AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **53** (2012), Pages 257–262

Partitions of $\mathbb{Z}_m$ with the same representation functions*

Quan-Hui Yang†

School of Mathematical Sciences
Nanjing Normal University
Nanjing 210046
P. R. China
yangquanhui01@163.com

Feng-Juan Chen‡

Department of Mathematics
Suzhou University
Suzhou 215006
P. R. China

Abstract

For $A \subseteq \mathbb{Z}_m$ and $n \in \mathbb{Z}_m$, let $\sigma_1(A,n), \sigma_2(A,n), \sigma_3(A,n)$ denote the number of solutions of the equation $n = a + a'$ with ordered pairs $(a, a') \in A^2$, unordered pairs $(a, a') \in A^2 (a \neq a')$ and unordered pairs $(a, a') \in A^2$, respectively. In this paper, for any $i \in \{1, 2, 3\}$, we determine all subsets A of $\mathbb{Z}_m$ such that $\sigma_i(A, n) = \sigma_i(\mathbb{Z}_m \backslash A, n)$ holds for all $n \in \mathbb{Z}_m$.

1 Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. For a set $A \subseteq \mathbb{N}$, let $R_1(A,n), R_2(A,n)$, $R_3(A,n)$ denote the number of solutions of

$$\begin{aligned} a + a' = n, &\quad a, a' \in A, \\ a + a' = n, &\quad a, a' \in A, a < a', \\ a + a' = n, &\quad a, a' \in A, a \leqslant a', \end{aligned}$$

* This work was supported by the National Natural Science Foundation of China, Grant No. 11071121, and the second author supported by the Project of Graduate Education Innovation of Jiangsu Province(CXZZ11-0868).
† Corresponding author.
‡ (Also at same address as first author.)

respectively. For $i \in \{1,2,3\}$, Sárközy asked ever whether there are sets A and B with infinite symmetric difference such that $R_i(A,n) = R_i(B,n)$ for all sufficiently large integer n. In [4], Dombi proved that the answer is negative for $i = 1$ and positive for $i = 2$. For $i = 3$, Chen and Wang [3] proved the set of nonnegative integers can be partitioned into two subsets A and B with $R_3(A,n) = R_3(B,n)$ for all $n \geqslant n_0$. Lev (see [8]) gave a simple common proof to the results by Dombi [4] and Chen and Wang [3]. In [9], using generating functions, Sándor established the two following stronger results. Recently, Tang [10] gave a more natural proof of them.

Theorem A. *Let N be a positive integer. Then $R_2(A,n) = R_2(\mathbb{N} \setminus A, n)$ for all $n \geqslant 2N-1$ if and only if $|A \cap [0, 2N-1]| = N$ and $2m \in A \Leftrightarrow m \in A, 2m+1 \in A \Leftrightarrow m \notin A$ for all $m \geqslant N$.*

Theorem B. *Let N be a positive integer. Then $R_3(A,n) = R_3(\mathbb{N} \setminus A, n)$ for all $n \geqslant 2N-1$ if and only if $|A \cap [0, 2N-1]| = N$ and $2m \in A \Leftrightarrow m \notin A, 2m+1 \in A \Leftrightarrow m \in A$ for all $m \geqslant N$.*

For other related results, the reader is referred to [1, 2, 5–7]. In this paper, we shall consider the analogue of the above two theorems in $\mathbb{Z}_m$.

For a positive integer $m \geq 2$, let $\mathbb{Z}_m = \{\bar{0}, \bar{1}, \ldots, \overline{m-1}\}$ be the set of residue classes mod m, with the ordering defined as $\bar{0} \prec \bar{1} \prec \cdots \prec \overline{m-1}$. Besides, $\overline{x} \preceq \overline{y}$ if and only if $\overline{x} \prec \overline{y}$ or $\overline{x} = \overline{y}$. For $A \subseteq \mathbb{Z}_m$ and $n \in \mathbb{Z}_m$, let

$$\begin{aligned}\sigma_1(A,n) &= \sharp\{(a,a') \in A^2 : a + a' = n\},\\ \sigma_2(A,n) &= \sharp\{(a,a') \in A^2 : a + a' = n, a \prec a'\},\\ \sigma_3(A,n) &= \sharp\{(a,a') \in A^2 : a + a' = n, a \preceq a'\}.\end{aligned}$$

It is easy to notice that $\sigma_1(A,n) = \sigma_2(A,n) + \sigma_3(A,n)$ for any $A \subseteq \mathbb{Z}_m$ and $n \in \mathbb{Z}_m$. In this paper, based on the method of Tang, the following results are proved.

