AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **56** (2013), Pages 225–234

The hardness of the functional orientation 2-color problem

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Abstract

We consider the Functional Orientation 2-Color problem, which was introduced by Valiant in his seminal paper on holographic algorithms [*SIAM J. Comput.* 37(5) (2008), 1565–1594]. For this decision problem, Valiant gave a polynomial time holographic algorithm for planar graphs of maximum degree 3, and showed that the problem is **NP**-complete for planar graphs of maximum degree 10. A recent result on defective graph coloring by Corrêa et al. [*Australas. J. Combin.* 43 (2009), 219–230] implies that the problem is already hard for planar graphs of maximum degree 8. Together, these results leave open the hardness question for graphs of maximum degree between 4 and 7.

We close this gap by showing that the answer is always yes for arbitrary graphs of maximum degree 5, and that the problem is **NP**-complete for planar graphs of maximum degree 6. Moreover, for graphs of maximum degree 5, we note that a linear time algorithm for finding a solution exists.

1 Introduction

In this paper we assume $G = (V, E)$ is an undirected multigraph without loops. The maximum degree of G is denoted Δ. An *isolated vertex* is a vertex with degree zero. A *functional orientation* of G is an assignment of directions to a set of edges such that every non-isolated vertex has exactly one edge directed away from it. A single edge may be assigned two opposite directions, or it may remain undirected. A *full functional orientation* is a functional orientation of G that leaves no edges of G undirected. A *k-coloring* of G is a partition of V into k sets $V_1, V_2, \ldots, V_k$. If the subgraphs induced by $V_1, V_2, \ldots, V_k$ contain no edges, we say that the coloring is a *proper k-coloring*. For a given k-coloring, an *induced monochromatic component* is a connected component of the subgraph induced by V_i for some $i = 1, \ldots, k$.

Functional orientations occur in many applications. They naturally capture deterministic transitional systems such as finite state machines and positional strategies in, e.g., stochastic games and Markov decision processes [21, 3]. Functional orientations also prove useful in the analysis of algorithms, for example, in a dictionary with cuckoo hashing a new element can be inserted without rehashing if and only if the resulting cuckoo graph has a full functional orientation [20]. The more general concept of *c-orientations*, where the edges are oriented such that every vertex has out-degree at most c, has previously been studied in connection with dynamic representations of sparse graphs [5].

The *Functional Orientation 2-Color problem* (FO-2-Color problem) is to determine whether G has a (not necessarily proper) 2-coloring of the vertices and a functional orientation, such that every edge between two vertices of the same color is directed in at least one direction by the functional orientation. Equivalently, a graph is FO-2-Colorable if and only if there is a 2-coloring of G such that every induced monochromatic component has a full functional orientation. See Figure 1 for an example.

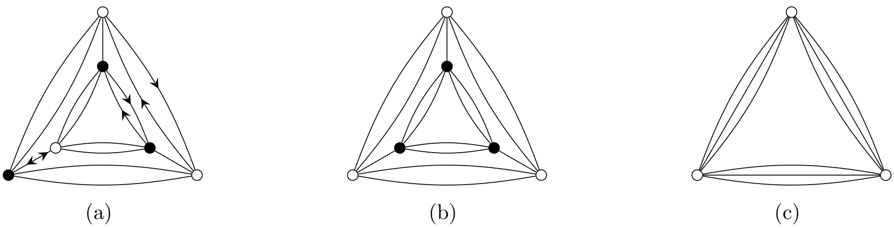


Figure 1: Two different 2-colorings of a graph with maximum degree 5 are shown in (a) and (b). The coloring in (a) admits a full functional orientation, such as the one shown, on the induced subgraphs. On the other hand, the subgraphs induced by the coloring in (b) have no full functional orientation. The graph in (c) is not FO-2-Colorable.

The FO-2-Color problem was one of the problems considered by Valiant in his seminal paper on holographic algorithms [23]. Valiant gave a polynomial time holographic algorithm for determining if a planar graph with $\Delta \leq 3$ is FO-2-Colorable. Essentially, the algorithm counts the number of possible FO-2-Colorings by transforming the problem to that of counting perfect matchings. However, each FO-2-Coloring may be counted multiple times, thus we are unable to exactly count the number of FO-2-Colorings. Even so, the input graph is FO-2-Colorable if and only if the sum is nonzero. Assuming that the input graph is planar, this sum can be computed in polynomial time using the FKT algorithm [16, 22, 17, 18].

