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# Almost resolvable minimum coverings of complete graphs with 4-cycles

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**Abstract**

If the complete graph $K_n$ has vertex set $X$, a minimum covering of $K_n$ with 4-cycles, $(X, C, P)$, is a partition of the edges of $K_n \cup P$ into a collection $C$ of 4-cycles, where $P$ is a subgraph of $\lambda K_n$ and the number of edges in $P$ is as small as possible. An almost parallel class of a minimum covering of $K_n$ with 4-cycles is a largest possible collection of vertex disjoint 4-cycles (so with $\lfloor n/4 \rfloor$ 4-cycles in it).

In this paper, for all orders $n$, except order 9 which does not exist, we exhibit a minimum covering of $K_n$ with 4-cycles so that the 4-cycles in the

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covering are resolvable into almost parallel classes, with any remaining 4-cycles being vertex disjoint.

We also complete the missing examples of order 23 for the same problem with almost resolvable *maximum packings* with 4-cycles, and orders 41 and 57 in the case of an exact decomposition into 4-cycles, partitioned into almost parallel classes.

## 1 Introduction

A 4-cycle system of order $n$ is a pair $(X, C)$, where $C$ is a collection of 4-cycles which partitions the edge set of $K_n$ with vertex set $X$. It is well-known [3] that the spectrum (that is, the set of admissible orders) for 4-cycle systems is precisely the set of all $n \equiv 1 \pmod 8$, and that if $(X, C)$ is a 4-cycle system of order $n$, then $|C| = n(n-1)/8$.

Clearly a 4-cycle system *cannot* contain a parallel class because its order is never 0 (mod 4). So we make the following definitions. An *almost parallel class* of 4-cycles of the complete graph $K_n$ is a largest possible collection of $\lfloor n/4 \rfloor$ vertex disjoint 4-cycles. An *almost resolvable* 4-cycle system is a pair $(X, C)$ where $C$ is partitioned into a maximum collection of almost parallel classes such that the remaining 4-cycles are vertex disjoint. This amounts to $(n-1)/2$ almost parallel classes (each consisting of $(n-1)/4$ 4-cycles) and a half parallel class consisting of $(n-1)/8$ vertex disjoint 4-cycles. In [2] the spectrum for almost resolvable 4-cycle systems was shown to be all $n \equiv 1 \pmod 8$, $n \geqslant 17$, with the two possible exceptions of $n = 41$ and 57.

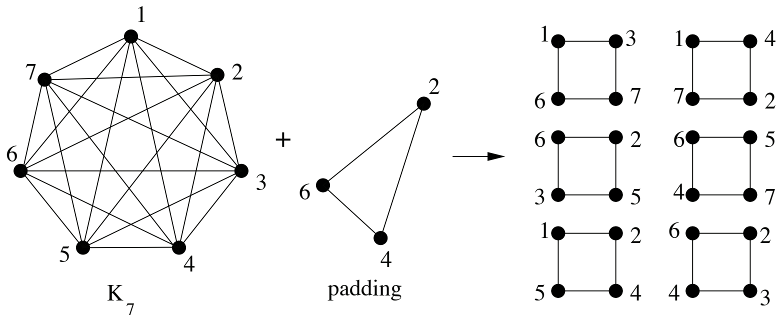
An almost resolvable 4-cycle system of order 9 does not exist [2]. However, a resolvable *covering* with padding of one 4-cycle does exist; see the Appendix. In this paper we completely settle the existence problem for almost resolvable 4-cycle systems by giving examples for $n = 41$ and 57 (in the Appendix).

We now turn attention to other orders $n$ which are not 1 (mod 8). It is reasonable to see how close to a 4-cycle system we can get; so *maximum packings* are considered. A packing of $K_n$ with 4-cycles is a triple $(X, C, L)$, where $C$ is a collection of edge disjoint 4-cycles of $K_n$, with vertex set $X$, and $L$ is the collection of edges not belonging to any of the 4-cycles in $C$. The set $L$ is called the *leave*. If $C$ is as large as possible (or if $|L|$ is as small as possible) then $(X, C, L)$ is said to be a *maximum packing* of $K_n$ with 4-cycles. (We remark that if $n \equiv 1 \pmod 8$, then $L = \emptyset$ and $(X, C, L)$ is a 4-cycle system.) The problem of constructing maximum packings of $K_n$ with 4-cycles was settled in [4]. The problem of constructing almost resolvable maximum packings was settled in [1] with the possible exception of order $n = 23$. We fill in this possible exception for $n = 23$ with an example (in the Appendix), thereby completing the problem of providing almost resolvable maximum packings of $K_n$ for all $n$.

Having dealt with almost resolvable maximum packings, the obvious case to consider is that of almost resolvable minimum *coverings*.

