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The modular product and existential closure II

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Abstract

In this article we study the modular graph product, $\Diamond$, that is known to preserve the property of being 3-existentially closed (i.e., 3-e.c.). We produce new families of 3-e.c. graphs $G\Diamond H$ such that neither G nor H is required to be 3-e.c. Assuming that G is weakly 3-existentially closed with certain adjacency properties, we find the sufficient conditions on the adjacency properties of H such that $G\Diamond H$ is 3-e.c. The graph G can have as few as four vertices, and it is settled in this article that H can have as few as 24 vertices. These altogether present an improvement in comparison to when at least one of G or H were required to be 3-e.c.

1 Introduction

A graph G with vertex set $V(G)$ is said to be n-existentially closed, or n-e.c., if for each $S \subset V(G)$ with $|S| = n$ and each subset T of S, there exists some vertex $x \in V(G) \setminus S$ that is adjacent to each vertex in T but to none of the vertices in $S \setminus T$. Although the property is straightforward to define, it is not easy to find n-e.c. graphs. Since the property was introduced in 1963 [11], only a handful of classes of finite graphs have been shown to be n-e.c. for arbitrary (but fixed) values of n. The countably infinite random graph is known to be n-e.c. for every positive integer n [11].

Sufficiently large Paley graphs were the first families that were discovered to contain n-e.c. members for every n [6]. Since then, the search for n-e.c. graphs has led to a few families of such graphs. Cameron and Stark [9] have presented a family of strongly regular n-e.c. graphs. Also, Hadamard matrices and combinatorial structures such as affine planes, resolvable designs, Steiner triple systems, balanced

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incomplete block designs, and infinite combinatorial designs have been used to construct n-e.c. graphs [4, 5, 13, 15, 16, 17].

Even for $n = 3$, it is not easy to find explicit examples of n-e.c. graphs. Every 3-e.c. graph has at least 24 vertices and examples of 3-e.c. graphs of order 28 have been found [7, 14]. New 3-e.c. graphs of order $16m^2$ were constructed from Hadamard matrices of order $4m$ with odd $m > 1$ [8]. Baker et al. presented new 3-e.c. graphs arising from affine planes [3]. Most recently, another construction of 3-e.c. graphs of order at least p^d for prime $p \geq 7$ and $d \geq 5$ was presented [19]. Also, the only two Steiner triple systems with 3-e.c. block intersection graphs were identified [10, 13].

Binary graph operations can be used to construct more examples of 3-e.c. graphs. Bonato and Cameron examined several common binary graph operations to see which operations preserve the n-e.c. property for $n \geq 1$. Although a few of the operations are 2-e.c. preserving, only the symmetric difference of two 3-e.c. graphs is a 3-e.c. graph [7]. Baker et al. subsequently introduced another graph construction which is 3-e.c. preserving [3].

In [18], the operation introduced in [3] was formulated as the non-commutative modular graph product denoted by $\Diamond$, and the necessary and sufficient conditions were found for the graph $G\Diamond H$ to be 3-e.c., given that H is a 3-e.c. graph. This operation then was used to construct new classes of 3-e.c. graphs of the form $G\Diamond H$ when G is not necessarily 3-e.c. and can have as few as four vertices. In this present article, we will use the modular graph product as described in [18] to construct new 3-e.c. graphs such that none of the graphs G and H used to construct the 3-e.c. graph $G\Diamond H$ is necessarily 3-e.c. Assuming that G is a weakly 3-existentially closed graph with prescribed adjacency properties, we find exactly what adjacencies are required for H such that $G\Diamond H$ is 3-e.c.; such graphs are called pseudo 3-existentially closed. The graph G can have as few as four vertices and H can be as small as $18 \leq |V(H)| \leq 24$.

2 Terminology and Preliminary Results

We will use the same terminology as was used in [18] which we review briefly here. We define the modular product of two graphs G and H, $G\Diamond H$, to be the graph with vertex set $V(G) \times V(H)$ in which two vertices (x, u) and (y, v) are adjacent if

(a) $xy \in E(G)$ and $uv \in E(H)$, or

(b) $xy \notin E(G)$ and $uv \notin E(H)$.