Theorem 1. *For $i \in \{2,3\}$, the equality $\sigma_i(A,n) = \sigma_i(\mathbb{Z}_m \backslash A, n)$ holds for all $n \in \mathbb{Z}_m$ if and only if m is even and $t \in A \Leftrightarrow t + \overline{m/2} \notin A$ for $t = \bar{0}, \bar{1}, \ldots, \overline{m/2-1}$.*

Theorem 2. *The equality $\sigma_1(A,n) = \sigma_1(\mathbb{Z}_m \backslash A, n)$ holds for all $n \in \mathbb{Z}_m$ if and only if m is even and $|A| = m/2$.*

Currently we have no answers for the following problems.

Problem 1. Given a positive integer $k (k \geqslant 3)$. For $i \in \{1,2,3\}$, does there exist a partition

$$\mathbb{Z}_m = \bigcup_{t=1}^{k} A_t, \quad A_u \cap A_v = \emptyset, \quad u \neq v$$

such that $\sigma_i(A_t, n) = \sigma_i(A_{t'}, n)$ $(1 \leqslant t < t' \leqslant k)$ for all $n \in \mathbb{Z}_m$?

Problem 2. For $i \in \{1,2,3\}$, determine all pairs of subsets $A, B \subseteq \mathbb{Z}_m$ such that $\sigma_i(A,n) = \sigma_i(B,n)$ for all $n \in \mathbb{Z}_m$.

2 Proofs

Before the proof, we give the following lemma.

Lemma 1. *For any $i \in \{1,2,3\}$, if the equality $\sigma_i(\mathbb{Z}_m\backslash A, n) = \sigma_i(A, n)$ holds for all $n \in \mathbb{Z}_m$, then m is even and $|A| = m/2$.*

Proof. If $\sigma_2(\mathbb{Z}_m\backslash A, n) = \sigma_2(A, n)$ holds for all $n \in \mathbb{Z}_m$, then we have

$$\binom{|\mathbb{Z}_m\backslash A|}{2} = \sum_{n\in\mathbb{Z}_m} \sigma_2(\mathbb{Z}_m\backslash A, n) = \sum_{n\in\mathbb{Z}_m} \sigma_2(A, n) = \binom{|A|}{2}.$$

If $\sigma_3(\mathbb{Z}_m\backslash A, n) = \sigma_3(A, n)$ holds for all $n \in \mathbb{Z}_m$, then we have

$$\binom{|\mathbb{Z}_m\backslash A|+1}{2} = \sum_{n\in\mathbb{Z}_m} \sigma_3(\mathbb{Z}_m\backslash A, n) = \sum_{n\in\mathbb{Z}_m} \sigma_3(A, n) = \binom{|A|+1}{2}.$$

If $\sigma_1(\mathbb{Z}_m\backslash A, n) = \sigma_1(A, n)$ holds for all $n \in \mathbb{Z}_m$, then we have

$$\begin{aligned}\binom{|\mathbb{Z}_m\backslash A|}{2} + \binom{|\mathbb{Z}_m\backslash A|+1}{2} &= \sum_{n\in\mathbb{Z}_m} \sigma_1(\mathbb{Z}_m\backslash A, n)\\ &= \sum_{n\in\mathbb{Z}_m} \sigma_1(A, n)\\ &= \binom{|A|}{2} + \binom{|A|+1}{2}.\end{aligned}$$

Hence we get $|A| = |\mathbb{Z}_m\backslash A|$. This completes the proof of Lemma 1. □

Proof of Theorem 1. Assume that if $n \in \mathbb{Z}_m$, then $0 \leqslant n \leqslant m-1$. Write

$$\chi(n) = \begin{cases} 1 & \text{if } \ n \in A,\\ 0 & \text{otherwise.}\end{cases}$$

