x| + \sum_{y \in Y} |S_y| \geq |X||S_u| + |Y||S_v| = |S'|. \qquad (1)$$

If $\max\{|X|, |Y|\} \leq 1$, then $X\Box\{v\}$ and $\{u\}\Box Y$ are isolated vertices in G_1^u and G_2^v respectively. Let $X = \{a\}$, $S_a = [\{a\}\Box Z, \{a\}\Box(V(G_2) - Z)]$ and $S' = [\{a\}\Box Z, V(G_1 \Box G_2) - \{a\}\Box Z]$. Then

$$|S| \geq |S_a| + \sum_{y \in Z} |S_y| \geq |S_a| + |Z||S_v| = |S'|.$$

Since S is a restricted edge cut, it follows that Z induces a connected subgraph of order at least two. And so S' is a restricted edge cut in either of these two subcases and the lemma follows in both cases. When $r = 0$, we have $s = |V(G_1)|$. If we let $Y = G_2$ and X be as before, then formula (1) is still true. This method also works when $s = 0$ and so the lemma follows. □

Theorem 2.5 *Let G_i be connected k_i-regular graphs with $k_i \geq 2, i = 1, 2$. If $\lambda'(G_1), \lambda'(G_2) \geq 2$, then $\lambda'(G_1 \Box G_2) = \min\{|G_1|\lambda(G_2), |G_2|\lambda(G_1), \xi(G_1 \Box G_2)\}$.*

Proof If $S = [A, \overline{A}]$ is a minimum edge cut of G_1, then $[A\Box V(G_2), \overline{A}\Box V(G_2)]$ is a restricted edge cut of $G_1 \Box G_2$. It follows that $\lambda'(G_1 \Box G_2) \leq |G_2|\lambda(G_1)$ and $\lambda'(G_1 \Box G_2) \leq |G_1|\lambda(G_2)$ by the symmetry of G_1 and G_2 in $G_1 \Box G_2$. Therefore,

$$\lambda'(G_1 \Box G_2) \leq \min\{|G_1|\lambda(G_2), |G_2|\lambda(G_1), \xi(G_1 \Box G_2)\}.$$

To prove the converse of the above inequality, let S be a minimum restricted edge cut of $G_1 \square G_2$. By Lemma 2.4, we may assume that $S = [X \square Y, \overline{X \square Y}]$ with $X \subset V(G_1)$ and $Y \subset V(G_2)$. Define r, s as in the proof of Lemma 2.4.

If $r = 0$, then $s = |G_2|$ and $|S| \geq |G_2|\lambda(G_1)$; if $s = 0$, then $r = |G_1|$ and $|S| \geq |G_1|\lambda(G_2)$; if $r = 1$, then by Lemma 2.3 we have $s = 2$, and so $|S| = 2(k_1+k_2)-2 = \xi$; if $s = 1$, similarly we have $r = 2$ and $|S| = \xi$. Hence, assume in what follows that $|G_1| - 2 \geq r = |X| \geq 2$ and $|G_2| - 2 \geq s = |Y| \geq 2$.

Case 1. $|X| \leq k_1, |Y| \leq k_2$.

In this case, noticing that $2 \leq |X| \leq k_1$, we have $|[X, \overline{X}]| \geq |X|k_1 - |X|(|X|-1) \geq k_1$. Similarly, $|[Y, \overline{Y}]| \geq k_2$. These observations show that $|S| \geq |Y|k_1 + |X|k_2 \geq 2k_1 + 2k_2 > \xi$.

Case 2. $|X| \leq k_1$ and $|Y| \geq k_2 + 1$, or $|Y| \leq k_2$ and $|X| \geq k_1 + 1$.

In the first subcase, we have $|[X, \overline{X}]| \geq k_1$ and $|[Y, \overline{Y}]| \geq 2$. Hence $|S| \geq (k_2 + 1)k_1 + 2|X| \geq 2k_1 + 2(k_2 - 1) + 2|X| > \xi$. The second subcase will result in the same inequality.

Case 3. $|X| \geq k_1 + 1$ and $|Y| \geq k_2 + 1$.