Unless stated otherwise, we shall assume that G has a loop at each vertex. When describing $G\Diamond H$, for each vertex $x \in V(G)$ let H_x be the subgraph of $G\Diamond H$ that is isomorphic to H and consists of all vertices (x, u) where $u \in V(H)$. Since the vertices of H_x can be considered to be indexed by $x \in V(G)$, we will use u_x to denote the vertex (x, u). Two vertices $u_x \in H_x$ and $v_y \in H_y$ will be said to be congruent if $u = v$; otherwise they are incongruent.

Baker et al. showed that the modular product produces a 3-e.c. graph if applied on two 3-existentially closed graphs.

Theorem 2.1 *[3] If the graphs G and H are both* 3*-e.c., then the graph $G\Diamond H$ is also* 3*-e.c.*

For a graph G, given a set $S \subset V(G)$ and a subset T of S, we say a vertex $x \in V(G) \setminus S$ is a T-solution with respect to S if x is adjacent to every vertex in T and to none in $S \setminus T$. The graph G is then n-e.c. if for any n-set $S \subset V(G)$ there is a T-solution for each $T \in P(S)$ where $P(S)$ denotes the power set of S.

A graph G is said to be weakly n-existentially closed, or n-w.e.c., if for any n-set $S \subset V(G)$ and any $T \subseteq S$, there exists a vertex in $V(G)$ that is adjacent to each vertex in T and to no vertex in $S \setminus T$, or there exists a vertex in $V(G)$ that is adjacent to each vertex in $S \setminus T$ and to no vertex in T [18]. The following characterisation theorem enables the construction of new families of 3-e.c. graphs using the modular product applied on a 3-e.c. graph and a 3-w.e.c. graph:

Theorem 2.2 *[18] Let G be a graph with $|V(G)| \geq 4$ and with loops at every vertex of $V(G)$ and let H be a* 3*-e.c. graph. The graph G is* 3*-w.e.c. if and only if $G\Diamond H$ is* 3*-e.c.*

For a graph G and a vertex $x \in V(G)$ we define $N(x) = \{y \in V(G) \mid xy \in E(G)\}$ and $N[x] = N(x) \cup \{x\}$, and for a set X of vertices we let $N(X) = \bigcup_{x\in X} N(x)$, $N'(X) = \bigcap_{x\in X} N(x)$, $N[X] = \bigcup_{x\in X} N[x]$, and $N'[X] = \bigcap_{x\in X} N[x]$. It is clear that if G has a loop at each vertex, then $N[x] = N(x)$. To avoid confusion, we may use N_G to show that the neighbourhood is in graph G. By $G[X]$ we mean the subgraph of G induced by the set $X \subset V(G)$. With these notations, a graph G is 3-w.e.c. if for every 3-subset $A \subset V(G)$, the following hold:

(1) $N'[A] \neq \emptyset$ or $V(G) \setminus N[A] \neq \emptyset$, and

(2) for every vertex $t \in A$, $N[t] \setminus N[A \setminus \{t\}] \neq \emptyset$ or $N'[A \setminus \{t\}] \setminus N[t] \neq \emptyset$.

In the next section, we apply the modular product to produce a 3-e.c. graph while none of the graphs in the operation is necessarily 3-e.c.; one of the graphs is 3-w.e.c.

3 New Families of 3-existentially Closed Graphs

We begin this section by introducing some new terminology. We call a graph H pseudo n-existentially closed, or n-p.e.c., if for any n-subset $Y \subset V(H)$ the following hold:

Ⓐ $V(H) \setminus N[Y] \neq \emptyset$,

Ⓑ $N'(Y) \neq \emptyset$, and

Ⓒ for every vertex $r \in Y$, $N(r) \setminus N[Y \setminus \{r\}] \neq \emptyset$.

In this article we focus on the 3-p.e.c. property. Note that the notation p.e.c. has previously been used by Andrzejczak and Gordinowicz to denote a concept called perturbed existential closure [2].