Then for $0 \leqslant n \leqslant m-1$, we have

$$\begin{aligned}&\sigma_2(\mathbb{Z}_m\backslash A, n)\\ &= \sharp\{(a,a') : a, a' \in \mathbb{Z}_m\backslash A, a < a', a + a' = n \text{ or } a + a' = n + m\}\\ &= \sum_{0\leqslant i<n/2} (1-\chi(i))(1-\chi(n-i)) + \sum_{n+1\leqslant i<(n+m)/2} (1-\chi(i))(1-\chi(n+m-i))\\ &= \sum_{0\leqslant i<n/2} 1 - \sum_{0\leqslant i\leqslant n} \chi(i) + \chi(\tfrac{n}{2}) + \sum_{0\leqslant i<n/2} \chi(i)\chi(n-i) + \sum_{n+1\leqslant i<(n+m)/2} 1\\ &\quad - \sum_{n+1\leqslant i\leqslant m-1} \chi(i) + \chi(\tfrac{n+m}{2}) + \sum_{n+1\leqslant i<(n+m)/2} \chi(i)\chi(n+m-i)\\ &= \sum_{0\leqslant i<n/2} 1 + \sum_{n+1\leqslant i<(n+m)/2} 1 - |A| + \chi(\tfrac{n}{2}) + \chi(\tfrac{n+m}{2}) + \sigma_2(A, n).\end{aligned} \tag{1}$$

First, we prove the necessary part of Theorem 1. Suppose that $\sigma_2(A,n)=\sigma_2(\mathbb{Z}_m\backslash A,n)$ holds for all $n\in\mathbb{Z}_m$; by Lemma 1, we have m is even and $|A|=m/2$. Hence, by (1), we have that

$$\sum_{0\leqslant i<n/2}1+\sum_{n+1\leqslant i<(n+m)/2}1=m/2-\chi(\frac{n}{2})-\chi(\frac{n+m}{2}) \tag{2}$$

holds for all $n\in\mathbb{Z}_m$. If n is odd, by (2) we have $\chi(\frac{n}{2})+\chi(\frac{n+m}{2})=0$, which is obviously true. If n is even, by (2) we get $\chi(\frac{n}{2})+\chi(\frac{n+m}{2})=1$. Hence

$$t\in A\Leftrightarrow t+\overline{m/2}\notin A \text{ for } t=\bar{0},\bar{1},\ldots,\overline{m/2-1}.$$

Next, we prove the sufficient part of Theorem 1. Suppose that m is even, $|A|=m/2$ and

$$t\in A\Leftrightarrow t+\overline{m/2}\notin A \text{ for } t=\bar{0},\bar{1},\ldots,\overline{m/2-1},$$

we have

$$\sum_{0\leqslant i<n/2}1+\sum_{n+1\leqslant i<(n+m)/2}1+\chi(\frac{n}{2})+\chi(\frac{n+m}{2})=|A|.$$

By (1), we have

$$\sigma_2(\mathbb{Z}_m\backslash A,n)=\sigma_2(A,n)$$

holds for all $n\in\mathbb{Z}_m$. Noting that

$$\begin{aligned}
&\sigma_3(\mathbb{Z}_m\backslash A,n)\\
&=\sharp\{(a,a'):a,a'\in\mathbb{Z}_m\backslash A, a\leqslant a', a+a'=n \text{ or } a+a'=n+m\}\\
&=\sum_{0\leqslant i\leqslant n/2}(1-\chi(i))(1-\chi(n-i))+\sum_{n+1\leqslant i\leqslant(n+m)/2}(1-\chi(i))(1-\chi(n+m-i))\\
&=\sum_{0\leqslant i\leqslant n/2}1-\sum_{0\leqslant i\leqslant n}\chi(i)-\chi(\tfrac{n}{2})+\sum_{0\leqslant i\leqslant n/2}\chi(i)\chi(n-i)+\sum_{n+1\leqslant i\leqslant(n+m)/2}1\\
&\quad-\sum_{n+1\leqslant i\leqslant m-1}\chi(i)-\chi(\tfrac{n+m}{2})+\sum_{n+1\leqslant i\leqslant(n+m)/2}\chi(i)\chi(n+m-i)\\
&=\sum_{0\leqslant i\leqslant n/2}1+\sum_{n+1\leqslant i\leqslant(n+m)/2}1-|A|-\chi(\tfrac{n}{2})-\chi(\tfrac{n+m}{2})+\sigma_3(A,n).
\end{aligned} \tag{3}$$

