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A note on balanced independent sets in the cube

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Abstract

Ramras conjectured that the maximum size of an independent set in the discrete cube $\mathcal{Q}_n$ containing equal numbers of sets of even and odd size is $2^{n-1} - \binom{n-1}{(n-1)/2}$ when n is odd. We prove this conjecture, and find the analogous bound when n is even. The result follows from an isoperimetric inequality in the cube.

The discrete hypercube $\mathcal{Q}_n$ is the graph with vertices the subsets of $[n] = \{1, \ldots, n\}$ and edges between sets whose symmetric difference contains a single element. The cube $\mathcal{Q}_n$ is bipartite, with classes X_0 and X_1 consisting of the sets of even and odd size respectively. The maximum-sized independent sets in $\mathcal{Q}_n$ are precisely X_0 and X_1. Ramras [3] asked: how large an independent set can we find with half its elements in X_0 and half in X_1? Call such an independent set *balanced*. The following result verifies the conjecture made by Ramras for the case where n is odd.

Theorem 1. *The largest balanced independent set in $\mathcal{Q}_n$ has size*

$$2^{n-1} - 2\binom{n-2}{(n-2)/2} \qquad \text{if } n \text{ is even,}$$
$$2^{n-1} - \binom{n-1}{(n-1)/2} \qquad \text{if } n \text{ is odd.}$$

For a set A of vertices of $\mathcal{Q}_n$, write $N(A)$ for the set of vertices adjacent to an element of A. The maximal independent sets in $\mathcal{Q}_n$ all have the form $A \cup (X_1 \setminus N(A))$ for some $A \subseteq X_0$. So for a maximum-sized balanced independent set we seek the largest $A \subseteq X_0$ for which

$$|A| \leq |X_1 \setminus N(A)|.$$

We use the following isoperimetric theorem for even-sized sets, due independently to Bezrukov [1] and Körner and Wei [2] (see also Tiersma [4]). Recall that $x < y$ in the *simplicial* order on $\mathcal{Q}_n$ if either $|x| < |y|$, or $|x| = |y|$ and $x < y$ lexicographically.

Theorem 2 ([1], [2])**.** *Let $A \subseteq X_0$, and let B be the initial segment of the simplicial order restricted to X_0 with $|B| = |A|$. Then $|N(B)| \leq |N(A)|$, and $X_1 \setminus B$ is a terminal segment of the simplicial order restricted to X_1.*

Proof of Theorem 1. We will exhibit an initial segment A of the simplicial order restricted to X_0, and a terminal segment B of the simplicial order restricted to X_1, with $N(A) \cap B = \emptyset$ and $|A| = |B|$ as large as possible. It follows from Theorem 2 that $A \cup B$ will be a maximum-sized balanced independent set.

The form of A and B depends on the residue of n mod 4. For $n = 4k$ we take

$$\begin{aligned} A &= [n]^{(0)} \cup [n]^{(2)} \cup \cdots \cup [n]^{(2k-2)} \cup (12 + [3,n]^{(2k-2)}) \\ B &= (1 + [3,n]^{(2k)}) \cup [2,n]^{(2k+1)} \cup [n]^{(2k+3)} \cup \cdots \cup [n]^{(n-3)} \cup [n]^{(n-1)}, \end{aligned}$$

where, for instance,

$$12 + [3,n]^{(2k-2)} = \{\{1,2\} \cup x : x \subseteq \{3,4,\ldots,n\}, |x| = 2k-2\}\,.$$

For $n = 4k+1$ we take

$$\begin{aligned} A &= [n]^{(0)} \cup [n]^{(2)} \cup \cdots \cup [n]^{(2k-2)} \cup (1 + [2,n]^{(2k-1)}) \\ B &= [2,n]^{(2k+1)} \cup [n]^{(2k+3)} \cup \cdots \cup [n]^{(n-2)} \cup [n]^{(n)}. \end{aligned}$$

For $n = 4k+2$ we take

$$\begin{aligned} A &= [n]^{(0)} \cup [n]^{(2)} \cup \cdots \cup [n]^{(2k-2)} \cup (1 + [2,n]^{(2k-1)}) \cup (2 + [3,n]^{(2k-1)}) \\ B &= [3,n]^{(2k+1)} \cup [n]^{(2k+3)} \cup \cdots \cup [n]^{(n-3)} \cup [n]^{(n-1)}. \end{aligned}$$

Finally, for $n = 4k+3$ we take

$$\begin{aligned} A &= [n]^{(0)} \cup [n]^{(2)} \cup \cdots \cup [n]^{(2k)} \\ B &= [n]^{(2k+3)} \cup \cdots \cup [n]^{(n-2)} \cup [n]^{(n)}. \end{aligned}$$

Verifying that these sets have the claimed sizes, and that $|A| = |B|$ in each case, is a simple application of the identities $\binom{m}{r} = \binom{m-1}{r-1} + \binom{m-1}{r}$, $\binom{m}{r} = \binom{m}{m-r}$ and $\sum_{r=0}^{m} \binom{m}{r} = 2^m$. ☐

The maximum-sized balanced independent sets constructed above are also maximal independent sets. For example, if $n = 4k+3$, then any set not in the family is adjacent to a complete layer; the other cases are similar, with slight complications in the middle layers of the cube.

References

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