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A note on the existence of factors in squares of graphs

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Abstract

Let G be a simple connected graph such that $\delta(G) \geq 3$. For every function $f : V(G) \to \{1,2\}$, where $\sum_{x \in V(G)} f(x)$ is even, the square graph G^2 has an f-factor.

All graphs considered are assumed to be simple and finite. We refer the reader to [2] for standard graph theoretic terms not defined in this paper.

Let G be a graph. The degree $d_G(u)$ of a vertex u in G is the number of edges of G incident with u. The minimum degree of G is denoted by $\delta(G)$. If X and Y are subsets of $V(G)$, we will write $E_G(X,Y)$ and $e_G(X,Y)$ for the set and the number, respectively, of the edges of G joining X to Y.

For any set X of vertices in G, we define the neighbour set of X in G to be the set of all vertices adjacent to vertices in X; this set is denoted by $N_G(X)$. A set of vertices in G is said to be independent if no two of them are adjacent. The edge analogue of an independent set is a set of edges in G no two of which have a common end.

Let X be a nonempty subset of $V(G)$. The subgraph of G whose vertex set is X and whose edge set is the set of those edges of G that have both ends in X is called the subgraph of G induced by X and is denoted by $G[X]$; we say that $G[X]$ is an induced subgraph of G.

The square G^2 is a graph having the same vertex set as G, and two vertices are adjacent in G^2 if they are at distance 1 or 2 apart in G. Given a function $f : V(G) \to Z^+$, we say that G has an f-factor if there exists a spanning subgraph H of G such that $d_H(x) = f(x)$ for every $x \in V(G)$. If f is the constant function taking the value r, then an f-factor is said to be an r-factor.

A necessary and sufficient condition for a graph to have an f-factor was obtained by Tutte [8] in 1952.

Tutte's f-factor Theorem [8]: *A graph G has an f-factor if and only if*

$$q_G(D,S;f) + \sum_{x\in S}(f(x) - d_{G-D}(x)) \leq \sum_{x\in D} f(x)$$

for all sets $D, S \subseteq V(G)$ with $D \cap S = \varnothing$ where $q_G(D,S;f)$ denotes the number of components C of $(G-D)-S$ such that

$$e_G(V(C), S) + \sum_{x\in V(C)} f(x)$$

is odd. (Sometimes we refer to these as odd components.)

Tutte also noted that

$$q_G(D,S;f) + \sum_{x\in S}(f(x) - d_{G-D}(x)) - \sum_{x\in D} f(x) \equiv \sum_{x\in V(G)} f(x) (\mathrm{mod}\ 2). \qquad (1)$$

Several authors have shown that if G is a connected graph of even order, then its square G^2 has a 1-factor ([7], [3]).

In 1973, Hobbs obtained the following result.

Theorem 1 [5]: *If $\delta(G) \geq 2$, then G^2 contains a 2-factor.*

An extension of Theorem 1 was also obtained in [1].

Fleischner used a stronger condition and obtained the following theorem.

Theorem 2 [4]: *If G is 2-connected, then G^2 is hamiltonian.*

A generalization of Hobbs's result was obtained recently by Fourtounelli and Katerinis.

Theorem 3 [6]: *Let G be a connected graph and k a positive integer such that (i) $k|V(G)|$ is even, (ii) $\delta(G) \geq k$. Then G^2 contains a k-factor.*

The main purpose of this paper is to present the following theorem which fits into the above-mentioned literature.

Theorem 4 : *Let G be a simple connected graph such that $\delta(G) \geq 3$. For every function $f : V(G) \to \{1,2\}$, where $\sum_{x\in V(G)} f(x)$ is even, G^2 has an f-factor.*

For the proof of Theorem 4, we shall need the following lemma.

