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Signed total (j,k)-domatic numbers of graphs

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Abstract

Let G be a finite and simple graph with vertex set $V(G)$, and let $f : V(G) \to \{-1, 1\}$ be a two-valued function. If $k \geq 1$ is an integer and $\sum_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the neighborhood of v, then f is a signed total k-dominating function on G. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leq j$ for each $x \in V(G)$, is called a signed total (j,k)-dominating family (of functions) on G, where $j \geq 1$ is an integer. The maximum number of functions in a signed total (j,k)-dominating family on G is the signed total (j,k)-domatic number of G, denoted by $d_{st}^{(j,k)}(G)$.

In this paper we initiate the study of the signed total (j,k)-domatic number. We present different bounds on $d_{st}^{(j,k)}(G)$, and we determine the signed total (j,k)-domatic number for special graphs. Some of our results are extensions of well-known properties of different other signed total domatic numbers.

1 Terminology and introduction

Various numerical invariants of graphs concerning domination were introduced by means of dominating functions and their variants (see, for example, Haynes, Hedetniemi and Slater [1, 2]). In this paper we define the *signed total* (j,k)*-domatic number* in an analogous way as Henning [4] has introduced the signed total domatic number.

We consider finite, undirected and simple graphs G with vertex set $V(G) = V$ and edge set $E(G) = E$. The cardinality of the vertex set of a graph G is called the *order* of G and is denoted by $n(G) = n$. If $v \in V(G)$, then $N_G(v) = N(v)$ is the *open neighborhood* of v, i.e., the set of all vertices adjacent to v. The *closed neighborhood* $N_G[v] = N[v]$ of a vertex v consists of the vertex set $N(v) \cup \{v\}$. The number $d_G(v) = d(v) = |N(v)|$ is the *degree* of the vertex v. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$. The *complement* of a graph G is denoted by $\overline{G}$. A *fan* is a graph obtained from a path by adding a new vertex and edges joining it to all the vertices of the path. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ into some set of numbers, then $f(A) = \sum_{x \in A} f(x)$.

If $k \geq 1$ is an integer, then the *signed total k-dominating function* (STkD function) is defined in [7] as a two-valued function $f : V(G) \to \{-1, 1\}$ such that $\sum_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$. The sum $f(V(G))$ is called the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed total k-dominating functions f on G, is called the *signed total k-domination number* of G, denoted by $\gamma_{st}^k(G)$. A $\gamma_{st}^k(G)$ *function* is a STkD-function on G of weight $\gamma_{st}^k(G)$. As the assumption $\delta(G) \geq k$ is necessary, we always assume that when we discuss $\gamma_{st}^k(G)$, all graphs involved satisfy $\delta(G) \geq k$ and thus $n(G) \geq k+1$. The function assigning $+1$ to every vertex of G is a STkD function, called the function ϵ, of weight n. Thus $\gamma_{st}^k(G) \leq n$ for every graph of order n with $\delta(G) \geq k$. Moreover, the weight of every STkD function different from ϵ is at most $n-2$ and more generally, $\gamma_{st}^k(G) \equiv n \pmod 2$. Hence $\gamma_{st}^k(G) = n$ if and only if ϵ is the unique STkD function of G. The special case $k = 1$ was defined by Zelinka in [8], and has been studied by several authors (see for example, Henning [3]). We make use of the following result.

Observation 1. *([6])* Let G be a graph of order n and minimum degree $\delta(G) \geq k$. Then $\gamma_{st}^k(G) = n$ if and only if for each $v \in V(G)$, there exists a vertex $u \in N(v)$ such that $k \leq d(u) \leq k+1$.

