PREMATURE SETS OF ONE-FACTORS

L. Caccetta and S. Mardiyono

School of Mathematics and Statistics
Curtin University of Technology
GPO Box U1987
PERTH WA 6001

ABSTRACT:

A **1-factor** of a graph G is a 1-regular spanning subgraph of G. A **1-factorization** of G is a decomposition of the edge set E(G) into edge-disjoint 1-factors. A set S of edge-disjoint 1-factors in G is said to be **maximal** if there is no 1-factor of G which is edge-disjoint from S, and if the union of S is not all of G. A set F of edge-disjoint 1-factors is **premature** if $\bar{F}$, the complement in G of the union of members of F, is non-empty and has no 1-factorization. If $\bar{F}$ has at least one 1-factor, then F is called a **proper premature** set of one-factors. Maximal sets of 1-factors in K_{2n} have been investigated. In this paper we investigate the existence of proper premature sets of 1-factors in K_{2n}. In particular, we establish that the existence of a proper premature set of k 1-factors in K_{2n} implies the existence of a proper premature set of (2n + k - 2t) 1-factors in K_{4n-2t} for $0 \le t \le \lfloor \frac{1}{2}k \rfloor$. We apply this result to construct proper premature sets of aspecific size.

1. INTRODUCTION

We consider graphs which are undirected, finite, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [2]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices and $\varepsilon(G)$ edges. K_n denotes the complete graph on n vertices and $K_{n,m}$ denotes the complete bipartite graph with bipartitioning sets of size n and m.

A **k-factor** of a graph G is a k-regular spanning subgraph of G. A **k-factorization** of G is a set of (pairwise) edge-disjoint k-factors which between them contain every edge of G.

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Graph factors have been studied for well over one hundred years (see Biggs et al [3]). Much of the work has focussed on 1-factorizations of graphs; for a comprehensive survey on 1-factorizations we refer to the paper of Mendelsohn and Rosa [8]. In 1891, Petersen proved that every even-regular graph has a 2-factorization. Over the past twenty years or so there has been considerable interest in the problem of partitioning the edge set of a graph into disjoint Hamilton cycles (ie. a 2-factorization in which each two factor is a Hamilton cycle. A recent survey of results is the paper by Alspach et al [1]. For the general problem of k-factors, some results have been obtained for bipartite graphs (see Enomoto et al [5]). In this paper we shall focus on a specific problem concerning 1-factors of K_{2n}.

A set S of edge-disjoint 1-factors in a graph G is said to be **maximal** if there is no 1-factor which is edge-disjoint from S and the union of S is not all of G. Thus if we write $\bar{S}$ for the complement in G of the union of members of S, then S is maximal if and only if $\bar{S}$ is a non-empty graph with no 1-factor. A set F of edge-disjoint 1-factors in G is **premature** if $\bar{F}$, the complement of F in G, is non-empty and has no 1-factorization. If $\bar{F}$ has at least one 1-factor, then F is called a **proper premature** set of 1-factors. We call $\bar{F}$ the **leave** of F. Observe that if G is regular, then $\bar{F}$ is also regular.

Maximal sets of 1-factors exist in K_{2n}. For example, for odd n, K_{2n} has a maximal set S whose leave consist of two odd components, each a K_n. Maximal sets of 1-factors have been studied by many authors (see Caccetta and Mardiyono [4] and Rees and Wallis [9]). The problem of determining the spectrum of maximal sets of 1-factors in K_{2n} has recently been completely solved by Rees and Wallis [9]. The corresponding question for 2-factors has also been resolved (Hoffman et al [7]).

Premature sets of 1-factors were first considered by Rosa and Wallis [10] who established the existence of large premature sets in K_{2n}. In particular, they proved by construction, that there is a premature set of k 1-factors in K_{2n} whenever k is even and $n < k < 2n - 4$, and for $k = 2n - 4$ when n is odd, $n \geq 5$. The non-existence of premature sets of three 1-factors was also shown.

Wallis [13] introduced the idea of what we call proper premature sets. He established the existence of a proper premature set of (2n - 4) 1-factors in K_{2n} for every $2n \geq 10$. His method was to reduce the problem to one of finding proper premature sets of (2n - 4) 1-factors in K_{2n} for $10 \leq 2n \leq 16$, and then exhibiting the required sets. The reduction was achieved by establishing that if K_{2n} contains a proper premature set of k 1-factors then K_{4n} and K_{4n-2} contain proper premature sets of (2n + k) and (2n + k - 2) 1-factors, respectively. In this paper we will generalize this result by proving that the existence of a proper premature set of k 1-factors in K_{2n} implies the existence of a proper premature set (2n + k - 2t) 1-factors in K_{4n-2t} for $0 \leq t \leq \lfloor\frac{1}{2}k\rfloor$. We will apply this result to construct proper premature sets of a specific size.