Lemma 1: *Let G be a graph and let function $f : V(G) \to \{1,2\}$, where $\sum_{x\in V(G)} f(x)$ is even. Suppose that there exist $D, S \subseteq V(G)$ with $D \cap S = \varnothing$ such that*

$$q_G(D,S;f) + \sum_{x\in S}(f(x) - d_{G-D}(x)) \geq \sum_{x\in D} f(x) + 2. \qquad (2)$$

If S is minimal with respect to (2), then *(a)* $E_G(S, S) = \varnothing$;

(b) For every $x \in S$, $N_{G-D}(x)$ *contains* $d_{G-D}(x)$ *vertices belonging to* $d_{G-D}(x)$ *different odd components of* $(G - D) - S$.

Proof. Define $W = (G - D) - S$, $L = G[S]$.

(a) Suppose that $E_G(S, S) \neq \varnothing$, that is to say S is not an independent set. Then there exists a vertex $u \in S$ such that $d_L(u) \geq 1$.

Define $S' = S - \{u\}$. Then

$$\begin{aligned} q_G(D, S'; f) &\geq q_G(D, S; f) - (d_{G-D}(u) - d_L(u)) \\ &\geq q_G(D, S; f) - d_{G-D}(u) + 1 \end{aligned}$$

and

$$\sum_{x \in S'} (f(x) - d_{G-D}(x)) \geq \sum_{x \in S} (f(x) - d_{G-D}(x)) - (f(u) - d_{G-D}(u)).$$

Hence

$$q_G(D, S'; f) + \sum_{x \in S'} (f(x) - d_{G-D}(x)) \geq q_G(D, S; f) + \sum_{x \in S} (f(x) - d_{G-D}(x)) - f(u) + 1$$

and by using (2), we have

$$\begin{aligned} q_G(D, S'; f) + \sum_{x \in S'} (f(x) - d_{G-D}(x)) &\geq \sum_{x \in D} f(x) + 2 - f(u) + 1 \\ &\geq \sum_{x \in D} f(x) + 1 \text{ since } f(u) \leq 2. \end{aligned} \quad (3)$$

But using (1), (3) yields

$$q_G(D, S'; f) + \sum_{x \in S'} (f(x) - d_{G-D}(x)) \geq \sum_{x \in D} f(x) + 2,$$

which contradicts the minimality of S with respect to (2). So S is an independent set in G.

(b) Suppose that there exists $u \in S$ such that $N_{G-D}(u)$ contains at least two vertices belonging to the same odd component of W or $N_{G-D}(u)$ contains one vertex belonging to a non-odd component of W.

Define $S' = S - \{u\}$. Then

$$q_G(D, S'; f) + \sum_{x \in S'} (f(x) - d_{G-D}(x)) \geq \sum_{x \in D} f(x) + 2 \quad (4)$$

since

$$q_G(D,S^{'};f) \geq q_G(D,S;f) - (d_{G-D}(u)-1),$$

$$\begin{aligned}\sum_{x\in S'}(f(x)-d_{G-D}(x)) &\geq \sum_{x\in S}(f(x)-d_{G-D}(x)) - (f(u)-d_{G-D}(u)) \\ &\geq \sum_{x\in S}(f(x)-d_{G-D}(x)) + d_{G-D}(u) - 2 \text{ since } f(u)\leq 2\end{aligned}$$

and by using (1). But (4) contradicts the minimality of S with respect to (2). So this case cannot occur. □

Proof of Theorem 4:

Suppose that G^2 has no f-factor. Then by Tutte's f-factor theorem, there exist $D, S \subseteq V(G^2)$ with $D\cap S=\varnothing$ such that

$$q_{G^2}(D,S;f) + \sum_{x\in S}(f(x)-d_{G^2-D}(x)) \geq \sum_{x\in D} f(x) + 2. \tag{5}$$

We assume that S is minimal with respect to (5). Then S is an independent set in G^2 by Lemma 1(a).