Lemma 3.1 *If H is a 3-p.e.c. graph, then H has no isolated and no universal vertices.*

Proof Let $u \in V(H)$. By Ⓐ there is a vertex non-adjacent to u and by Ⓑ there is a vertex adjacent to u. ■

The following theorem ensures that we can construct a 3-e.c. graph by applying the modular product on a 3-w.e.c. graph and a 3-p.e.c. graph.

Theorem 3.1 *For two graphs G and H, $G\Diamond H$ is 3-e.c. if H is a 3-p.e.c. graph and G is a 3-w.e.c. graph such that for each set of three vertices $X = \{x, y, z\} \subset V(G)$:*

① *$N'[X] \neq \emptyset$, and*

② *for every vertex $t \in X$, $N'[X \setminus \{t\}] \setminus N[t] \neq \emptyset$.*

Proof Note that neither G nor H have an isolated or a universal vertex. To show that $G\Diamond H$ is 3-e.c., for an arbitrary 3-set $S = \{u_x, v_y, w_z\} \subset V(G\Diamond H)$ we find a T-solution for each $T \in P(S)$. Let $X = \{x, y, z\} \subset V(G)$ and $Y = \{u, v, w\} \subset V(H)$; hence $1 \leq |X|, |Y| \leq 3$. We consider the three possible values of $|X|$ in separate cases. When considering $|X| \in \{2, 3\}$, there are two scenarios: either S includes some congruent vertices, or the vertices of S are all incongruent.

Case 1. If $|X| = 1$, say $X = \{x\}$, by Ⓐ if $r \in V(H) \setminus N[Y]$, then r_x is an $\emptyset$-solution and by Ⓑ if $r \in N'(Y)$, then r_x is an S-solution. Let $a \in V(G)$ be a vertex non-adjacent to x. By Ⓒ if $s \in N(u) \setminus N[Y \setminus \{u\}]$, then s_x is a $\{u_x\}$-solution and s_a is a $\{v_x, w_x\}$-solution. Similarly we can find T-solutions for $T \in \{\{v_x\}, \{u_x, w_x\}, \{w_x\}, \{u_x, v_x\}\}$.

Case 2. Suppose that $|X| = 2$ and $S = \{u_x, v_y, w_y\}$.

Case 2.a. First assume that the vertices of S are incongruent. For $T \in \{\emptyset, S\}$, by ① if $a \in N'[X]$, then by Ⓐ if $s \in V(H) \setminus N[Y]$, then s_a is an $\emptyset$-solution, and by Ⓑ if $s \in N'(Y)$, then s_a an S-solution. By Ⓒ, if $s \in N(u) \setminus N[\{v, w\}]$ then s_a is a $\{u_x\}$-solution, and if $s \in N(v) \setminus N[\{u, w\}]$ then s_a is a $\{v_y\}$-solution, and if $s \in N(w) \setminus N[\{u, v\}]$ then s_a is a $\{w_y\}$-solution. Also, by ② if $a \in N[y] \setminus N[x]$ and if by Ⓒ $s \in N(v) \setminus N[\{u, w\}]$, then s_a is a $\{u_x, v_y\}$-solution. Similarly we can find a $\{u_x, w_y\}$-solution. To find a $\{v_y, w_y\}$-solution, note that by ② if $a \in N'[x] \setminus N[y]$ and by Ⓐ if $s \in V(H) \setminus N[Y]$, then s_a is such a solution.

Case 2.b. If $S = \{u_x, u_y, w_y\}$ such that u_x and u_y are congruent, then similar to the Case 2.a. we can find a T-solution for $T \in \{\emptyset, S\}$. By ② if $a \in N[x] \setminus N[y]$, then by Ⓐ if $s \in V(H) \setminus N[Y]$, then s_a is a $\{u_y, w_y\}$-solution and by Ⓑ if $s \in N'(Y)$, then s_a is a $\{u_x\}$-solution. Also, by Ⓒ if $s \in N(u) \setminus N[w]$, then s_a is a $\{u_x, w_y\}$-solution. By Ⓒ if $s \in N(w) \setminus N[u]$, then s_a is a $\{u_y\}$-solution. Also, by ① if $a \in N'[X]$ and by Ⓒ if $s \in N(u) \setminus N[w]$, then s_a is a $\{u_x, u_y\}$-solution, and if $s \in N(w) \setminus N[u]$ by Ⓒ, then s_a is a $\{w_y\}$-solution.