Let $j \geq 1$ be an integer. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leq j$ for each $x \in V(G)$, is called a *signed total (j,k)-dominating family* on G. The maximum number of functions in a signed total (j,k)-dominating family on G is the *signed total (j,k)-domatic number* of G, denoted by $d_{st}^{(j,k)}(G)$. The signed total (j,k)-domatic number is well-defined and $d_{st}^{(j,k)}(G) \geq 1$ for all graphs G, since the set consisting of any STkD function, for instance the function ϵ, forms a signed total (j,k)-dominating family of G. A $d_{st}^{(j,k)}(G)$*-family* of a graph G is a signed total (j,k)-dominating family containing $d_{st}^{(j,k)}(G)$ STkD functions. The special cases $j = k = 1$ or $j = 1$ or $j = k$ of this paprameter are investigated by Henning [4] or Khodkar and Sheikholeslami [5] or Sheikholeslami and Volkmann [6].

Observation 2. Let G be a graph of order n and $\delta(G) \geq k$. If $\gamma_{st}^k(G) = n$, then ϵ is the unique STkD function of G and so $d_{st}^{(j,k)}(G) = 1$.

The following observation is a consequence of Observations 1 and 2.

Observation 3. If G is a graph of order $n \geq 3$ and $k = n-1$ or $k = n-2$, then $\gamma_{st}^k(G) = n$ and thus $d_{st}^{(j,k)}(G) = 1$.

2 Properties of the signed total (j,k)-domatic number

In this section we present basic properties of $d_{st}^{(j,k)}(G)$ and bounds on the signed total (j,k)-domatic number of a graph. Some of our results are extensions of these given by Henning [4], Khodkar and Sheikholeslami [5] as well as Sheikholeslami and Volkmann [6].

Theorem 4. If G is a graph of order n with minimum degree $\delta(G) \geq k$, then

$$\gamma_{st}^k(G) \cdot d_{st}^{(j,k)}(G) \leq j \cdot n.$$

Moreover, if $\gamma_{st}^k(G) \cdot d_{st}^{(j,k)}(G) = j \cdot n$, then for each $d_{st}^{(j,k)}(G)$-family $\{f_1, f_2, \ldots, f_d\}$ with $d = d_{st}^{(j,k)}(G)$ on G, each function f_i is a $\gamma_{st}^k(G)$-function and $\sum_{i=1}^d f_i(x) = j$ for all $x \in V(G)$.

Proof. If $\{f_1, f_2, \ldots, f_d\}$ is a signed total (j,k)-dominating family on G such that $d = d_{st}^{(j,k)}(G)$, then the definitions imply

$$\begin{aligned} d \cdot \gamma_{st}^k(G) &= \sum_{i=1}^{d} \gamma_{st}^k(G) \leq \sum_{i=1}^{d} \sum_{x \in V(G)} f_i(x) \\ &= \sum_{x \in V(G)} \sum_{i=1}^{d} f_i(x) \leq \sum_{x \in V(G)} j = j \cdot n. \end{aligned}$$

If $\gamma_{st}^k(G) \cdot d_{st}^{(j,k)}(G) = j \cdot n$, then the two inequalities occurring in the proof become equalities. Hence for the $d_{st}^{(j,k)}(G)$-family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{x \in V(G)} f_i(x) = \gamma_{st}^k(G)$, and thus each function f_i is a $\gamma_{st}^k(G)$-function and $\sum_{i=1}^d f_i(x) = j$ for all $x \in V(G)$. □

Theorem 5. If G is a graph with minimum degree $\delta(G) \geq k$, then

$$d_{st}^{(j,k)}(G) \leq \frac{j \cdot \delta(G)}{k}.$$

Moreover, if $d_{st}^{(j,k)}(G) = j \cdot \delta(G)/k$, then for each function of any signed total (j,k)-dominating family $\{f_1, f_2, \ldots, f_d\}$ with $d = d_{st}^{(j,k)}(G)$, and for all vertices v of degree $\delta(G)$, $\sum_{x \in N(v)} f_i(x) = k$ and $\sum_i^d f_i(x) = j$ for every $x \in N(v)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed total (j,k)-dominating family on G such that $d = d_{st}^{(j,k)}(G)$. If $v \in V(G)$ is a vertex of minimum degree $\delta(G)$, then it follows that

$$\begin{aligned} d \cdot k &= \sum_{i=1}^{d} k \leq \sum_{i=1}^{d} \sum_{x \in N(v)} f_i(x) \\ &= \sum_{x \in N(v)} \sum_{i=1}^{d} f_i(x) \leq \sum_{x \in N(v)} j = j\delta(G), \end{aligned}$$

and this implies the desired upper bound on the signed total (j,k)-domatic number.

