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Minimal dominating sets in maximum domatic partitions

S. Arumugam* K. Raja Chandrasekar

National Centre for Advanced Research in Discrete Mathematics (n-CARDMATH)
Kalasalingam University
Anand Nagar, Krishnankoil-626126
India
s.arumugam.klu@gmail.com rajmath84@yahoo.com

Abstract

The domatic number $d(G)$ of a graph $G = (V, E)$ is the maximum order of a partition of V into dominating sets. Such a partition $\Pi = \{D_1, D_2, \ldots, D_d\}$ is called a minimal dominating d-partition if Π contains the maximum number of minimal dominating sets, where the maximum is taken over all d-partitions of G. The minimal dominating d-partition number $\Lambda(G)$ is the number of minimal dominating sets in a minimal dominating d-partition of G. In this paper we initiate a study of this parameter.

1 Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set of G if every vertex not in S is adjacent to at least one vertex in S. A dominating set S is called a minimal dominating set in G if no proper subset of S is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. An excellent treatment of fundamentals of domination is given in the book by Haynes et al. [7] and survey papers on several advanced topics are given in the book edited by Haynes et al. [8]. A domatic partition of G is a partition of $V(G)$ into classes that are pairwise disjoint dominating sets.

* Also at School of Electrical Engineering and Computer Science, The University of Newcastle, NSW 2308, Australia.

The domatic number of G is the maximum cardinality of a domatic partition of G and it is denoted by $d(G)$. The domatic number was introduced by Cockayne and Hedetniemi [5]. For a survey of results on the domatic number and its variants we refer to Chapter 13 in [8] by Zelinka.

In the context of vertex coloring Arumugam et al. ([1] and [3]) investigated the problem of determining the maximum number of maximal independent sets (equivalently, independent dominating sets) in a minimum coloring of G. In this paper we investigate an analogous problem, namely, the maximum number of minimal dominating sets in a maximum domatic partition of G.

We need the following definitions and theorems.

Theorem 1.1. [5] *For any graph G of order n,*

1. $d(G) \leq \delta(G) + 1$, *where $\delta(G)$ denotes the minimum degree of G.*
2. $d(G) \leq n/\gamma(G)$.

Definition 1.2. A graph G for which $d(G) = \delta(G) + 1$ is called *domatically full.*

Definition 1.3. Let $S \subseteq V$ and $u \in S$. Then a vertex v is called a *private neighbor* of u with respect to S if $N[v] \cap S = \{u\}$. If further $v \in V - S$, then v is called an *external private neighbor* of u.

Definition 1.4. Let A and B be two disjoint nonempty subsets of V. We denote by $[A, B]$, the set of all edges in G with one end in A and other end in B. A graph $G = (V, E)$ is said to have a *bijective matching* if there exists a nonempty subset A of V such that $[A, V - A]$ is a perfect matching in G.

Clearly if G has a bijective matching, then n is even and $|A| = n/2$.

Definition 1.5. Given an arbitrary graph G, the *trestled graph of index k*, denoted $T_k(G)$, is the graph obtained from G by adding k copies of K_2 for each edge uv of G and joining u and v to the respective end vertices of each K_2.

Definition 1.6. The *corona* of two graphs G_1 and G_2 is defined to be the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

Theorem 1.7. [10] *For two arbitrary graphs G and H*, $d(G \circ H) = d(H) + 1$.

Definition 1.8. The *join graph* $G_1 + G_2$ is obtained from two graphs G_1 and G_2 by joining every vertex of G_1 with every vertex of G_2.

Definition 1.9. The *cartesian product* of G and H, written $G \Box H$, is the graph with vertex set $V(G \Box H) = \{(u, v) : u \in V(G) \text{ and } v \in V(H)\}$ and edge set $E(G \Box H) = \{(u, v)(u', v') : u = u' \text{ and } vv' \in E(H) \text{ or } v = v' \text{ and } uu' \in E(G)\}$.

Definition 1.10. The *n-cube Q_n* is the graph whose vertex set is the set of all n-dimensional boolean vectors, two vertices being joined if and only if they differ in exactly one coordinate. We observe that $Q_1 = K_2$ and $Q_n = Q_{n-1} \Box K_2$ if $n \geq 2$.

Theorem 1.11. [8] *For any positive integer k, the hypercube Q_{2^k-1} is a regular domatically full graph.*

Definition 1.12. A *decomposition* of a graph G is a family $\mathcal{C}$ of subgraphs of G such that each edge of G is in exactly one member in $\mathcal{C}$.

