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Equitable edge colored Steiner triple systems

Atif A. Abueida

Department of Mathematics
University of Dayton
300 College Park, Dayton, OH 45469-2316
U.S.A.
Atif.Abueida@notes.udayton.edu

James Lefevre   Mary Waterhouse

School of Mathematics and Physics
The University of Queensland
Brisbane, Qld. 4072
Australia
jgl@maths.uq.edu.au   maw@maths.uq.edu.au

Abstract

A k-edge coloring of G is said to be *equitable* if the number of edges, at any vertex, colored with a certain color differ by at most one from the number of edges colored with a different color at the same vertex. An STS(v) is said to be *polychromatic* if the edges in each triple are colored with three different colors. In this paper, we show that every STS(v) admits a 3-edge coloring that is both polychromatic for the STS(v) and equitable for the underlying complete graph. Also, we show that, for $v \equiv 1$ or 3 (mod 6), there exists an equitable k-edge coloring of K_v which does not admit any polychromatic STS(v), for $k = 3$ and $k = v - 2$.

1 Introduction

An *edge coloring* of a graph G is an assignment of colors to the edges of G. A *k-edge coloring* of G is an edge coloring of G in which k distinct colors are used, $c_1, c_2, \ldots, c_k$ say. We let $E(c_i)$ denote the set of edges that are assigned color c_i, for $i = 1, 2, \ldots, k$. Also, let $d_{(v)}$ denote the degree of a vertex v.

A *Steiner triple system of order* v, denoted STS(v), is an ordered pair (V, T), where V is a v-set of symbols and T is a set of 3-element subsets of V called *triples* such that any pair of symbols in V occurs together in exactly one triple. It is known that a Steiner triple system (STS) of order v exists if and only if $v \equiv 1$ or 3 (mod 6); see [11].

An STS(v) is said to be *resolvable* if its triples can be partitioned into sets, called *parallel classes*, such that every element of V appears exactly once within each parallel class.

Let G and H be graphs. An H-decomposition of G is a set $\mathcal{H} = \{H_1, H_2, \ldots, H_p\}$ such that H_i is isomorphic to H for $1 \leq i \leq p$ and $\mathcal{H}$ partitions the edge set of G. A K_3-decomposition of K_v is equivalent to an STS(v), and we will use the terms interchangeably. Similarly, we will often refer to K_3 as a triple.

We color the triple $\{x, y, z\}$ with the color triple $\langle c_i, c_j, c_k \rangle$ by assigning the colors c_i, c_j and c_k to the edges $\{x, y\}$, $\{y, z\}$ and $\{x, z\}$ respectively. If $i \neq j \neq k$ and $k \neq i$, then we say that $\{x, y, z\}$ is *polychromatic*. An STS(v) or K_3-decomposition of some graph G is said to be polychromatic if every triple in the STS(v) or decomposition is polychromatic.

In general, consider the two graphs H and G, and an edge coloring of G. The graph H is said to be polychromatic subgraph of G, if G contains a subgraph isomorphic to H, where all of whose edges are assigned distinct colors. Researchers devoted some effort to the study of determining the least number of colors used to color the $E(K_v)$ that forces a specified polychromatic subgraph to occur. Those problems are called *Anti-Ramsey* problems. For more details about Anti-Ramsey problems and polychromatic subgraphs, the reader is referred to [3, 4, 9, 12].

In [1], Bate studied complete graphs without polychromatic C_n for $n = 3$ and $n = 4$. A good survey about monochromatic and polychromatic subgraphs in edge colored graphs can be found in [10].

For each $v \in V(G)$ let $n_v(c_i)$ denote the number of edges in G of color c_i incident with vertex v, for $i = 1, 2, \ldots, k$. A k-edge coloring of G is said to be *equitable* if $|n_v(c_i) - n_v(c_j)| \leq 1$ for all $v \in V(G)$ and $1 \leq i < j \leq k$.

If there exists an equitable k-edge coloring of G, then G is said to be *equitably k-edge colorable*. The idea is that the number of edges of each color is as close as possible at any vertex of the graph G.

Hilton and de Werra [8] found a sufficient condition for a simple graph to admit an equitable edge coloring:

Theorem 1.1. *[8] Let G be a simple graph and let $k \geq 2$. If $k \nmid d_{(v)}$ (for all $v \in V(G)$) then G has an equitable edge coloring with k colors.*

Specialized colorings of cycle systems (C_3 and C_4), in which some condition on the coloring is satisfied, have received recent attention. We refer the interested reader to [2, 5, 6, 7].