If $d_{st}^{(j,k)}(G) = j \cdot \delta(G)/k$, then the two inequalities occuring in the proof become equalities, which leads to the two properties given in the statement. □

The special cases $j = k = 1$ or $j = 1$ or $j = k$ of Theorems 4 and 5 can be found in [4] or [5] or [6]. The upper bound on the product $\gamma_{st}^k(G) \cdot d_{st}^{(j,k)}(G)$ leads to a bound on the sum of these terms.

Corollary 6. If G is a graph of order n with minimum degree $\delta(G) \geq k$, then

$$\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) \leq jn + 1.$$

Proof. According to Theorem 4, we have

$$\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) \leq \frac{jn}{d_{st}^{(j,k)}(G)} + d_{st}^{(j,k)}(G).$$

Theorem 5 implies that $1 \leq d_{st}^{(j,k)}(G) \leq j\delta(G)/k \leq jn$. Using these inequalities, and the fact that the function $g(x) = x + jn/x$ is decreasing for $1 \leq x \leq \sqrt{jn}$ and increasing for $\sqrt{jn} \leq x \leq jn$, we deduce that

$$\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) \leq \max\{jn + 1, 1 + jn\} = jn + 1,$$

and the proof is complete. □

For $k \geq 2$ we will improve Corollary 6.

Theorem 7. Let $k \geq 2$ be an integer. If G is a graph of order $n \geq 3$ with minimum degree $\delta(G) \geq k$, then $\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) \leq n + 1$ when $j = 1$ and

$$\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) \leq \frac{jn}{2} + 2$$

when $j \geq 2$.

Proof. If $j = 1$, then it is just the same as Corollary 6. Assume next that $j \geq 2$.

Case 1: Assume that $\gamma_{st}^k(G) = 1$. Then the condition $k \geq 2$ and Theorem 5 lead to

$$\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) = 1 + d_{st}^{(j,k)}(G) \leq 1 + \frac{jn}{k} \leq \frac{jn}{2} + 2$$

as desired.

Case 2: Assume that $\gamma_{st}^k(G) \geq 2$. According to Theorem 4, we have

$$\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) \leq \gamma_{st}^k(G) + \frac{jn}{\gamma_{st}^k(G)}.$$

The assumption yields to $2 \leq \gamma_{st}^k(G) \leq n$. Using the fact that the function $g(x) = x + jn/x$ is decreasing for $2 \leq x \leq \sqrt{jn}$ and increasing for $\sqrt{jn} \leq x \leq n$, we observe that

$$\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) \leq \max\left\{2 + \frac{jn}{2}, n + j\right\} = \frac{nj}{2} + 2,$$

and the proof is complete. □

For the special case $j \le k$ we will improve Theorem 7.

Theorem 8. Let k and j be two integers such that $1 \le j \le k$. If G is a graph of order $n \ge 3$ with minimum degree $\delta(G) \ge k$, then

$$\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) \le n + j.$$

Proof. Assume first that $\gamma_{st}^k(G) \le j$. Then Theorem 5 leads to

$$\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) = j + \frac{j \cdot \delta(G)}{k} \le j + \delta(G) \le n + j.$$

Assume second that $\gamma_{st}^k(G) \ge j$. According to Theorem 4, we have

$$\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) \le \gamma_{st}^k(G) + \frac{jn}{\gamma_{st}^k(G)}. \tag{1}$$

