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3-Perfect hamiltonian decomposition of the complete graph

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Abstract

Let $n \geq 5$ be an odd integer and K_n the complete graph on n vertices. Let i be an integer with $2 \leq i \leq (n-1)/2$. A hamiltonian decomposition $\mathcal{H}$ of K_n is called *i-perfect* if the set of the chords at distance i of the hamiltonian cycles in $\mathcal{H}$ is the edge set of K_n. We show that there exists a 3-perfect hamiltonian decomposition of K_n for all odd $n \geq 7$.

1 Introduction

Let $n \geq 5$ be an odd integer and i an integer with $2 \leq i \leq (n-1)/2$. We consider the following problem:

> "Seat n persons at a round table on $(n-1)/2$ consecutive days so that every two persons sit as neighbours exactly once and sit at distance i exactly once."

In graph terminology, the problem asks for a hamiltonian decomposition $\mathcal{H}$ of K_n, the complete graph on n vertices, such that the set of all i-chords of the hamiltonian cycles in $\mathcal{H}$ is the edge set of K_n. (A *chord* of a cycle C is an edge not in the edge set of C whose endvertices are in the vertex set of C. An *i-chord* of a cycle C is a chord of C whose endvertices lie at distance i on C.) We call such a hamiltonian decomposition $\mathcal{H}$ an *i-perfect hamiltonian decomposition* of K_n, or an *i-perfect n-cycle system of order n.*

In general, an i-perfect m-cycle system of order n has been considered where n is any integer ≥ 1. An *m-cycle system of order n* is a set of m-cycles whose edges partition the edges of K_n. An m-cycle system $\mathcal{C}$ of order n is called *i-perfect* if the set of the i-chords of the cycles in $\mathcal{C}$ is the edge set of K_n $(2 \leq i \leq \lfloor (m-1)/2 \rfloor)$. A lot of work has been done on i-perfect m-cycle systems [2, 3, 9]. The most natural problem is the spectrum problem, that is finding the set of values of n for which there exists an i-perfect m-cycle system of order n. When m is a prime and $2m+1$ is a prime power, the spectrum problem for 2-perfect m-cycle system of order n has been solved with some possible exceptions ([9] p. 89). And for $m \leq 19$ and $2 \leq i \leq \lfloor (m-1)/2 \rfloor$ the spectrum problem has been solved with some possible exceptions [2].

In this paper, we consider the following problem.

Problem 1.1 *Let $i \geq 2$ be an integer. Construct an i-perfect hamiltonian decomposition of K_n for all odd n with $n \geq 2i+1$.*

This problem has been considered by Buratti, Rania and Zuanni ([4], p. 43). For composite n, they constructed 2-perfect hamiltonian decompositions of K_n when $n = 15, 21, 25, 27, 33, 35, 39$. It is known that there are no 2-perfect hamiltonian decompositions when $n = 9$ with the aid of a computer [8].

For odd m, an m-cycle system of order n is called *Steiner* if it is i-perfect for each i with $2 \leq i \leq (m-1)/2$. It is written that determining the spectrum of Steiner m-cycle systems is a very difficult problem [2, 9]. For recent results on Steiner m-cycle systems, see [4]. For an odd n, we call a Steiner n-cycle system of order n a *Steiner hamiltonian decomposition of K_n*. When n is an odd prime p, it is easy to see that K_p has a Steiner hamiltonian decomposition $\mathcal{P} = \{(0, i, 2i, \ldots, (p-1)i) \mid 1 \leq i \leq (p-1)/2\}$, where the vertex set is $\{0, 1, 2, \ldots, p-1\}$ and vertices are calculated modulo p. When n is not a prime, does there exist a Steiner hamiltonian decomposition of K_n?

The following lemma is known.