Furthermore we note that the next inequality follows easily from (5)

$$q_{G^2}(D,S;f) + \sum_{x\in S}(2-d_{G^2-D}(x)) \geq |D| + 2. \tag{6}$$

Define $W{=}(G^2-D)-S$ and let Q be the set of odd components of $(G^2-D)-S$ and let E be the set of components of $(G^2-D)-S$ which are not odd. Moreover define

$$Q_i = \{C\in Q | e_{G^2}(V(C),S)=i\}, \quad |Q_i| = q_i,$$
$$E_i = \{C\in E | e_{G^2}(V(C),S)=i\}, \quad |E_i| = e_i, \text{ for } i=0,1,2,\dots\ .$$

We will first prove that $d_{G-D}(x) = |N_G(x)\cap V(W)| \leq 1$ for every $x\in S$.

Suppose that there exists $u\in S$ such that $|N_G(u)\cap V(W)|\geq 2$ and let $u_1,u_2\in N_G(u)\cap V(W)$. Then u_1,u_2 are adjacent in G^2. This means that u is adjacent to two vertices belonging to the same component of W, contradicting Lemma 1(b). Thus for every $x\in S$, $d_{G-D}(x)=|N_G(x)\cap V(W)|\leq 1$.

Define $S_i=\{x\in S|d_{G-D}(x)=i\}$ for $i=0,1$. Clearly $|S|=|S_0|+|S_1|$.

For every element A_i of Q_0 there exists $v_i\in V(A_i)$ such that $N_G(v_i)\cap D\neq\varnothing$, since G (and hence G^2) is a connected graph. Let $M=\{v_1,v_2,\dots,v_{q_0}\}$.

Clearly $S\cup M$ is an independent set in G^2. We claim that $|N_G(x)\cap(S\cup M)|\leq 1$ for every $x\in D$. Indeed, assume the contrary and let u be a vertex of D such that

$|N_G(u) \cap (S \cup M)| \geq 2$. Let $u_1, u_2 \in N_G(u) \cap (S \cup M)$. Then u_1, u_2 are adjacent in G^2. But if this is the case, $S \cup M$ is not an independent set in G^2. So we obtain that

$$|N_G(x) \cap (S \cup M)| \leq 1 \text{ for every } x \in D$$

and thus

$$|D| \geq e_G(D, S \cup M) \geq q_0 + 3|S_0| + 2|S_1|$$

since $\delta(G) \geq 3$. Hence

$$q_0 + 2|S| \leq |D|. \tag{7}$$

Now (6) implies

$$q_0 + q_1 + q_2 + \cdots + 2|S| - q_1 - 2q_2 - \cdots - e_1 - 2e_2 - \cdots \geq |D| + 2$$

and thus

$$q_0 + 2|S| \geq |D| + 2,$$

contradicting (7).

This completes the proof of Theorem 4. $\square$

We next show that the minimum degree condition of Theorem 4 is in some sense best possible by describing a family of graphs having slightly lower minimum degree and having no f-factors.

We construct such graphs G as follows. We start from two copies of K_3. We next choose one vertex from each copy and we join them. Let H be the resulting graph. We take $n \geq 3$ copies of H and for every such copy H_i, we choose one vertex with degree 3. We join all these vertices having degree 3 to a new vertex u. For the resulting graph G we have that $\delta(G) = 2$. Furthermore define

$$S = \{v \in V(G) | d_G(v) = 2\}, \quad D = V(G) - S$$

and let f be a function $f : V(G) \to \{1, 2\}$ such that $f(x) = 2$ for every $x \in S$, $f(x) = 1$ for every $x \in D - \{u\}$, and $f(u) = 2$. Clearly $\sum_{x \in V(G)} f(x)$ is even, but G^2 has no f-factor, since

$$q_{G^2}(D, S; f) = 0, \quad \sum_{x \in S} (f(x) - d_{G^2 - D}(x)) = |S| = 2|D| - 2, \quad \sum_{x \in D} f(x) = |D| + 1$$

and thus

$$q_{G^2}(D, S; f) + \sum_{x \in S} (f(x) - d_{G^2 - D}(x)) \geq \sum_{x \in D} f(x) + 2,$$

contradicting Tutte's f-factor Theorem.

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