Observation 1.13. Any graph G admits a decomposition $\mathcal{C}$ of G such that each member of $\mathcal{C}$ is either a cycle or a path.

Theorem 1.14. [8] *For the Petersen graph P, $d(P) = 2$.*

2 Main Results

Let G be a graph with domatic number d and let $\Pi = \{D_1, D_2, \ldots, D_d\}$ be a domatic partition of G. If D_1 is not a minimal dominating set of G, then there exists $v \in D_1$ such that $D_1 - \{v\}$ is also a dominating set of G. We now replace D_1 by $D_1 - \{v\}$ and D_d by $D_d \cup \{v\}$. By repeating this process we obtain a domatic partition of G such that D_1 is a minimal dominating set of G. In fact we can apply the same process of transferring elements from $D_2, D_3, \ldots, D_{d-1}$ to D_d and obtain a domatic partition $\Pi^* = \{S_1, S_2, \ldots, S_d\}$ of G such that $S_1, S_2, \ldots, S_{d-1}$ are minimal dominating sets of G. This observation motivates the following definition.

Definition 2.1. Let G be a graph with domatic number $d(G)$. A domatic partition $\Pi = \{D_1, D_2, \ldots, D_d\}$ is called a *d-partition* of G. A d-partition Π is called a *minimal dominating d-partition* if Π contains maximum number of minimal dominating sets, where the maximum is taken over all d-partitions of G. The *minimal dominating d-partition number* $\Lambda(G)$ is the number of minimal dominating sets in a minimal dominating d-partition of G.

It follows immediately from the definition that $\Lambda(G) = d(G) - 1$ or $d(G)$.

Definition 2.2. A graph G is said to be *class 1* or *class 2* according as $\Lambda = d - 1$ or $\Lambda = d$.

A dominating set in a graph G can be thought of as a "secure set" in the sense that if a guard is placed at each vertex of a dominating set D of a graph G, then every vertex of G comes under the surveillance of at least one guard. If the dominating set D is minimal, then every guard in the troop is essential for the security of the graph. In this process of security, placing the guards always at the some set of vertices is not desirable. Hence if we can identify a collection $\mathcal{C}$ of disjoint minimal dominating sets in G, then at any point of time we can place a troop of guards, one at each vertex of a chosen dominating set from $\mathcal{C}$. In particular if we can find a domatic partition $\{D_1, D_2, \ldots, D_d\}$ of G such that every D_i is a minimal dominating set of G, then we obtain d different options for placing the guards in G. A graph that admits such a partition is of class 2. Otherwise we get a collection of $d - 1$ disjoint minimal dominating sets, each of which gives a possible options for the placement of a troop of guards.

Examples 2.3.

1. Any graph with domatic number n is isomorphic to K_n and is of class 2.

2. Any graph with domatic number $n-1$ is isomorphic to $K_n - e$ and is of class 2.

3. For the corona graph $H = G \circ K_1$, $d(H) = 2$. Also $\Pi = \{V(G), V(H) - V(G)\}$ is a d-partition of H and every member of Π is a minimal dominating set of H. Hence $\Lambda(H) = 2$ and H is of class 2.

4. The complete bipartite graph $K_{m,n}$ with $m \le n$ is of class 2 if and only if $m \in \{1, 2\}$ or $m = n$.

Theorem 2.4. *Let G be a graph of order $n \ge 3$ with $d(G) = n - 2$. Then G is of class 2 if and only if G is isomorphic to $\overline{K_3} + K_{n-3}$ or $H + K_{n-4}$, where $H \in \{P_4, C_4, 2K_2\}$.*

Proof. Suppose G is of class 2. Let $\{V_1, V_2, \ldots, V_{n-2}\}$ be a domatic partition of G such that each V_i is a minimal dominating set of G. Suppose $|V_1| = 3$. Then $|V_i| = 1$, for all $i = 2, 3, \ldots, n-2$. Hence $G = \langle V_1 \rangle + K_{n-3}$. Since V_1 is a minimal dominating set, it follows that V_1 is independent and hence G is isomorphic to $\overline{K_3} + K_{n-3}$. Suppose $|V_1| = |V_2| = 2$. In this case $G = H + K_{n-4}$, where $H = \langle V_1 \cup V_2 \rangle$. Since every vertex of V_1 is adjacent to some vertex of V_2 and vice versa, it follows that $\delta(H) \ge 1$. It follows from the minimality of V_1 and V_2 that $\Delta(H) \le 2$. Hence $H \in \{P_4, C_4, 2K_2\}$.