Lemma 1.2 ([6] p. 333) *Let $n \geq 5$ be odd. If there are two hamiltonian cycles in K_n such that all i-chords are distinct for each i $(2 \leq i \leq (n-1)/2)$, then we have $n \not\equiv 0 \pmod 3$.*

Proof. Let $V_n = \{0, 1, \ldots, n-1\}$ be the vertex set of K_n. Let H_1, H_2 be two hamiltonian cycles such that all i-chords are distinct for every i $(2 \leq i \leq (n-1)/2)$. We may put $H_1 = (0, 1, \ldots, n-1)$ without loss of generality. Put $H_2 = (y_0, y_1, \ldots, y_{n-1})$. Then we have $y_i - y_j \not\equiv \pm(i-j) \pmod n$ $(0 \leq i, j \leq n-1, i \neq j)$. So we have $y_i - i \not\equiv y_j - j$ and $y_i + i \not\equiv y_j + j \pmod n$ $(0 \leq i, j \leq n-1, i \neq j)$.

Therefore we have $\{y_i - i \mid 0 \le i \le n-1\} \equiv \{y_i + i \mid 0 \le i \le n-1\} \equiv V_n \pmod{n}$.
Put $w \equiv \Sigma_{i=0}^{n-1} i^2 \pmod{n}$. Then $2w \equiv \Sigma_{i=0}^{n-1}(y_i - i)^2 + \Sigma_{i=0}^{n-1}(y_i + i)^2 \equiv \Sigma y_i^2 + \Sigma i^2 + \Sigma y_i^2 + \Sigma i^2 \equiv 4w \pmod{n}$. Hence $2w \equiv n(n-1)(2n-1)/3 \equiv 0 \pmod{n}$, Therefore we have $n \not\equiv 0 \pmod{3}$. □

From Lemma 1.2, there exists no Steiner hamiltonian decompositions of K_n when $n \equiv 0 \pmod{3}$.

The smallest number $n \equiv 1, 2 \pmod{3}$ which is not a prime is $n = 25$. We found that there are no Steiner hamiltonian decompositions when $n = 25$ with the aid of a computer. And we found that for every prime $5 \le n \le 23$, $\mathcal{P}$ defined above is a unique Steiner hamiltonian decomposition of K_n. It seems that the condition having a Steiner hamiltonian decomposition is very strict. So it seems that there are no Steiner hamiltonian decompositions when n is not a prime. Then we propose the following conjecture.

Conjecture 1.3 *Let $n \ge 5$ be odd. If there exists a Steiner hamiltonian decomposition of K_n, then n is a prime.*

We note that Conjecture 1.3 is true in the cyclic case: if there exists a cyclic Steiner hamiltonian decomposition of K_n, then n is an odd prime ([4], Th. 7.1).

For applications of i-perfect m-cycle systems, it is known that 2-perfect m-cycle systems for odd m can be used to construct quasigroups ([3] p. 379). We point out here that 2-perfect hamiltonian decomposition is applied to construct a solution of Dudeney's round table problem. Dudeney's round table problem is an old famous problem which asks for a set of hamiltonian cycles having the property that each 2-path (a path of length 2) in K_n lies in exactly one of the cycles [7].

Theorem A ([8]) *Let $n \ge 5$ be an odd integer. A 2-perfect hamiltonian decomposition of K_n induces a solution of Dudeney's round table problem for $n + 1$ people.*

Thus the problem of constructing a 2-perfect hamiltonian decomposition of K_n is an interesting problem; however it is not settled.

In this paper, we will prove the following theorem which is the case $i = 3$ of Problem 1.1.

Theorem 1.4[1] *For any odd $n \ge 7$ there exists a 3-perfect hamiltonian decomposition of K_n.*

2 A proof of the theorem

Let $n \ge 7$ be an odd integer. Put $m = n - 1$, $r = m/2$, and $s = r/2$ (if $n \equiv 1 \pmod{4}$), $s' = (r-1)/2$ (if $n \equiv 3 \pmod{4}$). Let $K_n = (V_n, E_n)$ be the complete graph on n vertices. Put $V_n = \{\infty\} \cup \{0, 1, 2, \ldots, m-1\}$ and let σ be the vertex permutation $(\infty)(0\ 1\ 2\ 3\ \cdots\ m-1)$.