There are two natural questions one can ask about polychromatic Steiner triple systems and equitable edge colorings of the complete graph. One is: does every STS(v) admit a polychromatic k-edge coloring which is also an equitable k-edge coloring of the complete graph of order v, K_v? The second is: For $v \equiv 1$ or 3 (mod 6), does every equitable k-edge coloring of K_v admit a polychromatic STS(v)?

In the following section we settle the first question for $k = 3$. In the final section we settle the second question for $k = 3$ and $k = v - 2$.

Before proceeding, we define some notation. We let $K_{m,n}$ denote the complete bipartite graph with m and n vertices in each partite set. We let $K_x \setminus K_y$ denote the complete graph on x vertices with a hole of size y. When no confusion is likely to arise, we denote the edge $\{x, y\}$ by xy.

2 Polychromatic Steiner Triple Systems yielding equitable colorings

One relatively straightforward method for constructing a polychromatic STS(v) which yields an equitable 3-edge coloring of the underlying graph is based on cyclic systems. In fact this method may be generalized to certain higher numbers of colors.

Theorem 2.1. *Let $\mathcal{S}$ be a cyclic STS(v) on vertex set V, and let $k \geq 3$ and $k \mid (v-1)/2$. Then there exists a polychromatic coloring of $\mathcal{S}$ which is an equitable k-edge coloring of the complete graph on V.*

Proof. The number of base blocks is $(v-1)/6$, and they contain a total of $(v-1)/2$ edges. Given that $k \geq 3$ and $k \mid (v-1)/2$, we can assign these edges in equal numbers to k color classes so that every base block is polychromatic. When these base blocks are developed cyclically, with each triple assigned the coloring of the base block from which it was developed, we obtain the required coloring. □

In the case $k = 3$, we can prove a more general result—that in fact the required coloring is possible for *any* STS.

Theorem 2.2. *Every STS admits a 3-edge coloring which is both polychromatic (with respect to the triples) and equitable (with respect to the vertices).*

Proof. Given any STS on a vertex set V, we can trivially color the edges with colors c_1, c_2, c_3 so that each triple is polychromatic (in general this coloring is not equitable). Suppose we have an arbitrary such coloring, and let $n_x(c)$ denote the number of edges of color $c \in \{c_1, c_2, c_3\}$ incident with vertex $x \in V$ in this coloring. Suppose further that for some vertex $1 \in V$, we have $n_1(c_a) \geq n_1(c_b) + 2$. We will show that we can construct a second coloring with the following properties, where $n'_x(c)$ denotes the number of edges of color c incident with vertex x in this new coloring:

(1) Every triple is polychromatic.

(2) $|n'_1(c_a) - n'_1(c_b)| < |n_1(c_a) - n_1(c_b)|$.

(3) For every vertex x and any two colors c_i and c_j, we have
$|n'_x(c_i) - n'_x(c_j)| \leq |n_x(c_i) - n_x(c_j)|$.

From this, the result will follow inductively.

We prove the result for $a = 1$ and $b = 2$; the other cases follow similarly. Separate to the STS, we construct a digraph on the vertex set V as follows: For each triple $\{x, y, z\}$ in the STS with $\{x, z\} \in E(c_1)$, $\{y, z\} \in E(c_2)$, and $\{x, y\} \in E(c_3)$, we construct a directed edge (x, y) from vertex x to vertex y. Then for every $w \in V$ we have outdegree(w) − indegree$(w) = n_w(c_1) - n_w(c_2)$; note that any triple $\{w, p, q\}$ with $\{w, p\} \in E(c_1)$ and $\{w, q\} \in E(c_2)$ does not affect either side of this equation.

It follows that we have outdegree(1)−indegree(1) ≥ 2. We now wish to show that the digraph contains a directed path from 1 to some vertex for which the indegree exceeds the outdegree. Let $V' \subseteq V$ be the set of all vertices v for which there exists a directed path from 1 to v (including 1 itself). By definition, every edge directed away from a vertex of V' must be directed to another vertex of V', and hence the total indegree summed over the vertices of V' must be greater than or equal to the total outdegree. Since outdegree(1) $>$ indegree(1), it follows that there is a vertex $2 \in V'$ with outdegree(2) $<$ indegree(2), and hence with $n_2(c_1) < n_2(c_2)$. Since $2 \in V'$, there is a directed path from vertex 1 to 2.