The converse is obvious. □

Theorem 2.5. *Let G be a graph of order $n \ge 4$ with $d(G) = n - 3$. Then G is of class 2 if and only if G is isomorphic to one of the following graphs.*

(i) $\overline{K_4} + K_{n-4}$,

(ii) $H + K_{n-5}$, *where* $H \in \{K_{2,3}, P_5, P_3 \cup K_2, G_1, G_2, G_3\}$, *or*

(iii) $H + K_{n-6}$, *where H is a hamiltonian graph of order 6 with* $\Delta(H) \le 4$ *or* $H \in \{2K_3, G_4, G_5\}$,

where the graphs $G_1, G_2, \ldots, G_5$ are given in Figure 1.

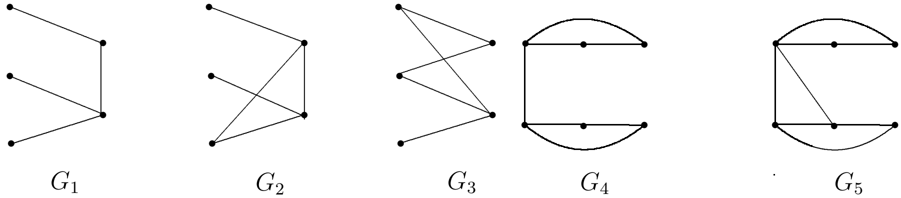

Figure 1.

Proof. Suppose G is of class 2. Let $\{V_1, V_2, \ldots, V_{n-3}\}$ be a domatic partition of G such that each V_i is a minimal dominating set of G.

Case i. $|V_1| = 4$.

In this case $|V_i| = 1$, for all $i = 2, 3, \ldots, n-3$. Hence $G = \overline{K_4} + K_{n-4}$.

Case ii. $|V_1| = 3, |V_2| = 2$.

In this case $G = H + K_{n-5}$, where $H = \langle V_1 \cup V_2 \rangle$.

Let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5\}$. If there exist two adjacent vertices say v_1, v_2 in V_1, with v_4 and v_5 as external private neighbors of v_1 and v_2 respectively with respect to V_1, then V_2 does not dominate v_3, which is a contradiction. Hence V_1 is independent. If $v_4v_5 \in E(H)$ with v_1 and v_2 as external private neighbors of v_4 and v_5 respectively with respect to V_2, then H is isomorphic to G_1 or G_2 according as v_3 is adjacent to one vertex or two vertices of V_2. If $v_4v_5 \notin E(H)$, then H is isomorphic to $G_1,\ G_3,\ P_5,\ P_3 \cup K_2$ *or* $K_{2,3}$.

Case iii. $|V_1| = |V_2| = |V_3| = 2$.

In this case $G = H + K_{n-6}$ where $H = \langle V_1 \cup V_2 \cup V_3 \rangle$.

Let $V_1 = \{v_1, v_2\}, V_2 = \{v_3, v_4\}$ *and* $V_3 = \{v_5, v_6\}$. Since each $V_i,\ i = 1, 2, 3$ is a dominating set of G, it follows that the edge induced subgraph of $[V_i, V_j],\ 1 \le i < j \le 3$ contains $2K_2$ as a subgraph. Hence it follows that either H is Hamiltonian with $\Delta(H) \le 4$ or H contains $2K_3$ as a subgraph. If there exist a $2K_2$ which is edge disjoint from $2K_3$ in H, then H is again Hamiltonian. Hence $|E(H)| = 6$ *or* 7 *or* 8 and hence H is isomorphic to either $2K_3$ *or* G_4 *or* G_5.

The converse is obvious. □

Proposition 2.6. *The cycle C_n is of class 1 if and only if n is odd and $n \not\equiv 0 \pmod 3$.*

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$.

Case i. n is odd and $n \not\equiv 0 \pmod 3$.