Since $\lambda'(G_1) \geq 2$ and $\lambda'(G_2) \geq 2$, it follows that $|S| \geq 2|X| + 2|Y| \geq 2(k_1 + 1) + 2(k_2 + 1) > \xi$. The theorem follows from these discussions. □

Corollary 2.6 *If G is a connected k-regular graph with $k \geq 2$, then $\lambda'(K_2 \square G) = \min\{2\lambda(G), |G|, 2k\}$.*

Proof This corollary follows directly from Lemma 2.4. □

Corollary 2.7 *If G_i are maximally edge connected k_i-regular graphs with $k_i \geq 2, i = 1, 2$, then $G_1 \square G_2$ is maximally restricted edge connected.*

Proof Since $|G_1|\lambda(G_2) = |G_1|k_2 \geq (k_1 + 1)k_2 \geq \xi(G_1 \square G_2)$ and $|G_2|\lambda(G_1) = |G_2|k_1 \geq (k_2 + 1)k_1 \geq \xi(G_1 \square G_2)$, the corollary follows from Theorem 2.5. □

Corollary 2.8 *If G_i are maximally edge connected k_i-regular graphs with $k_i > 2, i = 1, 2$, then $G_1 \square G_2$ is super restricted edge connected.*

Proof Since $k_1, k_2 > 2$, it follows that $|G_1|\lambda(G_2) = |G_1|k_2 \geq (k_1+1)k_2 > \xi(G_1 \square G_2)$ and $|G_2|\lambda(G_1) = |G_2|k_1 \geq (k_2 + 1)k_1 > \xi(G_1 \square G_2)$. Combining this observation and the proof of $\lambda'(G_1 \square G_2) \geq \min\{|G_1|\lambda(G_2), |G_2|\lambda(G_1), \xi(G_1 \square G_2)\}$ in the proof of Theorem 2.5, we see that $\lambda'(G_1 \square G_2) = \xi(G_1 \square G_2)$, and that it occurs only in the case when $r = 1$ and $s = 2$, or $s = 1$ and $r = 2$. The corollary follows. □

Corollary 2.9 *Let G be a maximally edge connected k-regular graph with $k \geq 2$. If $|G| \geq 2k$, then $K_2 \square G$ is maximally restricted edge connected; if $|G| > 2k$ and $k > 2$, then $K_2 \square G$ is super restricted edge connected.*

Proof The first statement follows directly from Corollary 2.6. If $|G| > 2k$ and $k > 2$, by Lemma 2.4, the normal fragment corresponding to any minimum restricted edge cut of $G_1 \square G_2$ is an isolated edge. The corollary follows as a result. □

References

[1] C. Balbuena, P. Garcia-Vázquez and X. Marcote, Sufficient conditions for λ'-optimality in graphs with girth g, *J. Graph Theory* 52 (2006), 73–86.

[2] C. Balbuena, X. Marcote and P. Garcia-Vázquez, On restricted connectivity of permutation graphs, *Networks* 45 (2005), 113–118.

[3] J. A. Bondy and U. S. R. Murty, *Graph Theory With Applications*, Macmillan Press, London, 1976.

[4] A. H. Esfahanian and S. L. Hakimi, On computing a conditional edge connectivity of a graph, *Inform. Proc. Lett.* 27 (1988), 195–199.

[5] J. Fábrega and M. A. Fiol, Extraconnectivity of graphs with large girth, *Discrete Math.* 127 (1994), 163–170

[6] A. Hellwig and L. Volkmann, Maximally edge-connected and vertex-connected graphs and digraphs, *Discrete Math.* 308 (2008), 3265–3296.

[7] Y. M. Hong, Q. H. Liu and Z. Zhang, Minimally restricted edge connected graphs, *Appl. Math. Lett.* 21 (2008), 820–823.

[8] Q. L. Li and Q. Li, Reliability analysis of circulant graphs, *Networks* 28 (1998), 61–65.

[9] J. X. Meng and Y. Ji, On a kind of restricted edge connectivity of graphs, *Discrete Appl. Math.* 117 (2002), 183–193.

[10] J. P. Ou and F. J. Zhang, Super restricted edge connectivity of regular graphs, *Graphs Combin.* 21 (2005), 459–467.

[11] J. P. Ou, Edge cuts leaving components of order at least m, *Discrete Math.* 305 (2005), 365–371.

[12] B. S. Shieh, Super edge- and point-connectivity of the Cartesian product of regular graphs, *Networks* 40 (2002), 91–96.

[13] N. Ueffing and L. Volkmann, Restricted edge connectivity and minimum edge-degree, *Ars Combin.* 66 (2003), 193–203.

[14] Z. Zhang and J. J. Yuan, Degree conditions for restricted-edge-connectivity and isoperimic-edge-connectivity to be optimal, *Discrete Math.* 307 (2007), 293–298.

(Received 14 Oct 2009; revised 30 July 2010)