Case 3. Suppose that $|X| = 3$ and $S = \{u_x, v_y, w_z\}$ consists of three vertices (possibly congruent) of $G \Diamond H$. By ① and ②, let $a, b \in V(G)$ be two vertices such that $a \in N'[X]$ and $b \in N'[\{y, z\}] \setminus N[x]$. By Ⓐ if $s \in V(H) \setminus N[Y]$, then s_a is an $\emptyset$-solution, and by Ⓑ if $s \in N'(Y)$, then s_a is an S-solution. By Ⓐ if $s \in V(H) \setminus N[Y]$, then s_b is a $\{u_x\}$-solution and by Ⓑ if $s \in N'(Y)$ then s_b is a $\{v_y, w_z\}$-solution. By a similar argument we can find T-solutions for $T \in \{\{v_y\}, \{u_x, w_z\}, \{w_z\}, \{u_x, v_y\}\}$.

Since $s \notin Y$, the proposed solutions are in $V(G \Diamond H) \setminus S$, and so $G \Diamond H$ is proved to be 3-e.c. ■

Remark. If H is a graph that satisfies Ⓐ, Ⓑ and Ⓒ, then its complement $\overline{H}$ is a graph for which for any 3-set Y of vertices, Ⓐ and Ⓑ are satisfied and also "for every $r \in Y$, $N'_{\overline{H}}(Y \setminus \{r\}) \setminus N_{\overline{H}}[r] \neq \emptyset$". Since $\overline{G \Diamond H} = G \Diamond \overline{H}$, then Theorem 3.1 still holds if we replace Ⓒ by Ⓒ′ below.

Ⓒ′ for every vertex $r \in Y$, $N'_H(Y \setminus \{r\}) \setminus N[r] \neq \emptyset$.

Remark. Note that C_4 satisfies ① and ②. So, it remains to find a graph satisfying properties Ⓐ, Ⓑ and Ⓒ; such a graph has at least eight vertices.

4 Minimum Order Pseudo 3-existentially Closed Graphs

In this section we will find a lower bound on the minimum order of a 3-p.e.c. graph H. A graph G is said to have property $P(m, n, k)$, $G \in \mathcal{G}(m, n, k)$, if for any set of $m + n$ distinct vertices there are at least k vertices each of which is adjacent to m first vertices but not adjacent to any of the latter n vertices [1]. For example $C_\ell \in \mathcal{G}(1, 1, 1)$ for any $\ell \geq 5$. Exoo and Harary studied the class of $\mathcal{G}(1, n, 1)$ and established that the Petersen graph is the smallest member of $\mathcal{G}(1, 2, 1)$ and any other graph in $\mathcal{G}(1, 2, 1)$ has at least 12 vertices [12]. Obviously, a 3-p.e.c. graph H is contained in $\mathcal{G}(1, 2, 1)$ and it is easy to see that the Petersen graph is not 3-p.e.c. As a result, any graph that is 3-p.e.c. must have at least 12 vertices.

For an arbitrary vertex $u \in V(H)$, we let $N_u = H[N(u)]$ and $N'_u = H[N^c(u)]$ where $N^c(u) = V(H) \setminus N[u]$. Theorems 4.1 and 4.2 below give lower bounds on $|V(N_u)|$ and $|V(N'_u)|$ which will give a lower bound on the order of a 3-p.e.c. graph. First we start with N_u.

Lemma 4.1 *Let H be a 3-p.e.c. graph and $u \in V(H)$. The graph $N_u = H[N(u)]$ is connected and for every pair of vertices v and w of N_u there are two additional vertices $p, q \in V(N_u)$ such that:*

(I) p is adjacent to both v and w, and

(II) q is adjacent to neither v nor w.