In this case $d(C_n) = 2$. Let $\{V_1, V_2\}$ be a domatic partition of G with $|V_1| \ge \lceil \frac{n}{2} \rceil$. We claim that V_1 is not a minimal dominating set of C_n. Suppose V_1 is a minimal dominating set of C_n. Then every component of $\langle V_1 \rangle$ is either K_1 or K_2 and at least one component of $\langle V_1 \rangle$ is K_2, say G_1. Let $V(G_1) = \{v_1, v_2\}$. Then v_n and v_3 are external private neighbors of v_1 and v_2 respectively with respect to V_1. Hence it follows that v_3, v_4, v_{n-1} *and* v_n are in V_2. Similarly if $\{v_i\}$ is a component of $\langle V_1 \rangle$, then v_{i-1} and v_{i+1} are in V_2. Hence it follows that $|V_2| > |V_1|$, which is a contradiction. Hence V_1 is not a minimal dominating set of C_n and C_n is of class 1.

Case ii. n is even and $n \not\equiv 0 \pmod 3$.

In this case $d(C_n) = 2$ and $\{V_1, V - V_1\}$, where $V_1 = \{v_i : i \text{ is odd}\}$ is a domatic partition of C_n and both V_1 and $V - V_1$ are minimal dominating sets of C_n. Hence C_n is of class 2.

Case iii. $n \equiv 0 \pmod 3$.

In this case $d(C_n) = 3$ and $\{V_0, V_1, V_2\}$, where $V_i = \{v_j : j \equiv i \pmod 3\}, i =$

$0, 1, 2$ is a domatic partition of C_n and each V_i is a minimal dominating set of C_n. Hence C_n is of class 2. □

We now proceed to consider graphs with domatic number 2. If $d(G) = 2$ and G is bipartite, then trivially G is of class 2. In particular all trees are of class 2. The following theorem gives a characterization of all nonbipartite connected graphs of class 2 with $d(G) = 2$ and $\delta(G) \geq 2$.

Theorem 2.7. *Let G be a nonbipartite connected graph with $\delta(G) \geq 2$ and $d(G) = 2$. Then G is of class 2 if and only if G has a bijective matching.*

Proof. If G has a bijective matching $[A, V - A]$, then A and $V - A$ are both minimal dominating sets of G and hence G is of class 2.

Conversely, suppose G is of class 2. Let $\{A, V - A\}$ be a domatic partition of G such that both A and $V - A$ are minimal dominating sets of G. Since G is not bipartite, we may assume that there exists a vertex x_1 in A such that x_1 is not an isolated vertex in $\langle A \rangle$. Since A is a minimal dominating set, it follows that there exists a vertex y_1 in $V - A$ such that y_1 is an external private neighbor of x_1 with respect to A. Since $\delta(G) \geq 2$, it follows that y_1 is not an isolated vertex in $\langle V - A \rangle$ and x_1 is the private neighbor of y_1 with respect to $V - A$. Hence if $A_1 = \{x \in A : x$ is not an isolated vertex in $\langle A \rangle\}$ and $B_1 = \{y \in V - A : y$ is not an isolated vertex in $\langle V - A \rangle\}$, then $[A_1, B_1]$ forms a perfect matching in $\langle A_1 \cup B_1 \rangle$. Since G is connected, it follows that $A_1 = A$ and $B_1 = V - A$ and hence G has a bijective matching. □

Corollary 2.8. *The Petersen graph is of class 2.*

Proof. The result follows from Theorem 1.14. □

Problem 2.9. *Characterize nonbipartite graphs of class 2 with $\delta = 1$ and $d = 2$.*

Theorem 2.10. *Let G and H be two connected graphs. Then H is of class 2 if and only if $G \circ H$ is of class 2.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $H_1, H_2, \ldots, H_n$ be n copies of H in $G \circ H$ such that v_i is adjacent to all the vertices in H_i. Suppose H is of class 2. Let $\{U_1, U_2, \ldots, U_{d(H)}\}$ be a domatic partition of H such that each U_i is a minimal dominating set of H. Let $\{U_1^i, U_2^i, \ldots, U_{d(H)}^i\}$ be the corresponding domatic partition of H_i and let $V_i = \bigcup_{j=1}^{n} U_i^j$. Then $\mathcal{P} = \{V_1, V_2, \ldots, V_{d(H)}, V(G)\}$ is a domatic partition of $G \circ H$ and each member of $\mathcal{P}$ is a minimal dominating set of $G \circ H$. Hence it follows from Theorem 1.7 that $G \circ H$ is of class 2.