A minimum covering of $K_n$ with 4-cycles is a triple $(X, C, P)$, where $K_n$ has vertex set $X$, the set $P$ is a subgraph of $\lambda K_n$ called the *padding* (where $\lambda$ indicates possible edge-multiplicity, $\lambda \geqslant 1$), the number of edges in $P$ is as small as possible, and $C$ is a partition of $K_n \cup P$ into 4-cycles.

**Example 1.1** *A minimum covering of $K_7$ with 4-cycles.*




□

Since an almost parallel class in the above example is simply a 4-cycle, Example 1.1 is also an almost resolvable minimum covering of $K_7$ with 4-cycles.

The object of this paper is threefold:

(i) the complete solution of the almost resolvable minimum covering of $K_n$ with 4-cycles;

(ii) examples of almost resolvable 4-cycle systems of orders 41 and 57 (thereby giving, with [2], a complete solution for 4-cycle systems); and

(iii) an example of an almost resolvable maximum packing of $K_{23}$ with 4-cycles, thus (with [1]) completing the solution for almost resolvable maximum packings.

The examples needed in cases (ii) and (iii) are given in the Appendix. From now on we concentrate on case (i).

Table 1 gives a summary of minimum coverings of $K_n$ with 4-cycles. We will produce minimum coverings in each case which are almost resolvable.

We will collect our results in five sections, followed by an Appendix.

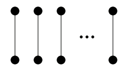
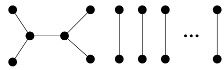
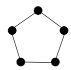
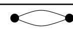
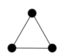
| Order $n$ | Padding |
|---|---|
| 1 (mod 8) | $\emptyset$ (4-cycle system) |
| 0, 4 (mod 8) |  1-factor |
| 2, 6 (mod 8) |  and others |
| 3 (mod 8) |  5-cycle |
| 5 (mod 8) |  double edge |
| 7 (mod 8) |  triangle |

Table 1: Minimum coverings of $K_n$ with 4-cycles: the paddings.

## 2 Orders 0, 2, 4 or 6 (mod 8)

In [1] an almost resolvable maximum packing of $K_n$ is given for all orders $n \equiv 0, 2, 4$ or 6 (mod 8). The leave in each case is a 1-factor of $K_n$. We will use these solutions to construct almost resolvable minimum coverings. There are two cases to consider here: (i) $n \equiv 0$ or 4 (mod 8), and (ii) $n \equiv 2$ or 6 (mod 8). We handle each in turn.

(i) $n \equiv 0$ or 4 (mod 8), $n \geqslant 4$

Let $(X, C, L)$ be an almost resolvable maximum packing of $K_n$ with 4-cycles where $L$ is the 1-factor consisting of the edges in Figure 1.

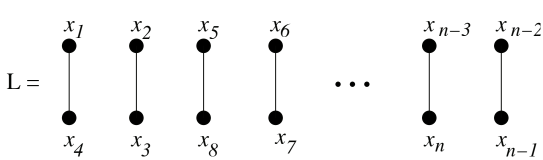



Figure 1.

Since $n \equiv 0$ or 4 (mod 8), $L$ consists of an even number of edges. Take $P$ as in Figure 2, and then let $C^*$ be as in Figure 3.

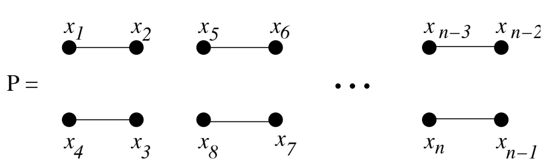



Figure 2.

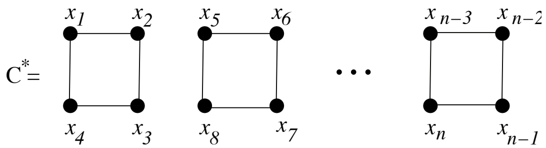



Figure 3.

Then $(X, C \cup C^*, P)$ is an almost resolvable minimum covering of $K_n$ with 4-cycles, with padding the 1-factor $P$.

(ii) $n \equiv 2$ or $6 \pmod 8$, $n \geqslant 6$

Let $(X, C, L)$ be an almost resolvable maximum packing of $K_n$ with 4-cycles where $L$ is the leave given in Figure 4.

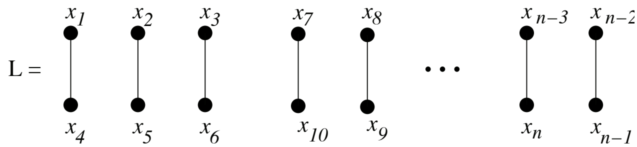



Figure 4.

Since $n \equiv 2$ or $6 \pmod 8$, $L$ consists of an odd number of edges. Take $P$ as in Figure 5, and then let $C^*$ be as in Figure 6.