Proof Consider the 3-set $S = \{u, v, w\}$ in H. By Ⓑ, there is a vertex p adjacent to all the vertices in S ($p \in V(N_u)$), and by Ⓒ, there is vertex q that is adjacent to u and is not adjacent to v or w ($q \in V(N_u)$). In order to show that N_u is connected, note that every two vertices of N_u are adjacent, or by (I) there is a path of length two that connects them. ■

Corollary 4.1 *If G is a graph such that $|V(G)| \geq 4$ and for any pair of vertices $v, w \in V(G)$ there are two additional vertices $p, q \in V(G)$ such that the adjacency properties (I) and (II) hold, then:*

(i) G is of diameter at most two (and so connected),

(ii) G has no universal vertices,

(iii) every vertex and edge of G is contained on some triangle,

(iv) $\overline{G}$ is a graph such that for any pair of vertices $v, w \in V(\overline{G})$ there are two additional vertices p and q such that the adjacency properties (I) and (II) hold.

(v) if $x \in V(G)$, then $|N(x)| \geq 2$ and $|N^c(x)| \geq 2$ (hence $|V(G)| \geq 5$), and

Lemma 4.2 *If G is a graph such that for any pair of vertices $v, w \in V(G)$ there are two additional vertices p and q such that the adjacency properties (I) and (II) hold, then $\delta(G) \geq 3$ and $\Delta(G) \leq |V(G)| - 4$.*

Proof (By contradiction.) Let $x \in V(G)$ with $\deg_G(x) = 2$. The vertex x is on some triangle; say xyz. Since $|V(G)| \geq 5$, we have $|V(G) \setminus \{x, y, z\}| \geq 2$. The distance between x and each vertex in $V(G) \setminus \{x, y, z\}$ is two. Hence, one of y or z, say y, is adjacent to at least $\lceil \frac{|V(G)|-3}{2} \rceil$ of the vertices in $V(G) \setminus \{x, y, z\}$ and z is adjacent to the rest of the vertices in $V(G) \setminus \{x, y, z\}$ (in other words $V(G) \setminus \{x, y, z\} \subset N(y) \cup N(z)$). This means that there is no vertex that is adjacent to neither y nor z which is a contradiction and therefore $\delta(G) \geq 3$.

Now we prove that $\Delta(G) \leq |V(G)| - 4$. For $x \in V(G)$, by Corollary 4.1 (iv) we have $\deg_{\overline{G}}(x) \geq 3$. Knowing that $\deg_G(x) + \deg_{\overline{G}}(x) = |V(G)| - 1$, we conclude that $\deg_G(x) \leq |V(G)| - 4$. ■

Corollary 4.2 *If G is a graph such that for any pair of vertices $v, w \in V(G)$ there are two additional vertices p and q such that the adjacency properties (I) and (II) hold, then $|V(G)| \geq 7$.*

Now we are ready to get the following theorem:

Theorem 4.1 *If G is the smallest graph that for any pair of vertices $v, w \in V(G)$ there are two additional vertices p and q such that the adjacency properties (I) and (II) hold, then $|V(G)| = 9$.*

Proof Every 2-e.c. graph has at least nine vertices (the Paley graph on nine vertices is, up to isomorphism, the smallest 2-e.c. graph [7]), and it satisfies the conditions of the theorem. So $7 \leq |V(G)| \leq 9$ by Corollary 4.2. Now we exclude the cases $|V(G)| \in \{7, 8\}$.

Case $|V(G)| = 7$: Let $x \in V(G)$. By Lemma 4.2, $\deg_G(x) = 3$ and G is 3-regular. This is a contradiction as the number of odd vertices must be even.

Case $|V(G)| = 8$: Let $x \in V(G)$. By Lemma 4.2, $\deg_G(x) \in \{3, 4\}$, and so G is 3-regular, 4-regular or has vertices of both degrees 3 and 4. We first show that G cannot be regular.

Suppose that G is 3-regular. Let $x \in V(G)$ and assume that $N(x) = \{y, z, t\}$. Since there is a vertex that is adjacent to both x and y, there is a vertex adjacent to both x and z, and there is a vertex adjacent to both x and t, the only possibility is that without loss of generality, we assume that $\{yz, yt\} \subset E(G)$. Since G is 3-regular, $N(y) = \{x, z, t\}$. Now, $|V(G) \setminus \{x, y, z, t\}| = 4$ and $d_G(x, i) = 2$ for each $i \in V(G) \setminus \{x, y, z, t\}$ by Corollary 4.1 (i). This implies that $\deg_G(z) \geq 4$ or $\deg_G(t) \geq 4$. This is a contradiction as G is supposed to be 3-regular. Since G cannot be 3-regular, G cannot be 4-regular either (by Corollary 4.1 (iv)).