2. PRELIMINARIES

In this section, we state previsely a number of results which we make use of in subsequent sections. We begin with the important theorem of Tutte :

Theorem 2.1: A nontrivial graph G has a one-factor if and only if for every proper subset S of V(G), the number of odd components of G-S does not exceed $|S|$. □

In the study of one-factors, it is useful to know the order of the smallest graph without a one-factor. The next result, due to Wallis [12], provides this information for regular graphs.

Theorem 2.2: A d-regular graph G with no one-factor and no odd component satisfies :

$$\nu(G) \geq \begin{cases} 3d + 7, & \text{for odd } d \geq 3 \\ 3d + 4, & \text{for even } d \geq 6 \\ 22, & \text{for } d = 4 \end{cases}$$

No such G exists for d = 1 or 2. □

A **matching** M in a graph G is a subset of E(G) in which no two edges have a common vertex. The following result was proved by Rees and Wallis [9].

Theorem 2.3: Let $K_{m,n}$ be the complete bipartite graph with bipartition (X,Y) where $|X| = m$, $|Y| = n$ and $m \leq n$. Let $Y_1, Y_2, \ldots, Y_n$ be any collection of m-subsets of Y such that each vertex $y \in Y$ is contained in exactly m of the Y_j's. Then there is an edge-decomposition of $K_{m,n}$ into matchings $M_1, M_2, \ldots, M_n$ where for each $j = 1,2,\ldots,n$ M_j is a matching with m edges from X to Y_j. □

The **edge-chromatic number** $\chi'(G)$ of a graph G is the minimum number of colours needed to colour the edges of G. Our next result is a special case of a theorem of Folkman and Fulkerson [6]; a proof of this was given in [4].

Theorem 2.4: If G is a graph with c.k edges and $c \geq \chi'(G)$, then the edge set of G admits a decomposition into c matchings, each with k edges. □

3. MAIN THEOREM

Our main theorem provides us with a recursive construction of proper premature sets of 1-factors in K_{2n}. The method of proof is analogous to that used in Caccetta and Mardiyono [4] to establish a result on maximal sets of 1-factors.

Theorem 3.1: If there exists a proper premature set of k 1-factors in K_{2n}, then there exists a proper premature set of $(2n + k - 2t)$ 1-factors in K_{4n-2t} for every integer t, $0 \leq t \leq \lfloor \frac{1}{2}k \rfloor$.

Proof: Using the join operation, we can write

$$K_{4n-2t} = K_{2n-2t} \vee K_{2n} .$$

Let X and Y denote the graphs K_{2n-2t} and K_{2n}, respectively. Let $F = \{F_1, F_2, \ldots, F_k\}$ be a proper premature set of 1-factors in Y. Then $\bar{F}$ has a 1-factor but no 1-factorization. We obtain the required proper premature set of 1-factors in K_{4n-2t} by extending the proper premature set F to a proper premature set F′ of 1-factors in $K_{2n-2t} \vee K_{2n}$; $\bar{F}'$ will contain $\bar{F}$ as a component.

Take 2t members of F and let H be the graph formed by the union of these 1-factors. Note that since $t \leq \lfloor \frac{1}{2}k \rfloor$ we can always do this. H is a 2t-regular graph on 2n vertices. Applying Theorem 2.4 (with $c = 2n$ and $k = t$) we decompose the edge-set of H into 2n matchings $M_1, M_2, \ldots, M_{2n}$, each with t edges. Let Y_i denote the vertices of H which are not saturated by M_i. Each vertex of H (and hence Y) is contained in exactly $2n - 2t$ of the Y_i's and $|Y_i| = 2n - 2t$ for each i.

Now consider the graph $K_{2n-2t,2n}$. Applying Theorem 2.3 we can decompose the edge-set of this graph into 2n disjoint matchings $N_1, N_2, \ldots, N_{2n}$, such that N_i joins the vertices of Y_i to the vertices of X. Let

$$L_i = M_i \cup N_i \qquad , \qquad i = 1,2,\ldots,2n.$$

Observe that each L_i is a 1-factor of $K_{2n-2t} \vee K_{2n}$.

There remains in Y a set S of $(k - 2t)$ 1-factors from the original premature set F. Construct $(k - 2t)$ 1-factors on X (such a set exists, since K_{2n-2t} has a 1-factorization) and pair these off with the 1-factors of S to form a set of $(k - 2t)$ 1-factors $\bar{L}_1, \bar{L}_2, \ldots, \bar{L}_{k-2t}$. Then the set

$$F' = \{L_1, L_2, \ldots, L_{2n}, \bar{L}_1, \bar{L}_2, \ldots, \bar{L}_{k-2t}\}$$

the leave $\bar{F}'$ of F' consists of two components one of which is $\bar{F}$ the leave of the premature set F of 1-factors in K_{2n}. This completes the proof of the theorem. □

As a corollary we have :

Corollary: If K_{2n} has a proper premature set of k 1-factors, then :

(a) for even $k \geq n - 1$, K_m has a proper premature set of $(m - 2n + k)$ 1-factors for every even integer $m \geq 4n - k$;

(b) for odd $k \geq n$, K_m has a proper premature set of $(m - 2n + k)$ 1-factors for every even integer $m \geq 4n - k + 1$.