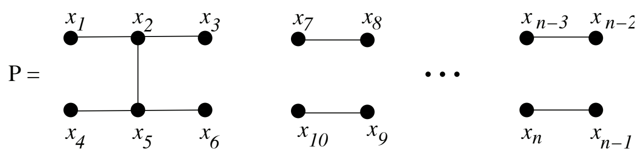



Figure 5.

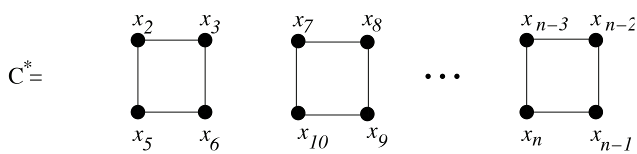



Figure 6.

Then $(X, C \cup C^* \cup \{(x_1, x_2, x_5, x_4)\}, P)$ is an almost resolvable minimum covering of $K_n$ with 4-cycles, with padding $P$. (We remark that the cycle $(x_1, x_2, x_5, x_4)$ is in a short class on its own.)

We have now proved:

**Lemma 2.1** *There exists an almost resolvable minimum covering of $K_n$ with 4-cycles for all $n \equiv 0 \pmod 2$, $n \geqslant 4$.* □

## 3 Order $n \equiv 3 \pmod 8$

In this case the padding is a 5-cycle. We begin with the following two examples.

**Example 3.1** *Order* $n = 11$.

Let $X = \{\infty, 0_1, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2\}$. Then $(X, C, P)$ is an almost resolvable covering, where the padding is $P = (0_2, 2_2, 4_2, 1_2, 3_2)$, and $C$ consists of the following seven partial parallel classes of two 4-cycles each, and one short class of one 4-cycle:

$\{\{(\infty, (3+i)_1, (0+i)_2, (1+i)_2),\ ((0+i)_1, (4+i)_2, (2+i)_2, (4+i)_1)\} \mid i \in \mathbb{Z}_5\}$;

$\{(0_1, 2_1, 3_2, 0_2),\ (1_1, 3_1, 4_2, 1_2)\}$; $\{(2_1, 4_1, 0_2, 2_2),\ (3_1, 0_1, 1_2, 3_2)\}$

Short class: $\{(1_1, 2_2, 4_2, 4_1)\}$.

□

**Example 3.2** *Order* $n = 19$.

Let $X = \{a, b, c\} \cup \{i_1, i_2 \mid i \in \mathbb{Z}_8\}$. Then $(X, C, P)$ is an almost resolvable covering of order 19, where the padding is $P = (a, 2_1, c, 3_1, 1_1)$ and $C$ consists of the following 11 almost parallel classes, each containing four 4-cycles.

$\{(a, b, c, 2_1), (0_2, 2_2, 4_2, 6_2), (1_1, 3_2, 5_1, 7_2), (3_1, 5_2, 7_1, 1_2)\}$;

$\{((a, c, 3_1, 1_1), (1_2, 3_2, 5_2, 7_2), (0_1, 2_2, 4_1, 6_2), (2_1, 4_2, 6_1, 0_2)\}$;

$\{(0_1, 4_1, 4_2, 0_2), (1_1, 5_1, 5_2, 1_2), (2_1, 6_1, 6_2, 2_2), (3_1, 7_1, 7_2, 3_2)\}$;

$$\{\{(a, (7+i)_1, (4+i)_1, (0+i)_2), (b, (5+i)_1, (3+i)_1, (2+i)_2), (c, (2+i)_1, (7+i)_2, (6+i)_2), ((0+i)_1, (1+i)_1, (4+i)_2, (1+i)_2)\} \mid i \in \mathbb{Z}_8\}.$$

□

We will now use the $8n+3$ Construction for almost resolvable maximum packings in [1] to obtain an almost resolvable minimum covering for orders $8n + 3 \geqslant 27$. Let $(X, C, L)$ be the almost resolvable maximum packing of order $8n + 3 \geqslant 27$, obtained in [1], where the leave $L$ is the 3-cycle $(\infty_1, \infty_2, \infty_3)$. Then $C$ contains an almost parallel class $\pi_1 = \{f_1, g_1, f_2, g_2, \ldots, f_n, g_n\}$ and a partial parallel class $\pi_2 = \{h_1, h_2, \ldots, h_n\}$ such that

(i) each of $\pi_1$ and $\pi_2$ miss the leave $(\infty_1, \infty_2, \infty_3)$; and

(ii) $v(h_i) \subseteq v(f_i) \cup v(g_i)$, where $v(f_i)$, $v(g_i)$ and $v(h_i)$ are the vertex sets of $f_i$, $g_i$ and $h_i$.