To construct the second coloring, we reverse the orientation of each edge in the directed path from 1 to 2, and make the corresponding change in the edge coloring of the STS (all other edge colorings are equal to the original coloring). That is, for each directed edge (x, y) in this path, we change the coloring of the triple $\{x, y, z\}$ in the STS. By definition this triple must have original coloring $\{x, z\} \in E(c_1)$, $\{y, z\} \in E(c_2)$, and $\{x, y\} \in E(c_3)$; we replace this with the coloring $\{x, z\} \in E(c_2)$, $\{y, z\} \in E(c_1)$, and $\{x, y\} \in E(c_3)$. Observe that for all $w \in V \setminus \{1, 2\}$ and $c \in \{c_1, c_2, c_3\}$, we have $n'_w(c) = n_w(c)$, and also $n'_1(c_3) = n_1(c_3)$ and $n'_2(c_3) = n_2(c_3)$. Note also that all triples are polychromatic in the new coloring. The difference between the two colorings is thus that $n'_1(c_1) = n_1(c_1) - 1$, $n'_1(c_2) = n_1(c_2) + 1$, $n'_2(c_1) = n_2(c_1) + 1$ and $n'_2(c_2) = n_2(c_2) - 1$. Since $n_1(c_1) \geq n_1(c_2) + 2$ and $n_2(c_1) < n_2(c_2)$, we have $|n'_1(c_1) - n'_1(c_2)| < |n_1(c_1) - n_1(c_2)|$ and $|n'_2(c_1) - n'_2(c_2)| \leq |n_2(c_1) - n_2(c_2)|$. The result follows. □

3 Equitable edge coloring with no polychromatic Steiner triple system

We now address the question of whether every equitable k-edge coloring of K_v (for a given $v \equiv 1$ or $3 \pmod 6$) admits a polychromatic STS(v) (that is, a decomposition of the edge colored K_v into polychromatic triples). An immediate condition for the existence of such a polychromatic STS(v) is that $k \geq 3$. When $k = 3$ it is also necessary that there are equal numbers of edges in each color class; this is forced (by the equitable condition) when $v \equiv 1 \pmod 6$ but not when $v \equiv 3 \pmod 6$.

We will show that even if this condition holds, in the cases $k = 3$ and $k = v - 2$ there exists an equitable k-edge coloring of K_v which does not admit a polychromatic STS(v), for every $v \equiv 1, 3 \pmod 6$.

Unless otherwise specified, we will let the vertex set of K_v be $\{1, 2, \ldots, v\}$, and when using three colors, we denote the colors by c_1, c_2 and c_3.

Lemma 3.1. *There exists an equitable 3-edge coloring of K_v, with color classes of equal size, that does not admit any polychromatic STS(v), for $v \in 7, 9$.*

Proof. The following is an equitable 3-edge coloring of K_7:

$$\begin{aligned} E(c_1) &= \{12, 17, 27, 35, 36, 45, 46\}, \\ E(c_2) &= \{15, 16, 25, 26, 34, 37, 47\}, \\ E(c_3) &= \{13, 14, 23, 24, 56, 57, 67\}. \end{aligned}$$

The following is an equitable 3-edge coloring of K_9:

$$\begin{aligned} E(c_1) &= \{14, 15, 24, 25, 36, 37, 39, 49, 58, 67, 68, 89\}, \\ E(c_2) &= \{13, 18, 19, 23, 28, 29, 46, 47, 48, 56, 57, 79\}, \\ E(c_3) &= \{12, 16, 17, 26, 27, 34, 35, 38, 45, 59, 69, 78\}. \end{aligned}$$

In either case, it is easy to check that the edge 12 cannot appear in any polychromatic triple; see Figure 1. □

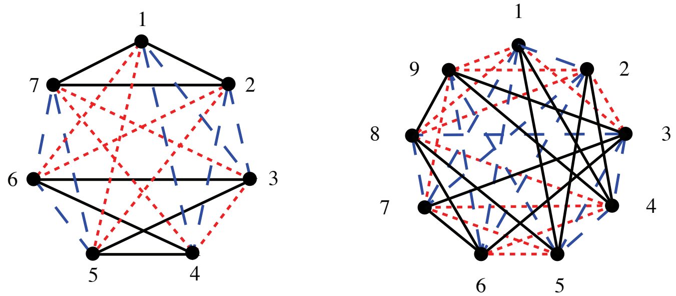

Figure 1: Equitably 3-edge colored copy of K_v which does not admit a polychromatic STS(v), for $v \in \{7, 9\}$.