Proof: The corollary is established by repeatedly applying Theorem 3.1. We illustrate the argument for the case when k is even; the case when k is odd is analogous.

Suppose K_{2n} has a premature set of k 1-factors and k is even. Then Theorem 3.1 implies that K_{4n-2t} has a premature set of $(2n + k - 2t)$ 1-factors for every $0 \leq t \leq \frac{1}{2}k$. Thus the assertion is true for even m, $4n - k \leq m \leq 4n$. Now consider the graph K_{4n-k} which has a proper premature set of 2n 1-factors. Applying Theorem 3.1 we can conclude that $K_{8n-2k-2t'}$ has a proper premature set of $(6n - k - 2t')$ 1-factors for every $0 \leq t' \leq n$. Observe that $k \geq n - 1$ implies $6n - 2k \leq 4n + 2$.

Consequently, repeated applications of Theorem 3.1 will indeed establish the Corollary. □

4. APPLICATION OF THEOREM 3.1

In this section we demonstrate the use of Theorem 3.1 for the construction of proper premature sets of 1-factors of specific size. In particular, we establish the existence of a proper premature set of $(2n - 4)$ 1-factors in K_{2n} for $2n \geq 10$ and the existence of a proper premature set of $(2n - 6)$ 1-factors in K_{2n} for $2n \geq 14$.

Observe that if K_{2n} contains a proper premature set F of k 1-factors, then the leave $\bar{F}$ of this set is a (2n - k - 1)-regular graph with at least one 1-factor but no 1-factorization. To apply Theorem 3.1 one needs to determine the smallest n for which such a graph exists. We need an extension of Theorem 2.2

We say a Graph G has **exactly t 1-factors** if the maximum cardinality of a set of edge-disjoint 1-factors in G is t. The problem that arises is that of determining the minimum order of a graph having exactly t 1-factors. For the results of this section we need to resolve this problem for the cases t = 1 and t = 3. We do this in the following two lemmas; the general problem remains open.

Lemma 4.1 : Let G be a d-regular graph on 2n vertices having exactly one one-factor. Then

$$n \geq \begin{cases} d + 2, & \text{if } d \text{ is odd,} \\ \frac{3}{2}d + 2, & \text{if } d \text{ is even} \end{cases}$$

Proof : Let F be the one-factor of G and G′ the subgraph obtained from G by deleting the edges of F. G′ is a (d-1)-regular graph without a one-factor. If G′ has no odd component, then Theorem 2.2 implies that d-1 ≥ 3 and

$$2n \geq \begin{cases} 3(d-1) + 7, & \text{for odd } d-1 \geq 3. \\ 3(d-1) + 4, & \text{for even } d-1 \geq 6. \\ 22, & d-1 = 4. \end{cases}$$

Thus the assertion clearly holds. So we may assume that G' has odd components. As G' is (d-1)-regular, each of its components must have at least d vertices. We need only consider the case when d is odd as d-1 is odd when d is even. So suppose d is odd.

A simple argument establishes that $n > d+2$ if G' has more than 2 components. Hence we can assume that G' consists of exactly two odd components, G'_1 and G'_2 say. Let $n_i = |V(G'_i)|$, so that $2n = n_1 + n_2$. Note that $n_1 \geq d$ and $n_2 \geq d$. Suppose without any loss of generality, that $n_1 \leq n_2$. Then the only case we need consider is that when $n_1 = d$. In this case $G'_1 = K_d$.

If $n_2 = d$, then $G'_2 = K_d$ and thus, in G, the edges of F join vertices in different components of G'. But then G would be Hamiltonian and hence have more than one 1-factor. Therefore $n_2 \geq d+2$. If $n_2 = d+2$, then $\delta(G_2) = d-1 \geq \frac{1}{2} n_2$ for $d \geq 5$. Thus for $d \geq 5$, G_2 has a Hamiltonian cycle. But then, since $G_1 = K_d$ and in G there are $d \geq 2$ edges going from $V(G_1)$ to $V(G_2)$, G is also Hamiltonian. This contradiction establishes the lemma for the case $d \geq 5$. For d=3, the only possibility is that G is the graph in Figure 4.1.