Now we show that G cannot be a graph having vertices of both degrees 3 and 4. To the contrary, assume that G has vertices of degrees 3 and 4, and let a be a vertex of degree 3 and $N(a) = \{b, c, d\}$. By Corollary 4.1 (iii), the edge ab is on some triangle, without loss of generality, say abc. Similarly, the edge ad is also on some triangle, say adc (without loss of generality). By (II), there is a vertex adjacent to neither b nor c. This must be a fifth vertex e. Since there is a path of length two that joins e to a, e must be adjacent to d. Now since the edge ed is on a triangle and e is not adjacent to any of a, b or c, this must be def for a sixth vertex f. The vertex d now has the maximum possible degree 4. Letting g be the seventh vertex, we must have that g is adjacent to b or c to have a path of length two to a.

As the first case, assume that g is adjacent to c. By (II), there is a vertex adjacent to neither c nor e; this must be the last vertex h. Since d and c both have the maximum possible degree 4, we must have that h is adjacent to b in order to get the path of length two to a guaranteed by (I). Now note that h is not adjacent to any of a, c, d or e. As h has degree (at least) three, it must be adjacent to b, f and g. But now there is no vertex adjacent to neither d nor h, contradicting (II).

The only other case is when g is adjacent to b. By (II), there is a vertex adjacent to neither b nor e. This must be the last vertex h which must now be adjacent to c to get a path of length two to a. Now, as h is not adjacent to any of a, b, d or e, it must be adjacent to c, f and g as it has degree at least 3. Now there is no vertex which is adjacent to neither d nor g, contradicting (II).

So, the smallest graph satisfying (I) and (II) has nine vertices. ■

By Lemma 4.1 and Theorem 4.1, it follows that:

Corollary 4.3 $|V(N_u)| \geq 9$.

Now we proceed to find a lower bound on the order of the graph N'_u.

Lemma 4.3 *If G is a 3-p.e.c. graph and $u \in V(G)$, then $N'_u = H[N^c(u)]$ has the property that for each pair of vertices $v, w \in V(N'_u)$, there are three more vertices $p, q, r \in V(N'_u)$ such that:*

(I′) p is adjacent to v and not adjacent to w,

(II′) q is adjacent to w and not adjacent to v,

(III′) r is adjacent to neither v nor w.

Proof Similar to the proof of Lemma 4.1. ■

Corollary 4.4 *If G is a graph such that for any pair of vertices v and w there are three vertices p, q and r such that (I′), (II′) and (III′) hold, then:*

(i′) G has no isolated or universal vertices,

(ii′) if $u \in V(G)$, then $|N(u)| \geq 2$ and $|N^c(u)| \geq 3$ (hence $|V(G)| \geq 6$).

Theorem 4.2 *If G is the smallest graph such that for any pair of vertices v and w there are three vertices p, q and r such that (I′), (II′) and (III′) hold, then $|V(G)| = 7$.*

Proof By Corollary 4.4 (*ii′*), $|V(G)| \geq 6$ and if $|V(G)| = 6$, then G is 2-regular and hence $G \in \{C_6, C_3 \cup C_3\}$. But, none of these graphs satisfy properties (I'), (II') and (III'). It can be easily seen that C_7 is a graph in which for any pair of vertices v and w there are three vertices p, q and r such that (I'), (II') and (III') hold. ■

By Lemma 4.3 and Theorem 4.2, it follows that:

Corollary 4.5 $|V(N'_u)| \geq 7$.