Lemma 3.2. *There exists an equitable 3-edge coloring of $K_9 \backslash K_3$, with 11 edges in each color class, that does not admit a polychromatic K_3-decomposition.*

Proof. Let the vertex set of $V(K_9 \backslash K_3)$ be $\{a, b, c, 1, 2, 3, 4, 5, 6\}$, where a, b and c correspond to the vertices in the hole. The following is an equitable 3-edge coloring of $K_9 \setminus K_3$:

$$\begin{aligned} E(c_1) &= \{a5, a6, b5, b6, c5, c6, 12, 13, 14, 23, 24\}, \\ E(c_2) &= \{a3, a4, b3, b4, c1, c2, 15, 16, 25, 26, 34\}, \\ E(c_3) &= \{a1, a2, b1, b2, c3, c4, 35, 36, 45, 46, 56\}. \end{aligned}$$

The edges 12, 34 and 56 cannot appear in any polychromatic triple; see Figure 2. □

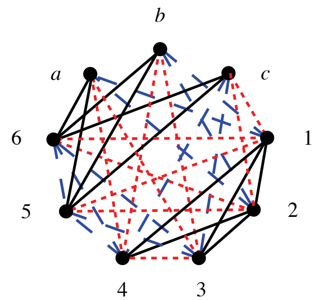

Figure 2: Equitably 3-edge colored copy of $K_9 \setminus K_3$ which does not admit a polychromatic K_3-decomposition.

Note that in the proof of Lemma 3.2, the edges 12, 34 and 56 are colored with c_1, c_2 and c_3 respectively and they are not in any polychromatic triple.

Lemma 3.3. *Let the vertex set of* $K_{6,6}$ *be* $\cup_{i=1,2}\{1_i, 2_i, \ldots, 6_i\}$*, with the obvious vertex partition. There exists an equitable* 3*-edge coloring of* $K_{6,6}$*, with* 12 *edges of each color, such that the edges* 1_1x *and* 2_1x *are assigned the same color for all* $x \in \{1_2, 2_2, \ldots, 6_2\}$*.*

Proof. The following is an equitable 3-edge colouring of $K_{6,6}$ which satisfies the specified constraint; see Figure 3.

$$\begin{aligned} E(c_1) &= \{1_11_2, 2_11_2, 1_16_2, 2_16_2, 3_14_2, 4_14_2, 3_15_2, 4_15_2, 5_12_2, 6_12_2, 5_13_2, 6_13_2\}, \\ E(c_2) &= \{1_12_2, 2_12_2, 1_13_2, 2_13_2, 3_11_2, 4_11_2, 3_16_2, 4_16_2, 5_14_2, 6_14_2, 5_15_2, 6_15_2\}, \\ E(c_3) &= \{1_14_2, 2_14_2, 1_15_2, 2_15_2, 3_12_2, 4_12_2, 3_13_2, 4_13_2, 5_11_2, 6_11_2, 5_16_2, 6_16_2\}. \end{aligned}$$

□

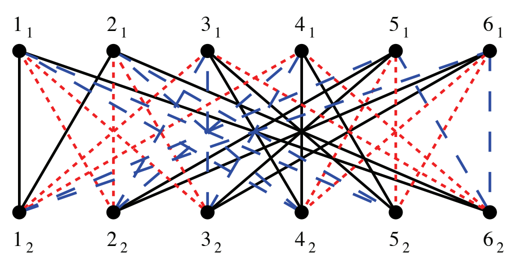

Figure 3: Equitably 3-edge colored copy of $K_{6,6}$.

Theorem 3.4. *For all* $v \equiv 1$ *or* $3 \pmod 6$*,* $v \geq 7$*, there exists an equitable* 3*-edge coloring of* K_v*, with* $(v-1)/6$ *edges in each color class, that does not admit any polychromatic* $STS(v)$*.*

Proof. We consider the cases $v \equiv 1 \pmod 6$ and $v \equiv 3 \pmod 6$ separately.