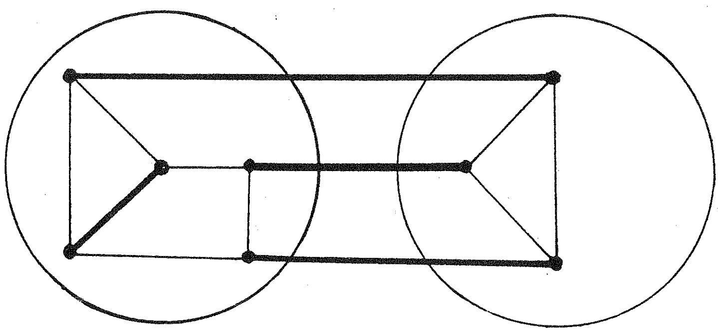


Figure 4.1

The edges drawn in solid lines indicate the edges of F. Clearly the graph has a one-factorization, again a contradiction. This completes the proof of the lemma. □

We demonstrate that the bounds on n given in the above lemma are best possible. For d=3 the graph displayed in Figure 4.2 is a 3-regular graph on 10 vertices having exactly one 1-factor (the edges on a 1-factor are drawn in solid lines).

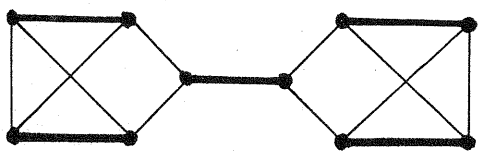


Figure 4.2

For odd d ≥ 5 the graph displayed in Figure 4.3 is a d-regular graph on 2(d + 2) vertices having exactly one one-factor.

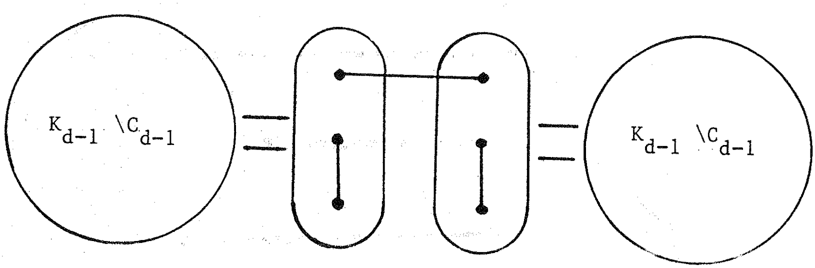


Figure 4.3

For even $d \geq 4$ the construction is a little more complicated and we describe it as follows. Our building block is the graph:

$$H(d+1,x) = (K_x \setminus \{\text{a maximum matching}\}) \vee K_{d+1-x}$$

on $d + 1$ vertices. Observe that for odd x this graph has $d + 2 - x$ vertices of degree d and $x - 1$ vertices of degree $d - 1$. Now consider three such graphs $H_1(d + 1,x)$, $H_2(d + 1,y)$ and $H_3(d + 1,z)$, where x,y and z are odd positive integers whose sum is $d - 1$. Identify a pair of adjacent vertices of degree d in each graph; call these pairs u,u'; v,v'; and w,w'. Consider the graph G on $3d + 4$ formed by taking the union of the graphs $H_1(d + 1,x)$, $H_2(d + 1,y)$ and $H_3(d + 1,z)$ and then adding a new vertex, α say and joining α to every vertex that has degree $d - 1$. Now form G' from G by deleting the edges uu', vv', ww' and adding the edges $u'w'$ and $v'\alpha$. The graph G' is displayed in Figure 4.4.

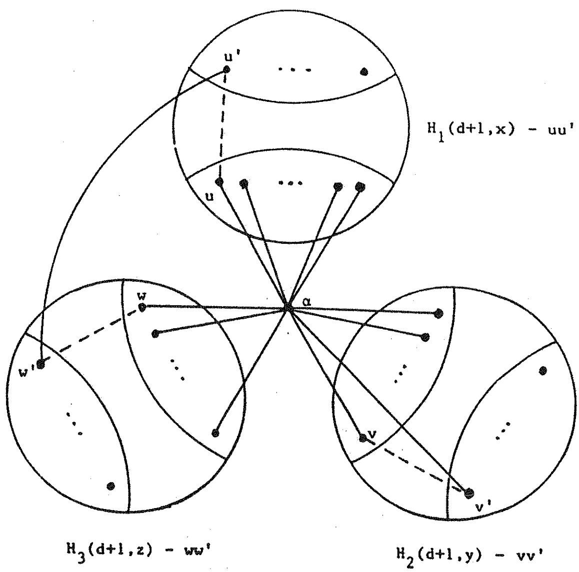


Figure 4.4

Note that the edge u'w' is the only edge between two vertices in different H_i's. It is easy to exhibit a one-factor F in G and as this one-factor must contain the edge u'w', the graph G\{F} has no one-factor. As G is d-regular this establishes the sharpness of the bound for even d. Figure 4.5 gives G when d = 4.