Corollary 4.6 *A* 3*-p.e.c. graph has at least* 18 *vertices.*

Proof Suppose H is a 3-p.e.c. graph and $u \in V(H)$. By Corollaries 4.3 and 4.5, $|N(u)| \geq 9$ and $|N^c(u)| \geq 7$. So, $|V(H)| \geq 17$. Consider a 3-p.e.c. graph H on 17 vertices. Since $|N(u)| = 9$ and $|N^c(u)| = 7$, H must be a 9-regular graph on 17 vertices which is impossible. So, $|V(H)| \geq 18$. ■

Note that any 3-e.c. graph is also 3-p.e.c., and observe that the smallest 3-e.c. graph has between 24 and 28 vertices [7, 14]. The graph in Figure 1 is a 3-p.e.c. graph on 24 vertices which is found by a computer search among the strongly regular graphs with at least 18 and at most 27 vertices; the adjacency list of this graph is given in Appendix A. Consequently, the smallest 3-p.e.c. graph has order between 18 and 24.

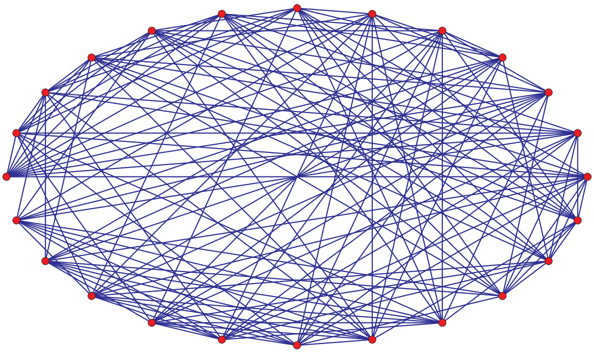

Figure 1: An example of a 3-p.e.c. graph on 24 vertices.

5 Discussion

In the statement of Theorem 3.1, we have considered a special class of 3-w.e.c. graphs, i.e., those that satisfy the conditions of the statement of the theorem. One may consider the other classes of 3-w.e.c. graphs G and determine the sufficient adjacency properties of H such that $G \Diamond H$ is 3-e.c.

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Appendix: Adjacency List of a 3-p.e.c. Graph

Here is the adjacency list of the 3-p.e.c. graph presented in Figure 1.

$1 : 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$

$2 : 1, 3, 5, 6, 7, 8, 9, 10, 16, 22, 23, 24$

$3 : 1, 2, 5, 6, 7, 11, 12, 13, 17, 22, 23, 24$

$4 : 1, 8, 9, 10, 11, 12, 13, 18, 19, 22, 23, 24$

$5 : 1, 2, 3, 8, 11, 14, 15, 18, 19, 20, 21, 22$

$6 : 1, 2, 3, 10, 12, 14, 15, 18, 19, 20, 21, 23$

$7 : 1, 2, 3, 9, 13, 14, 15, 18, 19, 20, 21, 24$

$8 : 1, 2, 4, 5, 12, 14, 15, 16, 17, 18, 20, 22$

$9 : 1, 2, 4, 7, 13, 14, 15, 16, 17, 18, 20, 23$

$10 : 1, 2, 4, 6, 11, 14, 15, 16, 17, 18, 20, 24$
$11 : 1, 3, 4, 5, 10, 14, 15, 16, 17, 19, 21, 24$
$12 : 1, 3, 4, 6, 8, 14, 15, 16, 17, 19, 21, 23$
$13 : 1, 3, 4, 7, 9, 14, 15, 16, 17, 19, 21, 22$
$14 : 5, 6, 7, 8, 9, 10, 11, 12, 13, 17, 19, 20$
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$16 : 2, 8, 9, 10, 11, 12, 13, 15, 21, 22, 23, 24$
$17 : 3, 8, 9, 10, 11, 12, 13, 14, 20, 22, 23, 24$
$18 : 4, 5, 6, 7, 8, 9, 10, 15, 19, 22, 23, 24$
$19 : 4, 5, 6, 7, 11, 12, 13, 14, 18, 22, 23, 24$
$20 : 5, 6, 7, 8, 9, 10, 14, 17, 21, 22, 23, 24$
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$22 : 2, 3, 4, 5, 8, 13, 16, 17, 18, 19, 20, 21$
$23 : 2, 3, 4, 6, 9, 12, 16, 17, 18, 19, 20, 21$
$24 : 2, 3, 4, 7, 10, 11, 16, 17, 18, 19, 20, 21$

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