The assumption yields to $j \le \gamma_{st}^k(G) \le n$. Using the fact that the function $g(x) = x + jn/x$ is decreasing for $j \le x \le \sqrt{jn}$ and increasing for $\sqrt{jn} \le x \le n$, we observe by (1) that

$$\gamma_{st}^k(G) + d_{st}^{(j,k)}(G) \le \max\{j + n, n + j\} = n + j,$$

and the proof is complete. □

Theorem 9. If the graph G contains a vertex v such that $d(v) \le k + 1$, then $d_{st}^{(j,k)}(G) \le j$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed total (j,k)-dominating family on G such that $d = d_{st}^{(j,k)}(G)$. Since $\sum_{x \in N(v)} f_i(x) \ge k$ and $|N(v)| \le k+1$, we deduce that $f_i(x) = 1$ for each $x \in N(v)$ and each $i \in \{1, 2, \ldots, d\}$. If x is an arbitrary neighbor of v, then it follows that

$$d_{st}^{(j,k)}(G) = d = \sum_{i=1}^{d} f_i(x) \le j,$$

and this is the desired upper bound. □

Let $j \ge 1$ be an integer, and let $n = j + 5$. Now let F_n be a fan with vertex set $\{x_1, x_2, \ldots, x_n\}$ such that $x_1 x_2 \ldots x_n x_1$ is a cycle of length n and x_n is adjacent to x_i for each $i = 2, 3, \ldots, n-2$. For $3 \le t \le n-3$ define $f_t : V(F_n) \to \{-1, 1\}$ by $f_t(x_t) = -1$ and $f_t(x) = 1$ for $x \in V(F_n) \setminus \{x_t\}$. Then it easy to see that $\{f_3, f_4, \ldots, f_{n-3}\}$ is a signed total $(j,1)$-dominating family on F_n. Therefore Theorem 9 implies that $d_{st}^{(j,1)}(F_{j+5}) = j$.

This example demonstrates that Theorem 9 is sharp, at least for $k = 1$.

As an application of Theorem 5, we will prove the following Nordhaus-Gaddum type result.

Theorem 10. If G is a graph of order n such that $\delta(G) \ge k$ and $\delta(\overline{G}) \ge k$, then

$$d_{st}^{(j,k)}(G) + d_{st}^{(j,k)}(\overline{G}) \le \frac{j(n-1)}{k}.$$

Moreover, if $d_{st}^{(j,k)}(G) + d_{st}^{(j,k)}(\overline{G}) = \frac{j(n-1)}{k}$, then G is regular.

Proof. Since $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$, it follows from Theorem 5 that

$$\begin{aligned} d_{st}^{(j,k)}(G) + d_{st}^{(j,k)}(\overline{G}) &\leq \frac{j\delta(G)}{k} + \frac{j\delta(\overline{G})}{k} \\ &= \frac{j}{k}(\delta(G) + \delta(\overline{G})) \\ &= \frac{j}{k}(\delta(G) + (n - \Delta(G) - 1)) \\ &\leq \frac{j}{k}(n-1), \end{aligned}$$

and this is the desired Nordhaus-Gaddum inequality. If G is not regular, then $\Delta(G) - \delta(G) \geq 1$, and the above inequality chain leads to the better bound $d_{st}^{(j,k)}(G) + d_{st}^{(j,k)}(\overline{G}) \leq \frac{j(n-2)}{k}$. This completes the proof. □

Theorem 11. If v is a vertex of a graph G such that $d(v)$ is odd and k is even or $d(v)$ is even and k is odd, then

$$d_{st}^{(j,k)}(G) \leq \frac{j \cdot d(v)}{k+1}.$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed total (j,k)-dominating family on G such that $d = d_{st}^{(j,k)}(G)$. Assume first that $d(v)$ is odd and k is even. The definition yields to $\sum_{x \in N(v)} f_i(x) \geq k$ for each $i \in \{1, 2, \ldots, d\}$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as k is even, we obtain $\sum_{x \in N(v)} f_i(x) \geq k+1$ for each $i \in \{1, 2, \ldots, d\}$. It follows that

$$\begin{aligned} j \cdot d(v) &= \sum_{x \in N(v)} j \geq \sum_{x \in N(v)} \sum_{i=1}^{d} f_i(x) \\ &= \sum_{i=1}^{d} \sum_{x \in N(v)} f_i(x) \\ &\geq \sum_{i=1}^{d} (k+1) = d(k+1), \end{aligned}$$

and this leads to the desired bound.