Conversely, suppose $G \circ H$ is of class 2. Let $\{M_1, M_2, \ldots, M_{d(H)+1}\}$ be a domatic partition of $G \circ H$ such that each M_i is a minimal dominating set of $G \circ H$. Let $v_1 \in M_1$. Then $M_2 \cap V(H_1), \ldots, M_{d(H)+1} \cap V(H_1)$ are minimal dominating sets of H_1. Thus H_1 and hence H is of class 2. □

Proposition 2.11. *Any r-regular domatically full graph G is of class 2.*

Proof. Let $C = \{V_1, V_2, \ldots, V_{r+1}\}$ be a domatic partition of G. Now let $v \in V_i$. Since each $V_j, j \neq i, 1 \leq j \leq r+1$, contains one neighbor of v and $|N(v)| = r$, it follows that $N(v) \cap V_i = \emptyset$. Thus V_i is independent. Hence each V_i is a minimal dominating set of G and G is of class 2. □

Corollary 2.12. *The hypercube Q_{2^k-1} is of class 2.*

Proof. The result follows from Theorem 1.11. □

Theorem 2.13. *The graph $G = K_r \Box K_{1,s}$ is of class 2.*

Proof. Let $V(K_r) = \{u_0, u_1, \ldots, u_{r-1}\}$ and let $V(K_{1,s}) = \{v_0, v_1, \ldots, v_s\}$ where v_s is the centre vertex of $K_{1,s}$. Clearly the induced subgraph $H_i = \langle\{(u_j, v_i) : 0 \leq j \leq r-1\}\rangle$ is isomorphic to K_r, for all i, $0 \leq i \leq s$.

Case i. $r > s$.

We claim that $\gamma(G) = s+1$. Clearly $S = \{(u_0, v_i) : 0 \leq i \leq s\}$ is a dominating set of G and hence $\gamma(G) \leq s+1$. Now, let S_1 be any γ-set of G. If $S_1 \cap H_i \neq \emptyset$ for all i, then $\gamma(G) = |S_1| \geq s+1$. Suppose $S_1 \cap H_i = \emptyset$ for some i. Since for every $u, v \in H_i$, $N(u) \cap N(v) \cap (V(G) - V(H_i)) = \emptyset$, it follows that S_1 contains r vertices to dominate the vertices of H_i and hence $|S_1| \geq r \geq s+1$. Thus $\gamma(G) = s+1$. Hence by Theorem 1.1, $d(G) \leq \frac{|V(G)|}{\gamma(G)} = r$. Now, let $A_i = \{(u_i, v_j) : 0 \leq j \leq s\}$. Clearly each A_i is a minimal dominating set of G and $\{A_0, A_1, \ldots, A_{r-1}\}$ is a domatic partition of G. Hence G is of class 2.

Case ii. $r \leq s$.

Clearly $S = V(H_s)$ is a dominating set of G and hence $\gamma(G) \leq r$. By using an argument similar to that of case i, it can be proved that $\gamma(G) = r$. Now, let $B_i = \{(u_j, v_{j+i}) : 0 \leq j \leq r-1\} \cup \{(u_i, v_j) : r \leq j \leq s-1\}$, $0 \leq i \leq r-1$, and let $B_{r+1} = V(H_s)$, where addition in the suffix is taken modulo r. Clearly each B_i is a minimal dominating set of G and $\{B_1, B_2, \ldots, B_{r+1}\}$ is a domatic partition of G. Further since $d(G) \leq \delta(G) + 1 = r+1$, it follows that G is of class 2. □

In the following theorem we characterize the class of trestled graphs of class 2.

Theorem 2.14. *Let G be a graph of order n. Then the trestled graph $T_k(G)$ is of class 2 if and only if one of the following holds.*

1. $k \geq 2$,
2. $k = 1$ *and* $\delta(G) \geq 2$, *and*
3. $k = 1$, $\delta(G) = 1$ *and* G *is bipartite.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. For each edge $v_iv_j \in E(G)$, let $v_{ij}^1 v_{ji}^1, v_{ij}^2 v_{ji}^2, \ldots, v_{ij}^k v_{ji}^k$ be the corresponding k edges in $T_k(G)$ with v_i adjacent to v_{ij}^r and v_j adjacent to v_{ji}^r, $1 \leq r \leq k$. It follows from Theorem 1.1 that $d(T_k(G)) \leq \delta(T_k(G)) + 1 = 3$.

Case i. $k \geq 2$.