Additionally, Valiant showed that the FO-2-Color problem is **NP**-complete for planar graphs with $\Delta \geq 10$, and as we will explain in the next section, a recent result by Corra et al. [11] implies the **NP**-completeness for $\Delta \geq 8$. This leaves open the hardness of the problem for graphs of maximum degree between 4 and 7.

In this paper we close the hardness gap by showing that the answer to the FO-2-Color problem is always yes for arbitrary graphs with $\Delta \leq 5$ and that the problem becomes **NP**-complete for planar graphs with $\Delta \geq 6$. We also observe that previous results imply that for graphs with $\Delta \leq 5$, an FO-2-Coloring can be generated efficiently using a simple greedy algorithm.

1.1 Related Work

It is easy to decide if a graph has a proper 2-coloring, i.e., is bipartite, since this is the case if and only if it has no cycles of odd length. However, the more general problem of deciding whether a graph has a k-coloring, such that the induced subgraphs all satisfy some given property π, is often significantly harder or intractable, even for $k = 2$. The FO-2-Color problem belongs to this class of problems, where π is the property that the induced subgraphs have a full functional orientation.

This class of graph coloring problems has been studied for arbitrary properties [7, 14], as well as specific properties, e.g., the induced subgraphs must be acyclic [10, 9] or complete [2]. A large number of these properties involve avoiding certain induced subgraphs. This has been studied for both finite and infinite families of forbidden graphs [6] as well as for some fixed graph [8, 1].

Another and widely studied problem of this type, which is more closely related to the FO-2-Color problem, is that of *defective coloring* [4, 15, 13, 12, 11]. This is the problem of coloring the vertices with k colors such that every induced subgraph has maximum degree d. If such a coloring exists we say that the graph is (k, d)-colorable. Cowen et al. [12] showed that deciding if a planar graph of maximum degree 5 is $(2, 1)$-colorable is **NP**-complete.

Valiant's proof of **NP**-completeness for FO-2-Color was based on the following reduction from $(2, 1)$-coloring: Let G be an instance for $(2, 1)$-coloring. Then the graph G', obtained from G by duplicating every edge in G, is FO-2-Colorable if and only if G is $(2, 1)$-colorable. This implies that FO-2-Color is **NP**-complete for planar graphs of maximum degree 10. Recently, Corra et al. [11] improved the work of Cowen et al. [12] by showing that $(2, 1)$-coloring is **NP**-complete already for planar graphs of maximum degree 4. By Valiant's reduction this implies that FO-2-Color is **NP**-complete for planar graphs of maximum degree 8.

The FO-2-Color problem is also related to the maximum cut problem, since for

$\Delta \leq 5$ a maximum cut is a $(2,2)$-coloring, and therefore also an FO-2-Coloring. This follows from the fact that the induced subgraphs are either paths or cycles, which trivially have a full functional orientation. However, unless G is planar, this does not imply an efficient algorithm for finding an FO-2-Coloring, since finding a maximum cut is known to be **NP**-hard even for simple graphs of maximum degree 3 [24].

Lovsz [19] showed that any graph of maximum degree Δ can be $(k, \lfloor \Delta/k \rfloor)$-colored, and Cowen et al. [12] noted that for graphs on v vertices, such a coloring can be found in $O(\Delta v)$ time using a simple greedy algorithm: Initially, let all vertices have the same color. Repeatedly, pick a vertex u having more than $\lfloor \Delta/k \rfloor$ neighbours with the same color as u. If there is no such u the coloring is a $(k, \lfloor \Delta/k \rfloor)$-coloring. Otherwise, there must exist a different color for u such that at most $\lfloor \Delta/k \rfloor$ of u's neighbours have this color.

Lovsz's result implies that $(2,1)$-coloring is not **NP**-complete for $\Delta = 3$, and hence the simple reduction, given by Valiant, will not work to prove **NP**-completeness of FO-2-Color for $\Delta \geq 6$.