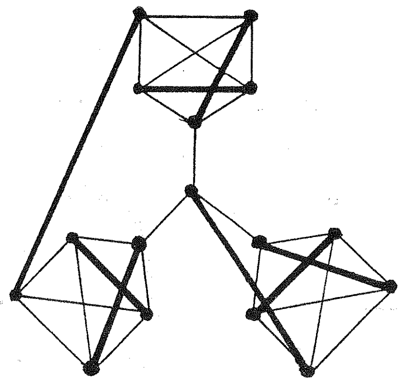


Figure 4.5

Lemma 4.2 : Let G be a 5-regular graph on 2n vertices containing exactly three 1-factors. Then $2n \geq 14$.

Proof: Suppose not and $2n \leq 12$. Let F_1, F_2 and F_3 be the three 1-factors of G. Then the subgraph $H = G \setminus \{F_1, F_2, F_3\}$ is 2-regular and hence is the union of cycles. Since H cannot have a 1-factor it must have at least two odd cycles. Since $H \cup F_i$, $1 \leq i \leq 3$, is a 3-regular graph with exactly one 1-factor, Lemma 4.1 implies that $2n \geq 10$. Hence either $2n = 10$ or $2n = 12$.

If $2n = 10$, then H consists of either two 3-cycles and a 4-cycle or of exactly two odd cycles, C_1 and C_2 say. Consider $H \cup F_i$, $1 \leq i \leq 3$. If H consists of two 3-cycles and a 4-cycle then it is easy to establish that $H \cup F_i$ is Hamiltonion. So suppose H consists of two odd cycles C_1 and C_2. If 2 or more edges of F_i join vertices in C_1 to vertices in C_2, then $H \cup F_i$ is Hamiltonian and hence has two 1-factors. As this is not possible, each $H \cup F_i$ has a cut edge (which necessarily belongs to F_i). Consequently C_1 and C_2 are each cycles of length 5. Further, C_1 has two edges of F_i, $1 \le i \le 3$. Hence $G[C_1]$ has 5 vertices and 11 edges, an impossibility. Hence $2n \ne 10$.

Suppose $2n = 12$. Then H contains either 2 or 4 odd cycles. If H contains 2 odd cycles it may contain an even cycle. We consider several cases separately.

Case 1 : Suppose H contains 4 odd cycles, C_1, C_2, C_3 and C_4.

As there are only 12 vertices each C_i must be a 3-cycle. Consider $H \cup F_i$, $1 \le i \le 3$. If $H \cup F_i$ has two edges between a pair of C_i's then it must be Hamiltonian (see Figure 4.6), a contradiction. Hence $H \cup F_i$ is connected and there is exactly one edge between every pair of C_i's. But then the only possibility is the graph of Figure 4.7 which is Hamiltonian.

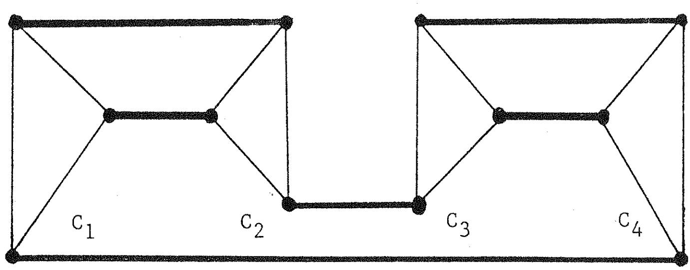


Figure 4.6

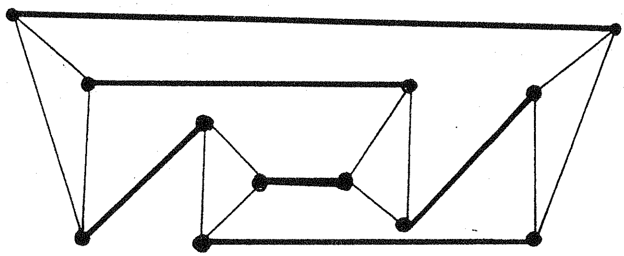


Figure 4.7

Case 2: H contains 2 odd cycles, C_1 and C_2 and one even cycle C_3.

Consider $H \cup F_i$, $1 \le i \le 3$. If there are 2 or more edges of F_i joining vertices in C_i to vertices in C_j, $j \ne i$, then $H \cup F_i$ is Hamiltonian as the only possibilities are the graphs displayed in Figure 4.8. But this is not possible.

Case 3: H consists of exactly 2 odd cycles, C_1 and C_1.