Assume next that $d(v)$ is even and k is odd. Note that $\sum_{x \in N(v)} f_i(x) \geq k$ for each $i \in \{1, 2, \ldots, d\}$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as k is odd, we obtain $\sum_{x \in N(v)} f_i(x) \geq k+1$ for each $i \in \{1, 2, \ldots, d\}$. Now the desired bound follows as above, and the proof is complete. □

The next result is an immediate consequence of Theorem 11.

Corollary 12. If G is a graph such that $\delta(G)$ is odd and k is even or $\delta(G)$ is even and k is odd, then

$$d_{st}^{(j,k)}(G) \leq \frac{j \cdot \delta(G)}{k+1}.$$

As an application of Corollary 12 we will improve the Nordhaus-Gaddum bound in Theorem 10 for some cases.

Theorem 13. Let G be a graph of order n such that $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$. If $\Delta(G) - \delta(G) \geq 1$ or k is odd or k is even and $\delta(G)$ is odd or k, $\delta(G)$ and n are even, then

$$d_{st}^{(j,k)}(G) + d_{st}^{(j,k)}(\overline{G}) < \frac{j(n-1)}{k}.$$

Proof. If $\Delta(G) - \delta(G) \geq 1$, then Theorem 10 implies the desired bound. Thus assume now that G is $\delta(G)$-regular.

Case 1: Assume that k is odd. If $\delta(G)$ is even, then it follows from Theorem 5 and Corollary 12 that

$$\begin{aligned} d_{st}^{(j,k)}(G) + d_{st}^{(j,k)}(\overline{G}) &\leq \frac{j\delta(G)}{k+1} + \frac{j\delta(\overline{G})}{k} \\ &= \frac{j\delta(G)}{k+1} + \frac{j(n-\delta(G)-1)}{k} \\ &< \frac{j(n-1)}{k}. \end{aligned}$$

If $\delta(G)$ is odd, then n is even and thus $\delta(\overline{G}) = n - \delta(G) - 1$ is even. Combining Theorem 5 and Corollary 12, we find that

$$\begin{aligned} d_{st}^{(j,k)}(G) + d_{st}^{(j,k)}(\overline{G}) &\leq \frac{j\delta(G)}{k} + \frac{j\delta(\overline{G})}{k+1} \\ &= \frac{j(n-\delta(\overline{G})-1)}{k} + \frac{j\delta(\overline{G})}{k+1} \\ &< \frac{j(n-1)}{k}, \end{aligned}$$

and this completes the proof of Case 1.

Case 2: Assume that k is even. If $\delta(G)$ is odd, then it follows from Theorem 5 and Corollary 12 that

$$d_{st}^{(j,k)}(G) + d_{st}^{(j,k)}(\overline{G}) \leq \frac{j\delta(G)}{k+1} + \frac{j\delta(\overline{G})}{k} < \frac{j(n-1)}{k}.$$

If $\delta(G)$ and n are even, then $\delta(\overline{G}) = n - \delta(G) - 1$ is odd, and we obtain the desired bound as above. □