1.2 Our Results

Our main result is to settle the hardness question of the FO-2-Color problem for graphs of maximum degree Δ. We show the following theorem

Theorem 1. *Let G be a multigraph with v vertices and maximum degree Δ.*

(i) If $\Delta \leq 5$ then G can be FO-2-Colored in $O(v)$ time.

*(ii) If $\Delta \geq 6$ the FO-2-Color problem is **NP**-complete, even for planar graphs.*

Theorem 1(i) follows immediately from the results of Lovsz [19] and Cowen et al. [12] for $k = 2$. Considering that Valiant [23] gave an involved decision algorithm for the case of planar graphs of maximum degree 3, it is perhaps surprising that arbitrary graphs of maximum degree 5 always have an FO-2-Coloring.

In the remaining part of the paper we prove Theorem 1(ii) by a reduction from 3-SAT. The **NP**-completeness for arbitrary graphs with $\Delta \geq 6$ is first established by a construction similar to those by Cowen et al. [12] and Corra et al. [11]. We extend the proof to hold for planar graphs by giving a planar crossover gadget, which we use to resolve any crossing edges.

2 NP-completeness of FO-2-Color

In this section we establish Theorem 1(ii) by a reduction from 3-SAT in conjunctive normal form (3-CNF). To do so, we will, given an instance Φ of 3-CNF, construct a graph G_Φ, which has an FO-2-Coloring if and only if Φ is satisfiable. An example of our construction is given in Figure 2. To construct such a graph we require OR-gadgets for choice, VAR-gadgets for consistency and EQ-gadgets to connect VAR-gadgets to OR-gadgets. The main challenge is to construct these gadgets such that G_Φ has maximum degree 6.

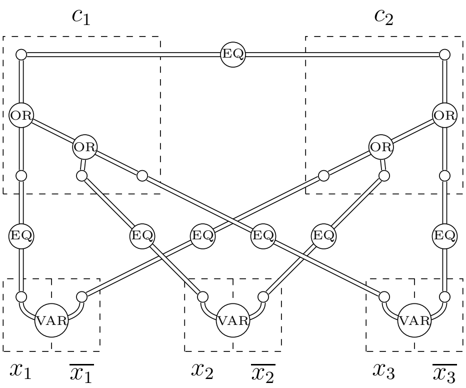


Figure 2: The graph G_Φ generated for the 3-SAT instance $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$, where x_1, x_2 and x_3 are the variables and c_1 and c_2 are the clauses.

First, in Section 2.1 we will characterize full functional orientations of graphs and prove the existence of the VAR, OR and EQ gadgets. In Section 2.2 we prove that the construction of G_Φ implies **NP**-completeness of FO-2-Color for arbitrary graphs of maximum degree 6. Finally, in Section 2.3 we provide a planar crossover gadget of maximum degree 6 to eliminate edge-crossings in G_Φ, thereby proving that the FO-2-Color problem is **NP**-complete for planar graphs of maximum degree 6.

2.1 Preliminaries

The following lemma will be useful.

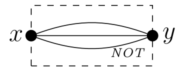
Lemma 1. *G has a full functional orientation if and only if it consists of acyclic and unicyclic components.*

Proof. It is easy to see that G has a full functional orientation if it consists of acyclic and unicyclic components. Conversely, suppose that G has a full functional orientation and a component containing two or more cycles. This component has strictly fewer vertices than edges and thus, by the pigeonhole principle, there is a vertex with two edges directed away from it, which is a contradiction. □

Corollary 1. *G is FO-2-Colorable if and only if it has a 2-coloring where every induced monochromatic component is acyclic or unicyclic.*

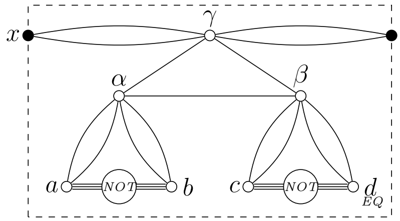
We now show how to construct the EQ, VAR and OR gadgets. Each gadget will be a planar embedded graph $G = (V, E)$ and its unique face of infinite area is called the *external face.* A subset of the vertices on the external face $V' \subseteq V$ will be called the *external vertices* of the gadget. The *connection degree* of a gadget is the maximum degree of its external vertices. Gadgets are combined by identifying the external vertices.