Consider $H \cup F_i$, $1 \le i \le 3$. If there is 2 or more edges of F_i joining vertices in C_1 to vertices in C_2, then using a case analysis similar to that used above we can conclude that $H \cup F_i$ is Hamiltonian. As this is not possible, each $H \cup F_i$ has a cut edge (which necessarily belongs to F_i). Consequently C_1 and C_2 are cycles of length 5 and 7 repectively. Suppose C_1 has length 5. Then C_1 has two edges of F_i, $1 \le i \le 3$. Hence $G[C_1]$ has 5 vertices and 11 edges, an impossibility. Hence $2n \ne 12$. This completes the proof of the lemma.

□

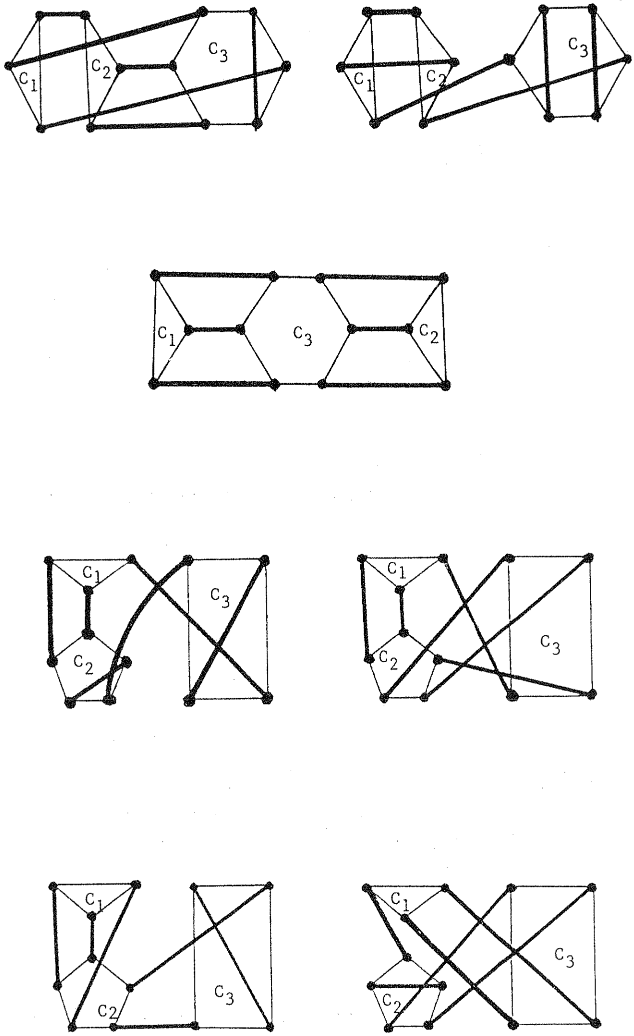


Figure 4.8

We now establish the existence of a proper premature set of (2n - 4) 1-factors in K_{2n} for $2n \geq 10$. Our proof which makes use of Theorem 3.1 is shorter than that given by Wallis [13] which explicitly constructed (2n - 4) 1-factors in K_{10}, K_{12}, K_{14} and K_{16}; application of Theorem 3.1 avoids the need to look at K_{14} and K_{16}.

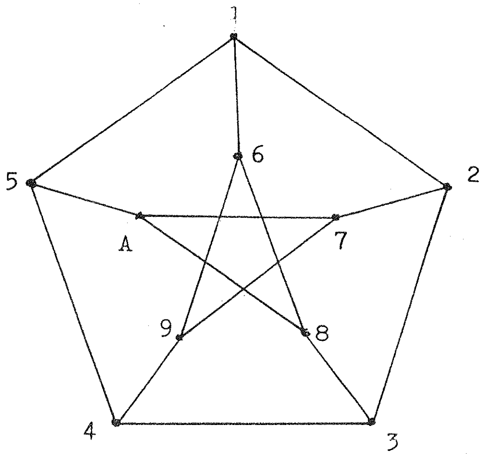
Theorem 4.1: There exists a proper premature set of (2n - 4) 1-factors in K_{2n} whenever $2n \geq 10$.

Proof : If F is a proper premature set of (2n-4) 1-factors in K_{2n}, then $\overline{F}$ is 3-regular and contains exactly one 1-factor. Lemma 4.1 implies that $2n \geq 10$. We now construct proper premature sets of (2n-4) 1-factors in K_{10} and K_{12}. Theorem 3.1 then guarantees the existence of a proper premature set of (2n-4) 1-factors in all the larger graphs of even order.

Consider the Petersen graph, P_{10} (see Figure 4.9) and the graph P_{12} drawn in Figure 4.10. It is well known that P_{10} has a 1-factor but no 1-factorization. The set $F_1, F_2, \ldots, F_6$ of 1-factors in K_{10} on vertices 1,2,..., 9,A defined by :

F_1	=	14	26	35	78	9A
F_2	=	1A	24	36	57	89
F_3	=	13	28	4A	59	67
F_4	=	18	2A	39	56	74
F	=	17	29	3A	46	58
F_6	=	19	25	37	48	6A

has as a leave the graph shown in Figure 4.9. Hence it forms a proper premature set of six 1-factors in K_{10}.