Case 1: $v \equiv 1 \pmod 6$
Let $v = 6n + 1$, for some positive integer n, and let the vertex set of K_v be $\{\infty\} \cup \bigcup_{i=1}^{n} V_i$, where $V_i = \{1_i, 2_i, \dots, 6_i\}$, for $i = 1, 2, \dots, n$.

For each $i = 1, 2, \dots, n$, place the equitable 3-edge coloring of K_7 given by Lemma 3.1 on $\{\infty\} \cup V_i$, such that vertex x in Lemma 3.1 corresponds to vertex $x_i \in V_i$, for $x \in \{1, 2, \dots, 6\}$, and vertex 7 corresponds to ∞.

Place the equitable 3-edge coloring of $K_{6,6}$ given by Lemma 3.3 on $V_i \cup V_j$, for $1 \le i < j \le n$, such that vertices x_1 and x_2 in Lemma 3.3 correspond to vertices $x_i \in V_i$ and $x_j \in V_j$, for $x \in \{1, 2, \dots, 6\}$.

Note that K_v is equitably 3-edge colored.

The edge $1_1 2_1$ cannot appear in any polychromatic triple. To prove this consider the triple $1_1 2_1 x$. For any $x \in V(K_v) \setminus \{1_1, 2_1\}$, the edges $1_1 x$ and $2_1 x$ are assigned the same colour. The result follows.

Case 2: $v \equiv 3 \pmod 6$
Let $v = 6n+3$ for some positive integer n, and let the vertex set of K_v be $\{a, b, c, 0, 1, \dots, 6n-1\}$.

If $n = 1$, then the result follows from Lemma 3.1. Suppose then that $n \ge 2$. Consider the following edge coloring: apply Lemma 3.2 to each of the induced graphs $K_9 \backslash K_3[a, b, c; 6t, 6t+1, \dots, 6t+5]$ for $0 \le t \le n-1$. Apply Lemma 3.3 to each of the bipartite graphs $K_{6,6}[6i, 6i+1, \dots, 6i+5; 6j, 6j+1, \dots, 6j+5]$ for $0 \le i \ne j \le n-1$. Finally, color the edges ab, ac and bc with the colors c_1, c_2 and c_3 respectively.

Then K_v is equitably 3-edge colored and the edges 12, 34 and 56 cannot be in any polychromatic triple. ☐

Theorem 3.5. *For all $v \equiv 1$ or $3 \pmod 6$, $v \ge 7$, there exists an equitable $(v-2)$-edge coloring of K_v that does not admit any polychromatic STS(v).*

Proof. Let $v \equiv 1$ or 3 (mod 6). Let $1, 2, \dots, v-2$ be the $v-2$ available colors. It is well known that there exists an idempotent commutative quasigroup of any odd order. Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $v-2$ where $Q = \{1, 2, \dots, v-2\}$. Consider an edge coloring of the complete graph K_{v-2} by coloring the edge $ij \in E(K_{v-2})$ with color $i \circ j$.

Hence vertex $i \in V(K_{v-2})$ is incident with exactly one edge of color $j \in \{1, 2, \dots, v-2\} \setminus \{i\}$, and no edges of color i.

Now, add two new vertices x and y and color the edges xi and yi with the color i. Finally, color the edge xy with any of the $v-2$ colors, say color 1.

Obviously, the coloring is equitable at all of the vertices $1, 2, \dots, v-2$ since each vertex z is incident with a single edge colored with one of the colors $1, 2, \dots, v-2$ except the color z that has been used twice to color the edges zx and zy. At vertex

x (and similarly vertex y) all of the $v-2$ colors are used once except color 1 which is used twice.

It is obvious that the edge xy cannot be in any polychromatic triple since for every vertex $z \notin \{x, y\}$, the edges xz and yz receive the same color. ☐

Lemma 3.6. *Let* $v \equiv 1$ *or* $3 \pmod 6$, $v \geq 7$. *Any equitably* $(v-1)$*-edge colored copy of* K_v *will admit a polychromatic* $STS(v)$.

Proof. This follows immediately by noting that the degree of every vertex in K_v is $v-1$. ☐

We conjecture the following:

Conjecture 1. *Let* $v \equiv 1$ *or* $3 \pmod 6$, $v \geq 7$. *There exists an equitable* k*-edge coloring of* K_v *that does not admit any polychromatic* $STS(v)$ *for* $2 \leq k \leq v-2$.

Obviously, the case when $k = 2$ is trivial.

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