Now let $a \in (v(f_1) \cup v(g_1)) \setminus v(h_1)$; $b, c \in (v(f_2) \cup v(g_2)) \setminus v(h_2)$, and let $P$ be the 5-cycle $(b, c, \infty_1, a, \infty_2)$. Then $(\infty_1, \infty_2, \infty_3) \cup (b, c, \infty_1, a, \infty_2)$ can be partitioned into the two 4-cycles $(b, c, \infty_1, \infty_2)$ and $(\infty_1, \infty_3, \infty_2, a)$.

Let

$\pi_1^* = (\pi_1 \setminus \{f_2, g_2\}) \cup \{h_2, (b, c, \infty_1, \infty_2)\}$ and $\pi_2^* = (\pi_2 \setminus h_2) \cup \{f_2, g_2, (\infty_1, \infty_3, \infty_2, a)\}$.

Then $(X, (C \setminus \{\pi_1, \pi_2\}) \cup \{\pi_1^*, \pi_2^*\}, P)$ is an almost resolvable minimum covering of $K_{8n+3}$ with 4-cycles.

So we have the following lemma.

**Lemma 3.1** *There exists an almost resolvable minimum covering of* $K_n$ *with* 4-*cycles for all* $n \equiv 3 \pmod 8$, $n \geqslant 11$. □

## 4 Order $n \equiv 5 \pmod 8$

In this case the padding is a double edge.

As in the case $n \equiv 3 \pmod 8$, we start with some examples.

**Example 4.1** *Order* $n = 5$

$(X, C, P)$ is an almost resolvable minimum covering of $K_5$ with 4-cycles, where $X = \{1, 2, 3, 4, 5\}$, $P$ is the twice repeated edge $\{1, 2\}$, and $C = \{(1, 2, 3, 4); (1, 2, 4, 5); (1, 2, 5, 3)\}$.

□

**Example 4.2** *Order* $n = 13$

$(X, C, P)$ is an almost resolvable minimum covering of $K_{13}$ with 4-cycles, where $X = \{i_j \mid i \in \mathbb{Z}_4,\ j \in \{1, 2, 3\}\} \cup \{\infty\}$, the padding $P$ is the twice repeated edge $\{0_3, \infty\}$, and $C$ consists of six almost parallel classes and one short class, as follows:

$\{(\infty, 0_3, 2_3, 1_3), (0_1, 2_2, 2_1, 0_2), (1_1, 3_2, 3_1, 1_2)\}$;

$\{(\infty, 0_3, 3_3, 2_3), (0_1, 2_1, 3_2, 1_2), (1_1, 3_1, 0_2, 2_2)\}$;

$\{\{(\infty, (3+i)_1, (2+i)_2, (3+i)_2), ((0+i)_1, (0+i)_3, (1+i)_1, (2+i)_3),$
$((0+i)_2, (1+i)_3, (1+i)_2, (3+i)_3)\} \mid i \in \mathbb{Z}_4\}$;

$\{(\infty, 0_3, 1_3, 3_3), (0_1, 1_1, 2_1, 3_1)\}$. □

**Example 4.3** *Order* $n = 21$

$(X, C, P)$ is an almost resolvable minimum covering of $K_{21}$ with 4-cycles, where $X = \mathbb{Z}_{21}$, the padding $P$ is the twice repeated edge $\{0, 1\}$, and $C$ consists of ten almost parallel classes and one short class, as follows:

$\{(0,1,2,3),\quad (4,5,6,7),\quad (8,9,10,11),\quad (12,13,14,15),\quad (16,17,18,19)\};$
$\{(0,1,4,2),\quad (3,5,8,6),\quad (7,9,12,10),\quad (11,13,16,14),\quad (15,17,20,18)\};$
$\{(0,1,3,4),\quad (2,5,9,6),\quad (7,8,12,11),\quad (10,13,17,14),\quad (15,16,20,19)\};$
$\{(0,5,1,6),\quad (2,7,3,8),\quad (4,9,15,10),\quad (11,16,12,18),\quad (13,19,14,20)\};$
$\{(0,7,1,8),\quad (2,9,3,10),\quad (4,6,18,14),\quad (5,13,15,20),\quad (11,17,12,19)\};$
$\{(0,9,1,10),\quad (2,11,3,12),\quad (4,13,18,16),\quad (5,7,19,17),\quad (6,14,8,20)\};$
$\{(0,11,1,12),\quad (2,13,6,17),\quad (3,15,8,19),\quad (5,14,7,18),\quad (9,16,10,20)\};$
$\{(0,13,1,16),\quad (2,14,9,19),\quad (4,12,7,20),\quad (5,11,6,15),\quad (8,17,10,18)\};$
$\{(0,14,1,18),\quad (2,15,11,20),\quad (3,13,9,17),\quad (4,8,10,19),\quad (5,12,6,16)\};$
$\{(0,15,1,17),\quad (3,14,12,20),\quad (4,11,9,18),\quad (5,10,6,19),\quad (7,13,8,16)\};$