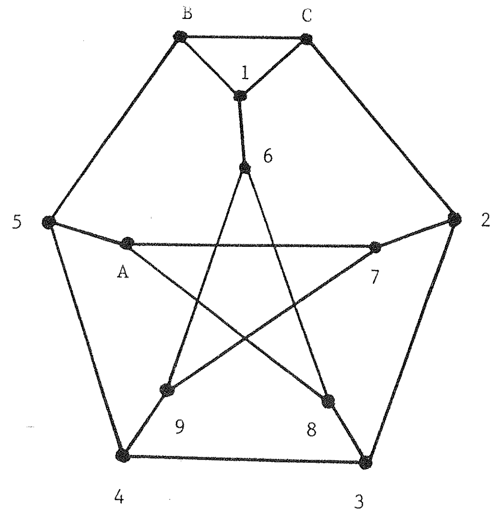
P_{10}

Figure 4.9

Wallis [13] proved that the graph P_{12} has a one-factor but no one-factorization. Its complement, $\overline{P}_{12}$, is a proper premature set of 8 one-factors in K_{12}. A complete 1-factorization $F_1, F_2, \ldots, F_8$ of $\overline{P}_{12}$ is

F_1	=	12	59	36	48	7B	AC
F_2	=	15	24	39	7C	8B	6A
F_3	=	14	2B	35	9A	78	6C
F_4	=	1A	26	3C	4B	57	89
F_5	=	13	28	4A	5C	9B	67
F_6	=	18	2A	3B	56	9C	47
F_7	=	17	29	3A	4C	58	6B
F_8	=	19	25	37	46	8C	AB

Hence, $F_1, F_2, \ldots, F_8$ form a proper premature set in K_{12}.
This completes the proof of the theorem. □



P_{12}

Figure 4.10

Next we apply Theorem 3.1 to establish the existence of a proper

premature set of (2n-6) 1-factors in K_{2n}. There cannot be any such sets for $2n \le 12$ (Theorem 2.2 and Lemma 4.2). In view of the Corollary to Theorem 3.1 we need to exhibit proper premature sets of (2n-6) 1-factors in K_{14}, K_{16} and K_{18}.

Consider the graph in K_{14} on vertices 1,2,...,9,A,...,E. If we take the 1-factors:

F_1	=	13	2E	4C	59	6B	7D	8A
F_2	=	1A	2C	3E	48	5B	6D	79
F_3	=	1E	24	3C	57	6A	8D	9B
F_4	=	1C	29	3B	4D	58	6E	7A
F_5	=	1B	2A	38	49	5D	6C	7E
F_6	=	19	2B	3D	4A	5E	68	7C
F_7	=	1D	28	3A	4E	5C	69	7B
F_8	=	16	2D	39	4B	5A	78	CE

then the leave of this set is the graph in Figure 4.11.

This graph contains exactly one 1-factor, and this 1-factor contains the edge 18.

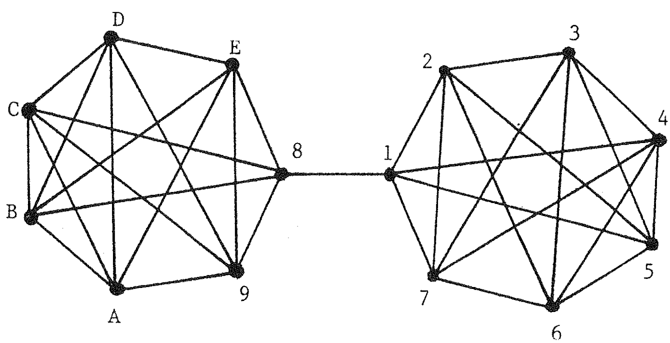


Figure 4.11

Consider the graph K_{16} on vertices 1,2, ... ,9,A, ...,G. If we take the 1-factors:

F_1	=	24	3G	5B	6D	7E	1F	8A	9C
F_2	=	2C	3E	4F	5D	6G	7A	19	8B
F_3	=	2D	13	4B	59	6C	7F	8E	AG
F_4	=	2B	39	48	5F	6E	7D	1G	AC
F_5	=	2A	3C	4E	5G	69	7B	1D	8F
F_6	=	2F	3A	4D	5C	16	78	EG	9B
F_7	=	2E	3B	4A	57	68	1C	9F	DG
F_8	=	29	38	4C	5E	6B	7G	1A	DF
F_9	=	2G	3F	49	58	6A	7C	1E	BD
F_{10}	=	28	3D	4G	5A	6F	79	1B	CE

then the leave of this set is shown in Figure 4.12. This graph has exactly one 1-factor. Hence the set $F_1, F_2, \ldots, F_{10}$ forms a proper premature set of 1-factors in K_{16}.