Theorem 14. If G is a graph such that j is odd and $d_{st}^{(j,k)}(G)$ is even or j is even and $d_{st}^{(j,k)}(G)$ is odd, then

$$d_{st}^{(j,k)}(G) \leq \frac{(j-1)\delta(G)}{k}.$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed total (j,k)-dominating family on G such that $d = d_{st}^{(j,k)}(G)$. Assume first that j is odd and d is even. If $x \in V(G)$ is an arbitrary vertex, then $\sum_{i=1}^{d} f_i(x) \leq j$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as j is odd, we obtain $\sum_{i=1}^{d} f_i(x) \leq j-1$ for each $x \in V(G)$. If v is a vertex of minimum degree, then it follows that

$$
\begin{aligned}
d \cdot k &= \sum_{i=1}^{d} k \leq \sum_{i=1}^{d} \sum_{x \in N(v)} f_i(x) \\
&= \sum_{x \in N(v)} \sum_{i=1}^{d} f_i(x) \\
&\leq \sum_{x \in N(v)} (j-1) \\
&= (j-1)\delta(G),
\end{aligned}
$$

and this yields to the desired bound. Assume second that j is even and d is odd. If $x \in V(G)$ is an arbitrary vertex, then $\sum_{i=1}^{d} f_i(x) \leq j$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as j is even, we obtain $\sum_{i=1}^{d} f_i(x) \leq j-1$ for each $x \in V(G)$. Now the desired bound follows as above, and the proof is complete. □

If we suppose in the case $j = 1$ that $d_{st}^{(1,k)}(G)$ is an even integer, then Theorem 14 leads to the contradiction $d_{st}^{(1,k)}(G) \leq 0$. Consequently, we obtain the next known result.

Corollary 15. *([5])* The signed total $(1,k)$-domatic number $d_{st}^{(1,k)}(G)$ is an odd integer.

The special case $k = 1$ in Corollary 15 can be found in [4].

Theorem 16. Let $j \geq 2$ and $k \geq 1$ be integers, and let G be a graph with minimum degree $\delta(G) \geq k$. Then $d_{st}^{(j,k)}(G) = 1$ if and only if for every vertex $v \in V(G)$, there exists a vertex $u \in N(v)$ such that $k \leq d(u) \leq k+1$.

Proof. Assume that for every vertex $v \in V(G)$, there exists a vertex $u \in N(v)$ such that $k \leq d(u) \leq k+1$. Observation 1 implies that $\gamma_{st}^{k}(G) = n$ and thus we deduce from Observation 2 that $d_{st}^{(j,k)}(G) = 1$.

Conversely, assume that $d_{st}^{(j,k)}(G) = 1$. Suppose to the contrary that G contains a vertex w such $d(x) \geq k+2$ for each $x \in N(w)$. Then the functions $f_i : V(G) \to \{-1, 1\}$ such that $f_1(x) = 1$ for each $x \in V(G)$ and $f_2(w) = -1$ and $f_2(x) = 1$ for each vertex $x \in V(G) \setminus \{w\}$ are signed total k-dominating functions on G such that $f_1(x) + f_2(x) \leq 2 \leq j$ for each vertex $x \in V(G)$. Thus $\{f_1, f_2\}$ is a signed total (j,k)-dominating family on G, a contradiction to $d_{st}^{(j,k)}(G) = 1$. □

Now we present a lower bound on the signed total (j,k)-domatic number.

Theorem 17. Let $j,k \geq 1$ be integers such that $j \leq k+2$, and let G be a graph with minimum degree $\delta(G) \geq k$. If G contains a vertex $v \in V(G)$ such that all vertices of $N[N[v]]$ have degree at least $k+2$, then $d_{st}^{(j,k)}(G) \geq j$.

Proof. Let $\{u_1, u_2, \ldots, u_j\} \subseteq N(v)$. The hypothesis that all vertices of $N[N[v]]$ have degree at least $k+2$ implies that the functions $f_i : V(G) \to \{-1, 1\}$ such that $f_i(u_i) = -1$ and $f_i(x) = 1$ for each vertex $x \in V(G) \setminus \{u_i\}$ are signed total k-dominating functions on G for $i \in \{1, 2, \ldots, j\}$. Since $f_1(x) + f_2(x) + \ldots + f_j(x) \leq j$ for each vertex $x \in V(G)$, we observe that $\{f_1, f_2, \ldots, f_j\}$ is a signed total (j,k)-dominating family on G, and Theorem 17 is proved. □

Corollary 18. Let $j,k \geq 1$ be integers such that $j \leq k+2$. If G is a graph of minimum degree $\delta(G) \geq k+2$, then $d_{st}^{(j,k)}(G) \geq j$.