Lemma 2. *The* NOT*-gadget of connection degree 3, shown below, ensures that x and y have different colors in any FO-2-Coloring.*



Proof. This follows trivially from Lemma 1. ☐

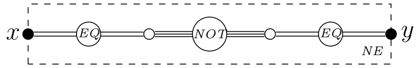
Lemma 3. *The* EQ*-gadget of connection degree* 2*, shown below, ensures that x and y have the same color in any FO-2-Coloring.*



Proof. Assume there exists a coloring where x and y have different colors. Assume without loss of generality that x is colored 0 and y is colored 1. Also assume, again without loss of generality, that a, c and γ are colored 0 and b and d are colored 1. All four possible colorings of α and β induce a monochromatic component with two cycles, hence violating Lemma 1. Conversely, assume that x and y have the same color, without loss of generality assume that color to be 0. Then γ, α, a and c may be colored 1 and β, b and d may be colored 0. This is a valid coloring. ☐

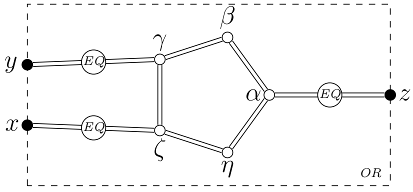
An essential property of the EQ-gadget is that in any FO-2-Coloring of the gadget it holds that γ has the opposite color of that of x and y. This allows us to connect arbitrary gadgets through intermediate EQ-gadgets, since the orientation of the external vertices are never used internally in the EQ-gadget.

Lemma 4. *The NE-gadget of connection degree* 2*, shown below, ensures that x and y have different colors in any FO-2-Coloring.*



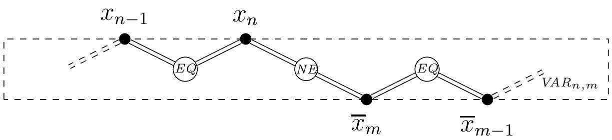
Proof. This follows trivially from Lemmas 3 and 4. ☐

Lemma 5. *The planar OR-gadget of connection degree* 2*, shown below, ensures that in any FO-2-Coloring the color of z is the same as the one of x or the one of y.*



Proof. Assume for the sake of a contradiction that the external vertices x and y have the same color and z the other color. Then γ, ζ and α have the same color as x, y and z, respectively. All four possible colorings of β and η induce a monochromatic component with two cycles, hence violating Lemma 1. It is easy to verify that any other coloring of the external vertices, x, y and z is consistent with an FO-2-Coloring. □

Lemma 6. *The VAR$_{n,m}$-gadget of connection degree 4, shown below, ensures that in any FO-2-Coloring $x_1, x_2, \ldots, x_n$ have the same color and $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m$ have the opposite color.*



Proof. This follows trivially from Lemmas 3 and 4. □

2.2 NP-completeness for Arbitrary Graphs

Lemma 7. *The FO-2-Color problem is **NP**-complete for graphs with $\Delta \geq 6$.*

Proof. We prove the **NP**-completeness of FO-2-Color by a reduction from 3-SAT in conjunctive normal form (3-CNF). First note that the problem is in **NP**, as the validity of an FO-2-Coloring can be verified in polynomial time.

Given an arbitrary instance Φ of 3-CNF, we will construct a graph G_Φ in polynomial time, such that G_ϕ is FO-2-Colorable if and only if Φ is satisfiable. To construct the graph G_Φ, we use the EQ-gadget of Lemma 3, the OR-gadget of Lemma 5 and the VAR$_{n,m}$-gadget of Lemma 6. The process is as follows: For every variable X instantiate a VAR$_{n,m}$-gadget with n being the number of clauses in which X occurs unnegated and m the corresponding negated occurrences. Then for every clause two OR-gadgets are instantiated and the output of one is connected to an input of the other, thus forming a three input OR-gadget. The remaining OR-gadget outputs are then connected together using EQ-gadgets, while the VAR-gadgets are connected with the OR-gadgets, using EQ-gadgets, such that the OR-gadget for clause C is connected with the unegated (negated) side of the VAR$_{n,m}$-gadget for variable X if and only if X occurs unnegated (negated) in C. An example construction may be seen in Figure 2.

