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Note on the bondage number of graphs on topological surfaces

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Abstract

The bondage number $b(G)$ of a graph G is the smallest number of edges whose removal from G results in a graph with larger domination number. In this paper, we present new upper bounds for $b(G)$ in terms of girth, order and Euler characteristic.

1 Introduction and main results

We shall consider graphs without loops and multiple edges. An orientable compact 2-manifold $\mathbb{S}_h$ or orientable surface $\mathbb{S}_h$ (see [8]) of genus h is obtained from the sphere by adding h handles. Correspondingly, a non-orientable compact 2-manifold $\mathbb{N}_k$ or non-orientable surface $\mathbb{N}_k$ of genus k is obtained from the sphere by adding k crosscaps. The Euler characteristic is defined by $\chi(\mathbb{S}_h) = 2 - 2h$, $h \geq 0$, and $\chi(\mathbb{N}_k) = 2 - k$, $k \geq 1$. Compact 2-manifolds are called simply surfaces throughout the paper. If a graph G is embedded in a surface $\mathbb{M}$ then the connected components of $\mathbb{M} - G$ are called the faces of G. If each face is an open disc then the embedding is called a 2-cell embedding. For such a graph G, we denote its vertex set, edge set, face set, maximum degree, and minimum degree by $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$, and $\delta(G)$, respectively. Set $|G| = |V(G)|$, $\|G\| = |E(G)|$, and $f(G) = |F(G)|$. We call $|G|$ and $\|G\|$ the order and the size of G. For a 2-cell embedding in a surface $\mathbb{M}$ the (generalized) Euler's formula states $|G| - \|G\| + f(G) = \chi(\mathbb{M})$ for any multigraph G that is 2-cell embedded in $\mathbb{M}$ [8, p. 85]. The Euclidean plane $\mathbb{S}_0$, the projective plane $\mathbb{N}_1$, the torus $\mathbb{S}_1$, and the Klein bottle $\mathbb{N}_2$ are all the surfaces of nonnegative Euler characteristic. For $i \geq 3$, let $f_i(G)$ be the number of faces with boundary walk of length i (if an edge is only on the boundary of a single face, then it should be counted twice). We say that two faces are intersecting or adjacent if they share a common vertex or a common edge, respectively. The girth of a graph G, denoted as $g(G)$, is the length of a shortest cycle in G. If G has no cycle then $g(G) = \infty$.

A dominating set for a graph G is a subset $D \subseteq V(G)$ of vertices such that every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set is called the domination number of G. The concept of domination in graphs has many applications in a wide range of areas within the natural and social sciences. One measure of the stability of the domination number of G under edge removal is the bondage number $b(G)$, defined in [2] (previously called the domination line-stability in [1]) as the smallest number of edges whose removal from G results in a graph with larger domination number. In general, it is NP-hard to determine the bondage number $b(G)$ (see Hu and Xu [6]), and thus useful to find bounds for it.

The main result of the paper is the following theorem.

Theorem 1. *Let G be a graph embeddable on a surface whose Euler characteristic χ is as large as possible and let $g(G) = g < \infty$.*

(i) Then $b(G) \leq 3 + \frac{8}{g-2} - \frac{4\chi g}{|G|(g-2)}$.

(ii) If G contains no intersecting g-faces, then $b(G) \leq 3 + \frac{8g+4}{g^2-g} - 4(1 + \frac{2}{g-1})\frac{\chi}{|G|}$.

(iii) If G contains no adjacent g-faces, then $b(G) \leq \frac{4g(g+1)}{g^2-g-1}(1 - \frac{\chi}{|G|}) - 1$.

Remark 1. *If G is a planar graph with girth $g \geq 4 + i$, $i \in \{0, 1, 2\}$ then Theorem 1(i) leads to $b(G) \leq 6 - i$. This observation was first proved by Fischermann et al. [3].*

Recently, the following results on bondage number of graphs on surfaces were obtained.