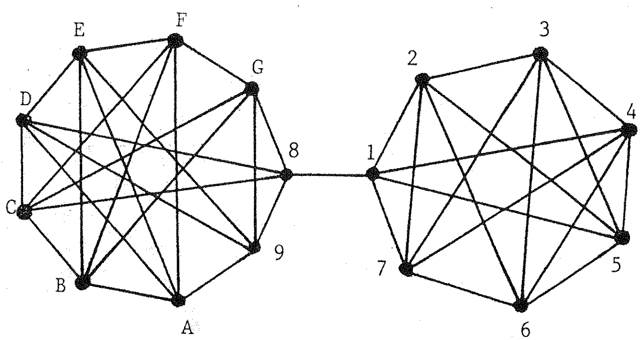


Figure 4.12

Consider the graph K_{18} on vertices 1,2,...,9,A,...,I. If we take the 1-factors:

F_1 =	24	3I	5E	6G	7B	8D	9H	1F	AC
F_2 =	25	3H	4B	6I	7F	8C	9A	1D	EG
F_3 =	28	35	4F	6E	7C	9G	1I	AD	BH
F_4 =	2A	3D	4E	5H	6C	17	8F	9B	GI
F_5 =	2B	39	4I	5C	68	7E	1G	AH	DF
F_6 =	2C	3G	4H	5F	6D	7A	18	9I	BE
F_7 =	2F	3A	14	5G	69	7H	8I	BD	CE
F_8 =	2D	3B	4G	5A	6H	79	8E	1C	FI
F_9 =	2E	3F	4D	5B	6A	7I	8G	9C	1H
F_{10} =	2G	3E	46	57	9D	1B	CI	8A	FH
F_{11} =	2I	3C	4A	5D	6B	7G	8H	9F	1E
F_{12} =	2H	13	4C	5I	6F	7D	8B	9E	AG

then the leave of this set is shown in Figure 4.13. This graph has exactly one 1-factor. Hence the above set forms a proper premature set of 1-factors in K_{18}.

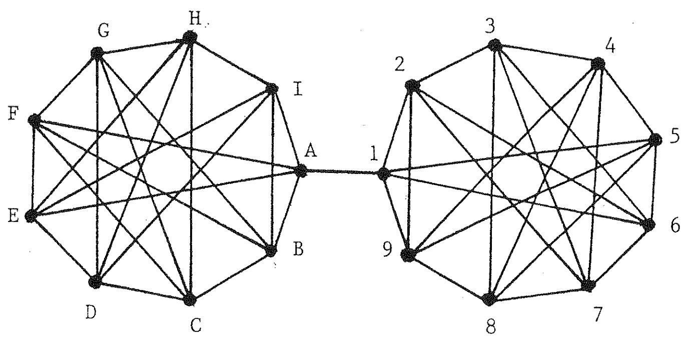


Figure 4.13

We have proved:

Theorem 4.2: There exists a proper premature set of (2n-6) one-factors in K_{2n} whenever $2n \geq 14$. □

5. SOME OPEN PROBLEMS

We conslude this paper by mentioning some open problems. The first problem concerns the order of a graph having exactly t 1-factors.

Problem 1: Let G be a d-regular graph with exactly t 1-factors, but no 1-factorization. Determine the minimum number of vertices of G.

Lemmas 4.1 and 4.2 resolve this problem for t = 1 and for t = 3 when d = 5, respectively. Solution of Problem 1 would assist in determining the spectrum of proper premature sets of 1-factors in K_{2n}.

We mentioned in the introduction that recently Hoffman, Rodger and Rosa [7] completely determined the spectrum of maximal sets of 2-factors and Hamiltonian cycles of K_n. Their approach is complicated and involves the application of Tutte's f-factor theorem [11]. It is natural to ask whether the approach adopted by Caccetta and Mardiyono [4] to maximal sets of 1-factors could be extended to maximal sets of Hamiltonian cycles. We can make progress on this provided the following is true.

Problem 2: Let G be a graph on 2n vertices formed by the union of k edge-disjoint Hamiltonian cycles $C_1, C_2, \ldots, C_k$. Suppose the edges of cycle C_i are coloured with colour i, $1 \leq i \leq k$. Does G contain a maximum matching consisting of k edges, each of a different colour.

Problem 2 is, of course, of interest in its own right. We conjecture that the answer to the question is yes.

Our final problem concerns maximal sets of 1-factors in graphs which are not complete.

Problem 3: Let G be a k-regular graph (k < 2n - 1) on 2n vertices having a 1-factorization. Determine the spectrum of maximal sets of 1-factors of G.

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