Let $D_1 = V(G)$. For each edge $e = v_iv_j \in E(G)$, *let* $D_e = \{v_{ij}^1, v_{ji}^2, v_{ij}^3, \ldots, v_{ij}^k\}$ and let $D_2 = \bigcup_{e \in E(G)} D_e$. Let $D_3 = V(T_k(G)) - (D_1 \cup D_2)$. Clearly each D_i is a minimal dominating set of $T_k(G)$ and $\{D_1, D_2, D_3\}$ is a domatic partition of $T_k(G)$. Hence $T_k(G)$ is of class 2.

Case ii. $k = 1$ and $\delta(G) \geq 2$.

Clearly there exists a decomposition $\mathcal{C}$ of G such that every member of $\mathcal{C}$ is a cycle or a path. We choose such a decomposition $\mathcal{C} = \{C_1, C_2, \ldots, C_r, P_1, P_2, \ldots, P_s\}$, where $r \geq 1, s \geq 0$ satisfying the following conditions.

(i) The number of cycles r is maximum.

(ii) Among all decompositions of G which use r cycles, $\mathcal{C}$ is such that the number of paths s is minimum.

It follows from (i) that the edge induced subgraph induced by $E(P_1) \cup E(P_1) \cup \cdots \cup E(P_s)$ is acyclic. Hence any two paths P_i, P_j in $\mathcal{C}$ have at most one common vertex. Also it follows from (ii) that if $v \in V(P_i) \cap V(P_j)$, then v is an internal vertex of at least one of the paths P_i and P_j. Hence every vertex of G lies on a cycle C_i or is an internal vertex of some path P_j. We now proceed to construct a domatic partition of G.

For any cycle $C_i = (v_1, v_2, \ldots, v_a, v_1)$ in $\mathcal{C}$, let $S_i = \{v_{12}^1, v_{23}^1, \ldots, v_{a1}^1\}$ and $S_i' = \{v_{21}^1, v_{32}^1, \ldots, v_{1a}^1\}$. Also for any path $P_j = (u_1, u_2, \ldots, u_b)$ in $\mathcal{C}$, let $T_j = \{u_{12}^1, u_{23}^1, \ldots, u_{b-1b}^1\}$ and $T_j' = \{u_{21}^1, u_{32}^1, \ldots, u_{bb-1}^1\}$. Let $D_1 = V(G), D_2 = \left(\bigcup_{i=1}^{r} S_i\right) \cup \left(\bigcup_{i=1}^{s} T_i\right)$ and $D_3 = \left(\bigcup_{i=1}^{r} S_i'\right) \cup \left(\bigcup_{i=1}^{s} T_i'\right)$. Obviously D_1 is a minimal dominating set of $T_k(G)$. Now we claim that D_2 and D_3 are minimal dominating sets of $T_k(G)$. It follows from the definition of D_2 and D_3 that if $v_{ij}^1 \in D_2$, then $v_{ji}^1 \in D_3$. Hence D_2 and D_3 dominate each other. Now let $v_r \in V(G)$. If v_r belongs to some cycle C_i in $\mathcal{C}$, let v_s and v_k be the two neighbors of v_r in C_i. Then $v_{rs}^1 \in D_2, v_{rk}^1 \in D_3$ and v_r is adjacent to v_{rs}^1 and v_{rk}^1. Similarly if v_r is an internal vertex of some path in $\mathcal{C}$, then v_r is adjacent to a vertex of D_2 and a vertex of D_3. Thus D_2 and D_3 are dominating sets of $T_k(G)$ and since D_2 and D_3 are independent, it follows that D_2 and D_3 are minimal dominating sets of $T_k(G)$. Thus $\{D_1, D_2, D_3\}$ is a maximum domatic partition of G into minimal dominating sets and hence $T_k(G)$ is of class 2.

Case iii. $k = 1$ and $\delta(G) = 1$ *and* G is bipartite.

Let $v_1 \in V(G)$ be such that $deg_G(v_1) = 1$ and $v_1v_2 \in E(G)$. Suppose $d(T_k(G)) = 3$. Let $\{D_1, D_2, D_3\}$ be a domatic partition of $T_k(G)$. Since $deg_{T_k(G)}(v_1) = 2$, we may assume that $v_2 \in D_1, v_1 \in D_2$, and $v_{12}^1 \in D_3$. Since the two neighbors of v_{21}^1 are in D_1 and D_3, it follows that $v_{21}^1 \in D_2$, but then v_{12}^1 is not dominated by D_1, which is a contradiction. Hence $d(T_k(G)) = 2$. Since G is bipartite, it follows that $T_k(G)$ is bipartite and hence $T_k(G)$ is of class 2.