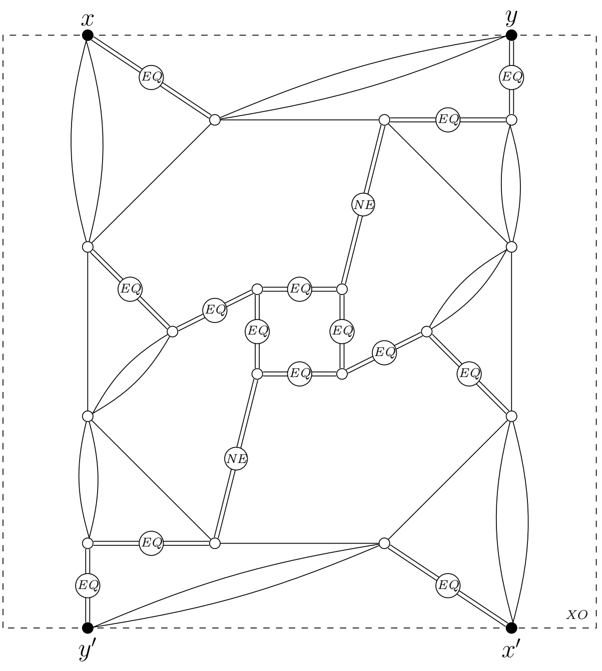
Consider the case where there exists an FO-2-Coloring of the graph. In this case the final outputs of the OR-gadgets will be identically colored. Let this color correspond to true. By Lemma 5 at least one of the three inputs has the same color, i.e., at least one literal in every clause is true. Also Lemma 6 ensures that the variables are assigned consistent values. Conversely, consider a satisfying assignment μ to Φ. From μ, it is possible to create an FO-2-Coloring in the following fashion: For every variable gadget, assign color 0 (1) to the n side and 1 (0) to the m side of the gadget if the variable is true (false) in μ. These are then propagated by the equality gadgets and, as the assignment was satisfying, every OR-gadget has the color 0 on at least one input, therefore the output may also be colored 0. This coloring is therefore valid.

Therefore each FO-2-Coloring of the graph G_Φ corresponds to a satisfying assignment of Φ. Additionally the existence of a satisfying assignment to Φ implies the existence of an FO-2-Coloring of G_Φ. Thus showing the **NP**-completeness of FO-2-Color for arbitrary graphs with $\Delta \geq 6$. □

2.3 NP-completeness for Planar Graphs

We extend the proof of Theorem 7 to planar graphs by the elimination of crossing edges using a crossover gadget. If such a gadget, of maximum degree 6, exists, then Theorem 1(ii) follows.

Lemma 8. *The XO-gadget of connection degree* 4*, shown below, ensures that* x *and* x' *as well as* y *and* y' *have the same colors in any FO-2-Coloring.*



Proof. By careful inspection along with Lemmas 3 and 4, it can be seen that there are only 4 FO-2-Colorings, all of which satisfy the lemma. □

Proof of Theorem 1(ii). Given a 3-CNF instance Φ, we apply the reduction of Theorem 7. The resulting graph G_Φ then contains at most $6\,|C|$ edges which connect OR-gadgets to VAR$_{n,m}$-gadgets. Each such edge can at most cross every other such edge once, except the corresponding parallel edge. These are the only crossings in G_Φ, thus there are at most $18\,|C|^2 - 6\,|C|$ crossings, each of which is replaced by an XO-gadget resulting in G'_Φ. The graph G'_Φ is planar and preserves the solutions of G_Φ by Lemma 8. Consequently, FO-2-Color is **NP**-complete for planar graphs with $\Delta \geq 6$. □

Acknowledgements

We are thankful to Philip Bille, Thore Husfeldt, Konstantin Kutzkov, Rasmus Pagh, Carsten Thomassen and the reviewers for their valuable comments on an earlier draft of this paper.

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(Received 9 Oct 2012; revised 24 Mar 2013)