Similarly, we have $\sigma_3(A,n)=\sigma_3(\mathbb{Z}_m\backslash A,n)$ holds for all $n\in\mathbb{Z}_m$ if and only if m is even and

$$\sum_{0\leqslant i\leqslant n/2}1+\sum_{n+1\leqslant i\leqslant(n+m)/2}1=|A|+\chi(\frac{n}{2})+\chi(\frac{n+m}{2})$$

holds for all $n\in\mathbb{Z}_m$.

This completes the proof of Theorem 1. □

Proof of Theorem 2. By Lemma 1, it is sufficient to prove that if m is even and $|A| = m/2$, then $\sigma_1(A, n) = \sigma_1(\mathbb{Z}_m \backslash A, n)$ holds for all $n \in \mathbb{Z}_m$. By (1) and (3), we have

$$\begin{aligned} \sigma_1(\mathbb{Z}_m \backslash A, n) &= \sigma_2(\mathbb{Z}_m \backslash A, n) + \sigma_3(\mathbb{Z}_m \backslash A, n) \\ &= \sum_{0 \leqslant i < n/2} 1 + \sum_{n+1 \leqslant i < (n+m)/2} 1 - 2|A| \\ &\quad + \sum_{0 \leqslant i \leqslant n/2} 1 + \sum_{n+1 \leqslant i \leqslant (n+m)/2} 1 + \sigma_1(A, n). \end{aligned} \tag{4}$$

Suppose that m is even; we have

$$\sum_{0 \leqslant i < n/2} 1 + \sum_{n+1 \leqslant i < (n+m)/2} 1 + \sum_{0 \leqslant i \leqslant n/2} 1 + \sum_{n+1 \leqslant i \leqslant (n+m)/2} 1 = m,$$

by (4) and $|A| = m/2$, so we have that $\sigma_1(A, n) = \sigma_1(\mathbb{Z}_m \backslash A, n)$ holds for all $n \in \mathbb{Z}_m$.

This completes the proof of Theorem 2. □

References

[1] Y. G. Chen, A. Sárközy, V. T. Sós and M. Tang, On the monotonicity properties of additive representation functions, *Bull. Austral. Math. Soc.* 72 (1) (2005), 129–138.

[2] Y. G. Chen and M. Tang, Partitions of nature numbers with the same representation functions, *J. Number Theory* 129 (2009), 2689–2695.

[3] Y. G. Chen and B. Wang, On additive properties of two special sequences, *Acta Arith.* 110 (3) (2003), 299–303.

[4] G. Dombi, Additive properties of certain sets, *Acta Arith.* 103 (2) (2002), 137–146.

[5] P. Erdős, A. Sárközy and V. T. Sós, Problems and results on additive properties of general sequences, IV, *Number Theory* (Ootacamund, 1984), 85–104, Lecture Notes in Math. 1122, Springer, Berlin, 1985.

[6] P. Erdős, A. Sárközy and V. T. Sós, Problems and results on additive properties of general sequences, *V, Monatsh. Math.* 102 (1986), no. 3, 183–197.

[7] P. Erdős, A. Sárközy and V. T. Sós, On additive properties of general sequences, *Discrete Math.* 136 (1994), 75–99.

[8] V. F. Lev, Reconstructing integer sets from their representation functions, *Electron. J. Combin.* 11 (2004), R78.

[9] C. Sándor, Partitions of natural numbers and their representation functions, *Integers* 4 (2004), A18.

[10] M. Tang, Partitions of the set of natural numbers and their representation functions, *Discrete Math.* 308 (2008), 2614–2616.

(Received 9 Oct 2011; revised 20 Dec 2011)