Conversely, suppose $T_k(G)$ is of class 2. Suppose $k = 1$, $\delta(G) = 1$ and G is not bipartite. Let $\{D_1, D_2\}$ be a domatic partition of $T_k(G)$ such that both D_1 and D_2 are minimal dominating sets in $T_k(G)$. Since G is not bipartite, it follows that there exist two adjacent vertices v_i and v_j in G such that $v_i, v_j \in D_1$. It follows from the minimality of D_1 and D_2 that $v_{ij}^1, v_{ji}^1 \in D_2$. Let $v_k \in V(G)$ be such that $v_k v_j \in E(G)$. Since v_j is an external private neighbor of v_{ji}^1 with respect to D_2, it follows that $v_k, v_{jk}^1 \in D_1$. Hence $v_{kj}^1 \notin D_1 \cup D_2$, which is a contradiction. Hence the proof. □

3 Complexity Results

Kaplan and Shamir [6] have proved that the domatic number problem is NP-complete for several families of perfect graphs, including chordal and bipartite graphs. In this section we prove that given a graph G, the problem of deciding whether G is of class 2 is NP-hard even when restricted to split graphs and bipartite graphs with $d(G) \geq 3$. We prove the result by a reduction from the 3-coloring problem and we use the proof technique given in [6].

DOMATIC PARTITION BASED CLASSIFICATION (DPBC)

Instance. A graph G.

Question. Is G of class 2?

We prove that DPBC is NP-hard even when restricted to split graphs and bipartite graphs.

We use the following well known NP-complete problem.

3-COLORING PROBLEM

Instance. A graph G.

Question. Is G 3-colorable?

Theorem 3.1. *DPBC is NP-hard for split graphs.*

Proof. The proof is by a reduction from the 3-coloring problem. Given a graph $G = (V, E)$ for the 3-coloring problem, construct a new graph $\tilde{G}$ by adding a new vertex on each of the original edges, and adding edges to form a clique on the original vertices. Thus $\tilde{G} = (\tilde{V}, \tilde{E})$ where $\tilde{V} = V' \cup V''$, $V' = V, V'' = \{v_{ij} : ij \in E\}$, and $\tilde{E} = E' \cup E'', E'$ is a clique on V' and $E'' = \{iv_{ij}, jv_{ij} : ij \in E\}$. The construction of $\tilde{G}$ is clearly polynomial. Also $\tilde{G}$ is a split graph, since V' induces a clique and V'' induces an independent set. Since $\delta(\tilde{G}) = 2$, it follows that $d(\tilde{G}) \leq 3$. We claim that G is 3-colorable if and only if $\tilde{G}$ is of class 2. Suppose G is 3-colorable. Let $\{V_1, V_2, V_3\}$ be a 3-coloring of G. We form a domatic partition $\{\tilde{D}_1, \tilde{D}_2, \tilde{D}_3\}$ as follows: Assign each $v \in V'$ to $\tilde{D}_i$ if $v \in V_i$. For $v_{ij} \in V''$, if $i \in V_l, j \in V_m$, then assign v_{ij} to the third class $\tilde{D}_k, k \neq l, k \neq m$. We claim that each $\tilde{D}_i$ is a minimal dominating set of $\tilde{G}$. Since $\tilde{D}_i \cap V' \neq \emptyset$ and V' induces a clique, $\tilde{D}_i$ dominates V'.

Also each triangle $\{k, j, v_{kj}\}$ contains one representative from each $\tilde{D}_i$, so that each $\tilde{D}_i$ dominates V''. Now let $r \in V' \cap \tilde{D}_i$ and let $rs \in E(G)$. Clearly $v_{rs} \notin \tilde{D}_i$. Hence v_{rs} is a private neighbor of r with respect to $\tilde{D}_i$. If $r \in V'' \cap \tilde{D}_i$, then r is an isolated vertex in $\left\langle \tilde{D}_i \right\rangle$ and hence each $\tilde{D}_i$ is a minimal dominating set in $\tilde{G}$. Thus $d(\tilde{G}) = 3$ and $\tilde{G}$ is of class 2.