[1]After acceptance of this paper, we have learned that the same result has been independently given also by Buratti, Rinaldi and Traetta ([5], Th. 2.2). Their solution is the same as that in this paper.

For any edge $\{a, b\} \in E_n$, define the length $d(a, b)$:

$$d(a,b) = \begin{cases} \min\{m - |b-a|, |b-a|\} & (\text{if } a, b \neq \infty) \\ \infty & (\text{otherwise}), \end{cases}$$

where additions of vertices ($\neq \infty$) are calculated modulo m.

Let i be an integer with $2 \leq i \leq (n-1)/2$. For a hamiltonian cycle H in K_n, define $E_i(H)$ to be the set of all i-chords of H, and for a hamiltonian decomposition $\mathcal{H} = \{H_t \mid 1 \leq t \leq r\}$ of K_n, put $E_i(\mathcal{H}) = \cup_{t=1}^{r} E_i(H_t)$.

Define a hamiltonian cycle H as follows (see Figures 2.1 and 2.2). When $n \equiv 1 \pmod 4$, put

$$H = (0, 1, -1, 2, -2, \cdots, s-1, -(s-1), s, \infty, -s, \\ s+1, -(s+1), \cdots, r-1, -(r-1), r).$$

When $n \equiv 3 \pmod 4$, put

$$H = (0, -1, 1, -2, 2, \cdots, -(s'-1), s'-1, -s', s', \infty, \\ -(s'+1), s'+1, -(s'+2), s'+2, \cdots, -(r-1), r-1, r).$$

Lemma 2.1 *The hamiltonian cycle H has a rotational symmetry of order 2, namely $\sigma^r H = H$.*

Lemma 2.2 *The lengths of the edges of H are $\infty, \infty, 1, 1, 2, 2, \cdots, r-1, r-1, r$.*

Put $\mathcal{H} = \{\sigma^j H \mid 0 \leq j \leq r-1\}$. Then we have the following lemma from Lemmas 2.1 and 2.2.

Lemma 2.3 *$\mathcal{H}$ is a hamiltonian decomposition of K_n.*

Next we consider $E_3(\mathcal{H})$. When $n \equiv 1 \pmod 4$, we have

$$E_3(H) = \{\{0,2\}, \{1,-2\}, \{-1,3\}, \{2,-3\}, \{-2,4\}, \{3,-4\}, \{-3,5\}, \cdots, \\ \{s-2, -(s-1)\}, \{-(s-2), s\}, \{s-1, \infty\}, \\ \{-(s-1), -s\}, \{s, s+1\}, \{\infty, -(s+1)\}, \\ \{-s, s+2\}, \{s+1, -(s+2)\}, \{-(s+1), s+3\}, \cdots, \\ \{r-2, -(r-1)\}, \{-(r-2), r\}, \{r-1, 0\}, \\ \{-(r-1), 1\}, \{r, -1\}\}.$$

When $n \equiv 3 \pmod 4$, we have

$$E_3(H) = \{\{0,-2\}, \{-1,2\}, \{1,-3\}, \{-2,3\}, \{2,-4\}, \{-3,4\}, \{3,-5\}, \cdots, \\ \{s'-2, -s'\}, \{-(s'-1), s'\}, \{s'-1, \infty\}, \\ \{-s', -(s'+1)\}, \{s', s'+1\}, \{\infty, -(s'+2)\}, \\ \{-(s'+1), s'+2\}, \{s'+1, -(s'+3)\}, \{-(s'+2), s'+3\}, \cdots, \\ \{r-3, -(r-1)\}, \{-(r-2), r-1\}, \{r-2, r\}, \\ \{-(r-1), 0\}, \{r-1, -1\}, \{r, 1\}\}.$$

We obtain Lemma 2.4 from Lemma 2.1.