$\{(0,19,1,20),\quad (2,16,3,18),\quad (4,15,7,17)\}.$ □

We will use the $8n+5$ Construction for almost resolvable maximum packings given in [1]. This is a lot easier for order $8n+5$ than for order $8n+3$. Let $(X,C,L)$ be the almost resolvable maximum packing of order $8n+5 \geqslant 29$ obtained in [1], where the leave $L$ is a 2-regular graph with 6 edges (a 6-cycle in [1]). Now replace the maximum packing on $\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\} \cup (h^* \times \{1,2,3,4\})$ with the minimum covering in Example 4.2. It is immediate that this produces a minimum covering containing $4n+2$ almost parallel classes and a partial parallel class consisting of $n+1$ 4-cycles, which is exactly the correct number. So we have:

**Lemma 4.1** *There exists an almost resolvable minimum covering of $K_n$ with 4-cycles for all $n \equiv 5 \pmod 8$, $n \geqslant 5$.* □

## 5 Order $n \equiv 7 \pmod 8$

In this case the padding is a 3-cycle. Once again we start with some necessary examples, noting that order 7 was dealt with back in Example 1.1.

**Example 5.1** *Order $n = 15$*

$(X,C,P)$ is an almost resolvable minimum covering of $K_{15}$ with 4-cycles, where $X = \{i_j \mid i \in \mathbb{Z}_3,\ j \in \{1,2,3,4,5\}\}$, the padding $P$ is the triangle $(0_1,1_1,2_1)$, and $C$ consists of nine almost parallel classes obtained from the following three starter classes, modulo 3 with subscripts fixed:

$\{(1_1,2_1,0_3,1_2),\quad (0_1,1_4,0_2,0_5),\quad (2_3,0_4,1_5,1_3)\};$
$\{(2_1,1_1,2_5,1_2),\quad (0_2,2_2,1_4,1_3),\quad (0_1,0_3,1_5,2_4)\};$
$\{(0_4,1_4,1_2,0_1),\quad (0_5,1_5,2_1,1_3),\quad (0_2,0_3,2_4,2_5)\}.$ □

**Example 5.2** *Order* $n = 23$

$(X, C, P)$ is an almost resolvable minimum covering of $K_{23}$ with 4-cycles, where $X = \mathbb{Z}_{23}$, the padding $P$ is the 3-cycle $\{(0, 1, 2)\}$, and $C$ consists of twelve almost parallel classes and one short class, as follows:

$$\begin{array}{lllll}
\{(0,1,2,3), & (4,5,6,7), & (8,9,10,11), & (12,13,14,15), & (16,17,18,19)\}; \\
\{(0,1,4,2), & (3,5,8,6), & (7,9,12,10), & (11,13,16,14), & (15,17,20,18)\}; \\
\{(0,2,6,4), & (1,3,7,5), & (8,10,14,12), & (9,11,15,13), & (16,18,22,21)\}; \\
\{(0,5,9,6), & (1,2,8,7), & (3,4,13,10), & (11,12,20,16), & (14,21,19,22)\}; \\
\{(0,7,13,8), & (1,6,14,9), & (2,5,15,10), & (3,11,19,17), & (4,21,20,22)\};
\end{array}$$

$$\begin{array}{lllll}
\{(0,9,4,10), & (1,8,17,11), & (2,7,18,12), & (3,13,20,19), & (5,21,6,22)\}; \\
\{(0,11,4,12), & (1,10,5,13), & (2,9,22,16), & (6,17,21,18), & (7,19,15,20)\}; \\
\{(0,13,2,14), & (1,12,16,15), & (3,9,17,22), & (4,18,5,19), & (8,20,10,21)\}; \\
\{(0,15,3,16), & (1,14,4,17), & (2,11,5,20), & (7,21,12,22), & (9,18,13,19)\}; \\
\{(1,16,8,19), & (2,15,9,21), & (3,12,6,20), & (5,14,7,17), & (10,18,11,22)\}; \\
\{(0,19,14,20), & (1,18,2,22), & (3,8,15,21), & (5,12,7,16), & (6,10,17,13)\}; \\
\{(0,21,13,22), & (2,17,12,19), & (3,14,8,18), & (4,16,9,20), & (6,11,7,15)\};
\end{array}$$

$$\{(0,17,14,18), \quad (1,20,11,21), \quad (4,8,22,15), \quad (6,16,10,19)\}.$$

□

**Example 5.3** *Order* $n = 31$

We will use the almost resolvable maximum packing of $K_{31}$ in Example 5.3 of [1] to obtain an almost resolvable minimum covering of $K_{31}$ with 4-cycles. So, let $(X, C, L)$ be the almost resolvable maximum packing of $K_{31}$ in [1], where $X = \{\infty_1, \infty_2, \infty_3\} \cup (Q \times \{1, 2\})$. In part (1) of this example we can take the maximum packing to be