Theorem 2 (Gagarin and Zverovich [4])**.** *Let G be a graph embeddable on an orientable surface of genus h and a non-orientable surface of genus k. Then $b(G) \leq \min\{\Delta(G) + h + 2, \Delta(G) + k + 1\}$.*

Theorem 3 (Jia Huang [7])**.** *Let G be a graph embeddable on a surface whose Euler characteristic χ is as large as possible. If $\chi \leq 0$ then $b(G) < \Delta(G) + \sqrt{12 - 6\chi} + 1/2$. If $\chi \leq 0$ then $b(G) \leq \Delta(G) + \frac{\sqrt{8g(2-g)\chi + (3g-2)^2} - g + 6}{2(g-2)}$.*

Since Theorem 1 does not involve $\Delta(G)$ while Theorems 2 and 3 do, Theorem 1 will provide an improvement as long as $\Delta(G)$ is sufficiently large. More specifically, we observe the following.

Remark 2. *In many cases, the bound stated in Theorem 1(i) is better than those given by Theorems 2 and 3. Indeed, it is easy to see that if $\chi \leq 0$ then:*

(a) $s(\chi, g, |G|) < z(\Delta, h, k)$ at least when both $\Delta(G) \geq \frac{8}{g-2}$ and $|G| > 8 + \frac{16}{g-2}$ hold;

(b) $s(\chi, 3, |G|) < j_1(\Delta, \chi)$ at least when both $\Delta(G) \geq 11$ and $-\frac{|G|^2}{24} \leq \chi$ hold;

(c) $s(\chi, g, |G|) < j_2(\Delta, \chi, g)$ *at least when both* $\Delta(G) \geq \frac{7}{2} + \frac{6}{g-2}$ *and* $-\frac{|G|^2}{8}(1-\frac{2}{g}) \leq \chi$ *hold,*

where (under the notation of Theorems 1, 2 and 3): $s(\chi, g, |G|) = 3 + \frac{8}{g-2} - \frac{4\chi g}{|G|(g-2)}$, $z(\Delta, h, k) = \min\{\Delta(G) + h + 2, \Delta(G) + k + 1\}$, $j_1(\Delta, \chi) = \Delta(G) + \sqrt{12 - 6\chi} + 1/2$ *and* $j_2(\Delta, \chi, g) = \Delta(G) + \frac{\sqrt{8g(2-g)\chi + (3g-2)^2} - g + 6}{2(g-2)}$ *are the bounds given in Theorems 1, 2 and 3 respectively.*

2 Proof of the main result

The average degree of a graph G is defined as $ad(G) = 2\|G\|/|G|$. For the proof of Theorem 1 we need the following lemmas.

Lemma 4 (Hartnell and Rall [5]). *For any graph* G, $b(G) \leq 2ad(G) - 1$.

Lemma 5. *Let* G *be a connected graph embeddable on a surface whose Euler characteristic* χ *is as large as possible and let* $g(G) = g < \infty$.

(i) Then $ad(G) \leq \frac{2g}{g-2}(1 - \frac{\chi}{|G|})$.

(ii) If G *contains no intersecting* g*-faces, then* $ad(G) \leq 2 + \frac{4g+2}{g^2-g} - 2(1 + \frac{2}{g-1})\frac{\chi}{|G|}$.

(iii) If G *contains no adjacent* g*-faces, then* $ad(G) \leq \frac{2g(g+1)}{g^2-g-1}(1 - \frac{\chi}{|G|})$.

Proof. Since χ is as large as possible, G will be 2-cell embedded. By Euler's formula, $f(G) = \chi - |G| + \frac{1}{2}ad(G)|G|$.

(i) We have $ad(G)|G| = 2\|G\| = \Sigma_{i \geq g} i f_i(G) \geq g f(G) = g(\chi - |G| + \frac{1}{2}ad(G)|G|)$ and the result easily follows.

(ii) Since G contains no intersecting g-faces, each vertex is incident to at most one g-face. This implies $g f_g(G) \leq |G|$. Hence, $ad(G)|G| = 2\|G\| = \Sigma_{i \geq g} i f_i(G) \geq (g+1)f(G) - f_g(G) \geq (g+1)(\chi - |G| + \frac{1}{2}ad(G)|G|) - \frac{|G|}{g}$. After a short computation, the result follows.

(iii) Since G contains no adjacent g-faces, it follows that $g f_g(G) \leq \|G\| = \frac{1}{2}ad(G)|G|$. Hence, $ad(G)|G| \geq (g+1)f(G) - f_g(G) \geq (g+1)(\chi - |G| + \frac{1}{2}ad(G)|G|) - \frac{1}{2g}ad(G)|G|$. After some obvious manipulations, we obtain the result. □

It is clear that Theorem 1 follows by combining Lemmas 4 and 5.

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