Lemma 2.4 *The set of edges* $E_3(H)$ *has rotational symmetry of order* 2*, namely* $\sigma^r E_3(H) = E_3(H)$.

Lemma 2.5 *The lengths of the edges in* $E_3(H)$ *are* $\infty, \infty, 1, 1, 2, 2, \cdots, r-1, r-1, r$.

We have the following lemma from Lemmas 2.4 and 2.5.

Lemma 2.6 *Rotating* $E_3(H)$ *by* σ*, we have all edges of* K_n*, i.e.,*

$$\bigcup_{j=0}^{r-1} \sigma^j E_3(H) = E_n.$$

Proposition 2.7 $\mathcal{H}$ *is an* 3*-perfect hamiltonian decomposition of* K_n.

Proof. To prove the proposition, we need only to show that $E_3(\mathcal{H}) = E_n$. By Lemma 2.6 we have

$$E_3(\mathcal{H}) = \bigcup_{j=0}^{r-1} E_3(\sigma^j H) = \bigcup_{j=0}^{r-1} \sigma^j E_3(H) = E_n.$$

This completes the proof of the proposition. Note that $\mathcal{H}$ is a modified Walecki decomposion [1]. □

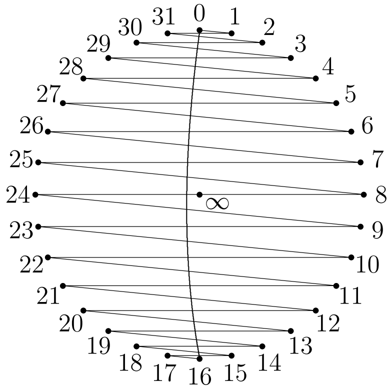

Figure 2.1 $n = 33$

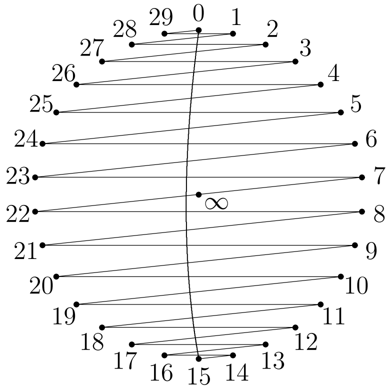

Figure 2.2 $n = 31$

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References

[1] B. Alspach, The wonderful Walecki construction, *Bull. Inst. Combin. Appl.* **52** (2008), 7–20.

[2] P. Adams and D. E. Bryant, i-Perfect m-cycle systems, $m \leq 19$, *Aeq. Math.* **53** (1997), 275–294.

[3] D. Bryant and C. A. Rodger, Cycle decompositions, in The CRC Handbook of Combinatorial Designs, 2nd edition (Eds. C. J. Colbourn and J. H. Dinitz), CRC Press, Boca Raton (2007), 373–382.

[4] M. Buratti, F. Rania and F. Zuanni, Some constructions for cyclic perfect cycle systems, *Discrete Math.* **299** (2005), 33–48.

[5] M. Buratti, G. Rinaldi and T. Traetta, Some results on 1-rotational hamiltonian cycle systems, *J. Combin. Des.* (to appear).

[6] A. Hedayat, A complete solution to the existence and nonexistence of Knut Vik designs and orthogonal Knut Vik designs, *J. Combin. Theory Ser. A* **22** (1977), 331–337.

[7] M. Kobayashi, B. D. McKay, N. Mutoh and G. Nakamura, Black 1-factors and Dudeney sets, *J. Combin. Math. Combin. Comput.* **75** (2010), 167–174.

[8] M. Kobayashi, N. Mutoh and G. Nakamura, Dudeney's round table problem and neighbour-balanced Hamilton decompositions, *J. Combin. Math. Combin. Comput.* (to appear).

[9] C. C. Lindner and C. A. Rodger, 2-Perfect m-cycle systems, *Discrete Math.* **104** (1992), 83–90.

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