Conversely suppose $\tilde{G}$ is of class 2. We claim that $d(\tilde{G}) = 3$. Suppose $d(\tilde{G}) = 2$. Since $\tilde{G}$ is nonbipartite and $\delta(\tilde{G}) = 2$, it follows from Theorem 2.7 that $\tilde{G}$ has a bijective matching, say V_1, V_2. Let $ij \in E(G)$. Since $\langle \{i, j, v_{ij}\} \rangle = K_3$, and $[V_1, V_2]$ is a bijective matching, we may assume that $\{i, j, v_{ij}\} \subseteq V_1$, but in this case v_{ij} is not adjacent to a vertex of V_2, which is a contradiction to $[V_1, V_2]$ is a bijective matching of $\tilde{G}$. Hence $d(\tilde{G}) = 3$. Let $\{\tilde{D}_1, \tilde{D}_2, \tilde{D}_3\}$ be a domatic partition of $\tilde{G}$. Since each triangle $\{i, j, v_{ij}\}$ intersects every $\tilde{D}_i$, it follows that no edge $ij \in E$ has both end points in the same set and hence the restricted coloring on $V' = V$ is a proper 3-coloring of G. □

Theorem 3.2. *DPBC is NP-hard for bipartite graphs G with $d(G) \geq 3$.*

Proof. The reduction is again from the 3-coloring problem. For an instance $G = (V, E)$ of 3-coloring problem, we form a bipartite graph $\tilde{G}$ by adding a vertex on each edge of the original graph. Thus $\tilde{G} = (\tilde{V}, \tilde{E})$ where $\tilde{V} = V' \cup V'', V' = V, V'' = \{v_{ij} : ij \in E\}$, and $\tilde{E} = \{iv_{ij}, jv_{ij} : ij \in E\}$. The reduction is clearly polynomial. Suppose G is 3-colorable. Let $\{V_1, V_2, V_3\}$ be a 3-coloring of G. Construct a domatic partition $\{\tilde{D}_1, \tilde{D}_2, \tilde{D}_3\}$ of $\tilde{G}$ as follows: Assign each $v \in V'$ in $\tilde{G}$ to $\tilde{D}_i$ if $v \in V_i$ in G. For each edge $ij \in E$, assign v_{ij} into the third set not assigned to either i or j. Since each V_k is independent, it follows that no edge $ij \in E$ has both end points in the same set and hence the third set is uniquely defined. We claim that each $\tilde{D}_i$ is a minimal dominating set of $\tilde{G}$. Let $r \in \tilde{D}_i \cap V'$ and let $rs \in E(G)$. Clearly $v_{rs} \notin \tilde{D}_i$ and hence v_{rs} is a private neighbor of r with respect to $\tilde{D}_i$. If $r \in \tilde{D}_i \cap V''$, then r is an isolated vertex in $\left\langle \tilde{D}_i \right\rangle$ and hence each $\tilde{D}_i$ is a minimal dominating set in $\tilde{G}$. Hence $\tilde{G}$ is of class 2.

Conversely suppose $\tilde{G}$ is of class 2. By using an argument similar to the proof of Theorem 3.1, $d(\tilde{G}) = 3$. Let $\{\tilde{D}_1, \tilde{D}_2, \tilde{D}_3\}$ be a domatic partition of $\tilde{G}$. Since each triangle $\{r, s, v_{rs}\}$ intersects every $\tilde{D}_i$, it follows that no edge $rs \in E$ has both end points in the same set and hence the restricted coloring on $V' = V$ is a proper 3-coloring of G. □

Problem 3.3. *Design an efficient algorithm for determining whether a given graph G is of class 2, when G is restricted to special families of graphs such as interval graphs and circular arc graphs.*

4 Conclusion and Scope

Let P be a graph theoretic property concerning subsets of the vertex set V. A subset S of V is called a P-set if S has the property P. A P-partition of G is a partition

$\{V_1, V_2, \ldots, V_k\}$ of V such that each V_i is a P-set. If the property P is hereditary (super hereditary), then the P-partition number $\Pi_P(G)$ is the minimum (maximum) cardinality of such a partition. If P is the property that S is an irredundant set, then $\Pi_P(G)$ is the irratic number $\chi_{ir}(G)$, which has been investigated in Hedetniemi et al. [9]. If P is the property that S is an open irredundant set, then $\Pi_P(G)$ is the open irratic number $\chi_{oir}(G)$. Several basic results concerning $\chi_{oir}(G)$ are given in Arumugam et al. [2] and they have obtained a characterization of all graphs G with $\chi_{oir}(G) = 2$. One could investigate the problem of finding the maximum number of minimal or maximal P-sets where the maximum is taken over all P-partitions of order $\Pi_P(G)$.

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