$$\begin{array}{ll}
\{(\infty_1, \infty_3, (2,2), (1,2)), & (\infty_1, (1,1), \infty_2, (2,2)), \\
((1,2), \infty_2, (2,1), \infty_3), & ((1,2), (2,1), (2,2), (1,1))\}
\end{array}$$

and $L = (\infty_1, \infty_2, \infty_3, (1,1), (2,1))$. In part (2) it is *extremely important* to note that the almost resolvable maximum packing of $K_{11}$ defined on $\{\infty_1, \infty_2, \infty_3\} \cup (h_i \times \{1, 2\})$ contains five 4-cycles each of which does not intersect $\{\infty_1, \infty_2, \infty_3\}$ and which can be partitioned into two partial parallel classes $\pi_{i1}$ and $\pi_{i2}$ with one 4-cycle $f_i$ left over. Then

$$\begin{array}{rcl}
A_1 & = & \{((1,2), (2,1), (2,2), (1,1)), \pi_{11}, \pi_{21}, \pi_{31}\} \quad \text{and} \\
A_2 & = & \{(\infty_1, \infty_3, (2,2), (1,2)), \pi_{12}, \pi_{22}, \pi_{32}\}
\end{array}$$

are almost parallel classes of $C$ and $F = \{f_1, f_2, f_3\}$ is a partial parallel class. Now let $x \in (h_1 \times \{1, 2\}) \setminus f_1$ and partition $(\infty_1, \infty_2, \infty_3, (1,1), (2,1)) \cup (x, \infty_1, \infty_3)$ into the two 4-cycles $((1,1), (2,1), \infty_1, \infty_3)$ and $(x, \infty_1, \infty_2, \infty_3)$. Then

$$A_1{}^* = (A_1 \setminus \pi_{11}) \cup \{f_1, (x, \infty_1, \infty_2, \infty_3)\}$$

is an almost parallel class and $F^* = \{((1,1),(2,1),\infty_1,\infty_3),\pi_{11},f_2,f_3\}$ is a partial parallel class consisting of five 4-cycles.

Let $C^* = (C \setminus \{A_1,F\}) \cup {A_1}^* \cup F^*$. Then $(X,C^*,P)$ with $P = (x,\infty_1,\infty_3)$ is an almost resolvable minimum covering of $K_{31}$ with 4-cycles. □

We can now give a construction for all orders $8n+7 \geqslant 39$. Again we use the almost resolvable maximum packing of order $8n+7$, given in [1]. When $n \geqslant 4$ we replace the order 15 design with the Example 5.1 above; the rest of the construction and resolution follows as in the case of the resolvable packing paper. The "short" almost parallel class contains two more 4-cycles this time, $n+2$ instead of $n$; these come from one of the almost parallel classes in Example 5.1, because the order 15 covering has all "full" almost parallel classes, of three 4-cycles, whereas the order 15 packing case had a short class consisting of one 4-cycle.

So we have:

**Lemma 5.1** *There exists an almost resolvable minimum covering of $K_n$ with 4-cycles for all $n \equiv 7 \pmod 8$, $n \geqslant 7$.*

□

## 6 Conclusion

Table 2 overleaf summarises the orders modulo 8 and the paddings and leaves for almost resolvable minimum coverings and maximum packings.

We remark that if a $C_4$ padding is adjoined to $K_9$, then an almost resolvable minimum covering of $K_9$ is possible; see Appendix C below.

This paper, with the missing cases for [1] and [2], completes the problem of finding almost resolvable 4-cycle systems, and almost resolvable maximum packing and minimum coverings with 4-cycles.

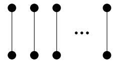
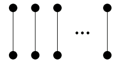
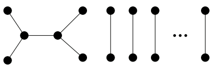
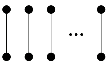
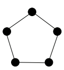
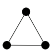
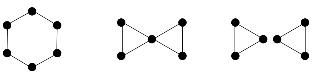
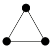
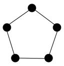
| Order (mod 8) | Covering: the padding | Packing: the leave | Almost-resolvable 4-cycles: the spectrum |
|---|---|---|---|
| 1 | ∅ | ∅ | all orders except 9 |
| 0, 4 | 1-factor  | 1-factor  | no exceptions |
| 2, 6 | 1-factor + 2 more edges spanning, of odd degree  | 1-factor  | no exceptions |
| 3 | 5-cycle  | 3-cycle  | no exceptions |
| 5 |  double edge |  6-cycle or bowtie or two 3-cycles | no exceptions |
| 7 | 3-cycle  | 5-cycle  | no exceptions |

Table 2: Almost-resolvable 4-cycle systems, coverings and packings.