Next we determime the signed total (j,k)-domatic number for some families of graphs.

Theorem 19. Let $j,k \geq 1$ be integers, and let G be a graph with minimum degree $\delta(G) \geq k+2$.

1. If $\delta(G) = k + 2t + 1$ with an integer $t \geq 1$ and $j < \min\left\{\frac{k+1}{t}, \frac{2k+2t+1}{2t+1}\right\}$, then $d_{st}^{(j,k)}(G) = j$.
2. If $\delta(G) = k + 2t$ with an integer $t \geq 1$ and $j < \frac{k}{t}$, then $d_{st}^{(j,k)}(G) = j$.

Proof. 1. Let $d = d_{st}^{(j,k)}(G)$, and let $\delta(G) = k + 2t + 1$ with an integer $t \geq 1$. Assume that $j < \min\left\{\frac{k+1}{t}, \frac{2k+2t+1}{2t+1}\right\}$. Since k and $\delta(G)$ are of different parity, we deduce from Corollary 12 that

$$d \leq \frac{j\delta(G)}{k+1} = \frac{j(k+2t+1)}{k+1}.$$

As $j < \frac{k+1}{t}$ this leads to

$$d \leq \frac{j(k+2t+1)}{k+1} < j+2$$

and so $d \leq j+1$. If we suppose that $d = j+1$, then we observe that d and j are of different parity. Using the condition $j < \frac{2k+2t+1}{2t+1}$ and Theorem 14, we arrive at the contradiction

$$j+1 = d \leq \frac{j-1}{k}(k+2t+1) < j+1.$$

Therefore $d \leq j$, and Corollary 18 yields to the desired result $d = j$.

2. Let $d = d_{st}^{(j,k)}(G)$, and let $\delta(G) = k + 2t$ with an integer $t \geq 1$. Assume that $j < \frac{k}{t}$. It follows from Theorem 5 and the condition $j < \frac{k}{t}$ that

$$d \leq \frac{j\delta(G)}{k} = \frac{j(k+2t)}{k} < j+2$$

and so $d \leq j+1$. If we suppose that $d = j+1$, then d and j are of different parity. Applying Theorem 14, we obtain the contradiction

$$j+1 = d \leq \frac{j-1}{k}(k+2t) < j+1.$$

Therefore $d \leq j$, and Corollary 18 yields to the desired result $d = j$. □

Theorem 19 demonstrates that the bound in Corollary 18 is sharp. Finally, we present a supplement to Theorems 10 and 13.

Theorem 20. Let $k \geq 2$ be an even integer, and let G be a $\delta(G)$-graph of odd order n such that $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$, $\delta(G)$ is even and $\delta(G) = k + 2t$ with an integer $t \geq 1$. If $j < k/t$, then

$$d_{st}^{(j,k)}(G) + d_{st}^{(j,k)}(\overline{G}) \leq \frac{j(n-2t-1)}{k} < \frac{j(n-1)}{k}.$$

Proof. Since $\overline{G}$ is $(n-k-2t-1)$-regular and $j < k/t$, we conclude from Theorems 5 and 19 that

$$\begin{aligned} d_{st}^{(j,k)}(G) + d_{st}^{(j,k)}(\overline{G}) &\leq j + \frac{j\delta(\overline{G})}{k} \\ &= j + \frac{j(n-k-2t-1)}{k} \\ &= \frac{j(n-2t-1)}{k} < \frac{j(n-1)}{k}. \end{aligned}$$

□

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