# Appendix

## A Almost resolvable 4-cycle systems of orders 41 and 57

These examples have no leave or excess; they use exact 4-cycle decompositions of each of these orders, since 41 and 57 are 1 (mod 8).

Order 41

The vertex set is taken as $\{i_j \mid i \in \mathbb{Z}_5,\ j \in \{1,2,3,4,5,6,7,8\}\} \cup \{\infty\}$.

There are four starters modulo 5 (all subscripts are fixed):

$\{(1_1,2_1,4_8,3_1),(4_1,0_2,3_8,1_2),(2_2,3_2,4_7,4_2),(0_3,1_3,2_8,2_3),(3_3,0_4,1_8,1_4),$
$(4_3,3_4,\infty,0_5),(2_4,4_4,3_7,1_5),(2_5,3_5,2_7,4_5),(0_6,1_6,1_7,2_6),(3_6,0_7,4_6,0_8)\};$

$\{(0_1,3_2,4_8,4_2),(1_1,0_3,3_8,1_3),(2_1,3_3,4_7,4_3),(3_1,0_4,3_7,1_4),(4_1,2_3,1_8,3_4),$
$(1_2,2_4,4_6,4_4),(2_2,0_5,2_8,1_5),(2_5,2_6,\infty,2_7),(3_5,0_6,0_8,1_6),(4_5,3_6,1_7,0_7)\};$

$\{(0_1,1_4,3_8,0_5),(1_1,2_5,2_8,3_5),(2_1,1_5,4_4,0_6),(3_1,2_6,4_3,3_6),(4_1,1_6,4_8,0_7),$
$(0_2,1_3,4_7,2_3),(1_2,3_7,1_8,0_8),(2_2,4_6,0_4,1_7),(3_2,2_4,3_4,4_5),(4_2,2_7,3_3,\infty)\};$

$\{(0_1,0_7,0_8,3_8),(1_1,3_7,4_8,\infty),(2_1,1_7,3_1,2_8),(4_1,2_5,0_2,0_6),(1_2,1_4,1_5,3_4),$
$(2_2,0_3,0_5,1_6),(3_2,3_5,4_3,4_6),(4_2,3_3,4_4,2_6),(1_3,4_5,2_3,3_6),(2_4,2_7,4_7,1_8)\}.$

Then a short parallel class:

$\{(0_1, 0_2, 0_3, 0_4), (1_1, 1_2, 1_3, 1_4), (2_1, 2_2, 2_3, 2_4), (3_1, 3_2, 3_3, 3_4), (4_1, 4_2, 4_3, 4_4)\}$.

Order 57

The vertex set is taken as $\{i_j \mid i \in \mathbb{Z}_7,\ j \in \{1, 2, 3, 4, 5, 6, 7, 8\}\} \cup \{\infty\}$.

There are four starters modulo 7 (all subscripts are fixed):

$\{(1_1, 2_1, 6_8, 3_1), (4_1, 0_2, 5_8, 1_2), (5_1, 3_2, 4_8, 4_2), (6_1, 0_3, 3_8, 1_3), (2_2, 5_2, \infty, 3_3),$
$(6_2, 2_3, 2_8, 4_3), (5_3, 6_3, 0_8, 0_4), (1_4, 2_4, 1_8, 3_4), (4_4, 0_5, 6_7, 1_5), (5_4, 3_5, 5_7, 4_5),$
$(6_4, 6_5, 3_7, 0_6), (2_5, 5_5, 1_7, 1_6), (2_6, 6_6, 0_7, 4_7), (3_6, 4_6, 2_7, 5_6)\}$;

$\{(0_1, 3_1, 4_8, 1_2), (1_1, 3_2, 6_7, 0_3), (2_1, 2_3, 5_7, 5_3), (4_1, 1_3, 3_7, 0_4), (5_1, 3_3, 4_7, 2_4),$
$(6_1, 1_4, 5_8, 4_4), (2_2, 4_2, 3_8, 4_3), (5_2, 6_2, 1_7, 3_4), (6_3, 5_4, \infty, 0_5), (6_4, 1_5, 6_8, 1_6),$
$(2_5, 4_6, 1_8, 5_6), (3_5, 4_5, 2_6, 5_5), (6_5, 0_6, 0_8, 6_6), (3_6, 2_7, 0_7, 2_8)\}$;

$\{(0_1, 1_4, 5_7, 6_4), (1_1, 0_5, 4_8, 1_5), (2_1, 3_5, 5_8, 4_5), (3_1, 6_5, 6_3, 0_6), (4_1, 2_5, 5_3, 2_6),$
$(5_1, 1_6, 5_4, 4_6), (6_1, 6_6, \infty, 0_7), (0_2, 4_3, 1_7, 0_4), (1_2, 2_4, 6_2, 3_4), (2_2, 1_3, 3_3, 3_6),$
$(3_2, 5_5, 2_3, 5_6), (4_2, 2_7, 6_8, 3_7), (5_2, 4_4, 0_8, 6_7), (4_7, 2_8, 1_8, 3_8)\}$;

$\{(0_1, 4_5, 2_3, 1_6), (1_1, 0_7, 4_1, 6_8), (2_1, 1_8, 4_8, \infty), (3_1, 1_7, 6_1, 3_7), (5_1, 0_6, 3_4, 5_8),$
$(0_2, 5_5, 4_4, 4_7), (1_2, 4_6, 6_3, 3_8), (2_2, 2_5, 2_8, 2_7), (3_2, 2_6, 5_2, 3_6), (4_2, 0_5, 6_2, 3_5),$
$(0_3, 6_5, 6_7, 5_7), (1_3, 4_3, 2_4, 5_4), (3_3, 1_5, 0_8, 5_6), (5_3, 1_4, 6_6, 6_4)\}$.

Then a short parallel class of seven 4-cycles:

$\{((0+i)_1, (0+i)_2, (0+i)_3, (0+i)_4) \mid i \in \mathbb{Z}_7\}$.

## B An almost resovable maximum packing of order 23

Letting $X = \mathbb{Z}_{23}$, we have an almost resolvable maximum *packing* $(X, C, L)$ where the leave $L$ is the 5-cycle $(17, 19, 20, 22, 21)$, and the almost resolvable classes $C$ are as follows (12 classes with five 4-cycles and one short class with two 4-cycles):

| | | | | |
|---|---|---|---|---|
| {(0, 13, 9, 20), | (1, 4, 16, 11), | (2, 18, 12, 21), | (5, 10, 19, 15), | (6, 17, 7, 22)}; |
| {(1, 7, 10, 14), | (2, 5, 17, 12), | (3, 19, 13, 21), | (6, 11, 20, 16), | (0, 18, 8, 22)}; |
| {(2, 8, 11, 15), | (3, 6, 18, 13), | (4, 20, 7, 21), | (0, 12, 14, 17), | (1, 19, 9, 22)}; |
| {(3, 12, 9, 16), | (4, 0, 19, 7), | (5, 14, 8, 21), | (1, 13, 15, 18), | (2, 20, 10, 22)}; |
| {(4, 10, 13, 17), | (5, 1, 20, 8), | (6, 15, 9, 21), | (2, 7, 16, 19), | (3, 14, 11, 22)}; |
| {(5, 11, 7, 18), | (6, 2, 14, 9), | (0, 16, 10, 21), | (3, 8, 17, 20), | (4, 15, 12, 22)}; |
| {(6, 12, 8, 19), | (0, 3, 15, 10), | (1, 17, 11, 21), | (4, 9, 18, 14), | (5, 16, 13, 22)}; |
| {(0, 1, 3, 2), | (4, 5, 7, 6), | (8, 9, 11, 10), | (12, 13, 14, 19), | (15, 16, 18, 17)}; |
| {(0, 5, 9, 7), | (1, 6, 8, 15), | (2, 11, 4, 13), | (3, 17, 10, 18), | (14, 16, 21, 20)}; |
| {(0, 6, 13, 8), | (1, 9, 17, 16), | (2, 4, 12, 10), | (5, 19, 18, 20), | (7, 14, 22, 15)}; |
| {(0, 14, 21, 15), | (2, 16, 22, 17), | (3, 5, 6, 10), | (4, 18, 11, 19), | (7, 12, 20, 13)}; |
| {(1, 2, 9, 10), | (3, 4, 8, 7), | (5, 12, 11, 13), | (6, 14, 15, 20), | (18, 21, 19, 22)}; |
| {(0, 9, 3, 11), | (1, 12, 16, 8)}. | | | |

## C An almost resovable "minimum covering" of order 9

It is impossible to take a 4-cycle system of order 9 and arrange the 4-cycles into almost parallel classes. However, by including a 4-cycle padding, we can cover $K_9$ almost resolvably with 4-cycles, as follows. With the element set $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and with a 4-cycle padding $(6, 7, 8, 9)$, we have five almost parallel classes:

$$\{(6,8,4,2),\quad (7,9,5,3)\}; \qquad \{(1,2,5,7),\quad (3,8,9,6)\};$$
$$\{(1,3,2,8),\quad (4,9,6,7)\}; \qquad \{(1,4,3,9),\quad (5,6,7,8)\};$$
$$\{(1,5,4,6),\quad (2,7,8,9)\}.$$

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