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A combinatorial proof for the enumeration of alternating permutations with given peak set

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Abstract

Using the correspondence between alternating permutations and pairs of matchings, we present a combinatorial proof for the enumeration of alternating permutations with given peak set. Moreover, we give a refinement according to the number of left to right maxima.

1 Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n] := \{1, 2, \ldots, n\}$. An alternating permutation on $[n]$ is defined to be a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$ satisfying $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$, etc., in an alternating way. Similarly, σ is reverse alternating if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots$, which is also referred to as an up-down permutation. The complement map $\sigma \mapsto \sigma^c$, defined by $\sigma^c_j = n+1-\sigma_j$ on $\mathfrak{S}_n$, shows that the number of alternating permutations equals the number of up-down permutations. Denote by $\mathcal{E}_n$ and $\mathcal{E}^c_n$ the sets of alternating permutations and reverse alternating permutations on $[n]$ respectively, and further let $E_n = |\mathcal{E}_n| = |\mathcal{E}^c_n|$. Note that E_n is called Euler number, and was shown by André [1, 2] to satisfy

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

Recently, Deutsch and Elizalde [6] introduced the concept of cycle up-down permutations. For a cycle $(a_1, a_2, a_3, \ldots)$ in standard form (that is, by requiring its smallest element lies in the first position), it is said to be up-down if $a_1 < a_2 > a_3 < a_4 > \cdots$, and a permutation σ is a cycle up-down permutation if it is a product of up-down cycles. They proved both bijectively and analytically that

Proposition 1 ([6], Lemma 2.2)**.** *The number of cycle up-down permutations of* $[2k]$ *all of whose cycles are even is* E_{2k}*.*

For our purposes, let us briefly recall the bijection τ developed in [6] to prove the above proposition. Given $\sigma = \sigma_1\sigma_2\cdots\sigma_{2k} \in \mathcal{E}^c_{2k}$, let $\sigma_{j_1} > \sigma_{j_2} > \cdots > \sigma_{j_m}$ be its left to right minima, the corresponding cycle up-down permutation $\tau(\sigma)$ with only even cycles is defined by

$$\tau(\sigma) = (\sigma_{j_1}, \ldots, \sigma_{j_2-1})(\sigma_{j_2}, \ldots, \sigma_{j_3-1})\cdots(\sigma_{j_m}, \ldots, \sigma_{2k}).$$

The element σ_j $(1 \leq j \leq n)$ is called a peak if $\sigma_{j-1} < \sigma_j > \sigma_{j+1}$, where we set $\sigma_0 = 0$ and $\sigma_{n+1} = 0$, and the peak set of σ is the set of elements that are peaks in σ. For other definitions of peaks, see [10, 11]. For even $n = 2k$, and for any sequence $2 \leq i_1 < i_2 < \cdots < i_k = n$, let $S_k(i_1, i_2, \ldots, i_k)$ denote the set of permutations in $\mathcal{E}_{2k}$ with peak set equal to $\{i_1, i_2, \ldots, i_k\}$, and set $s_k(i_1, i_2, \ldots, i_k) = |S_k(i_1, i_2, \ldots, i_k)|$. For odd $n = 2k + 1$, and for any sequence $2 \leq i_1 < i_2 < \cdots < i_k < i_{k+1} = n$, let $T_k(i_1, i_2, \ldots, i_{k+1})$ denote the set of permutations in $\mathcal{E}_{2k+1}$ with peak set equal to $\{i_1, i_2, \ldots, i_{k+1}\}$, and set $t_k(i_1, i_2, \ldots, i_{k+1}) = |T_k(i_1, i_2, \ldots, i_{k+1})|$. Strehl [12] derived the following enumeration formulas for s_k and t_k.

Theorem 2 ([12])**.** *For* $k \geq 1$*,*

$$s_k(i_1, i_2, \ldots, i_k) = \prod_{1 \leq j \leq k} (i_j - 2j + 1)^2, \tag{1.1}$$

$$t_k(i_1, i_2, \ldots, i_{k+1}) = \prod_{1 \leq j \leq k} (i_j - 2j + 2)(i_j - 2j + 1). \tag{1.2}$$

In [12], Strehl proved the above result by induction on n. In this note we will give bijective proofs for the identities (1.1) and (1.2), and moreover we give a refinement according to the number of left to right maxima. There is also a well-known bijection [7, 8] between permutations and paths diagrams, which explores the increasing binary tree as the intermediate structure and can be used to provide a combinatorial explanation for Theorem 2. This bijection is explained more explicitly in [9] for the special case of alternating permutations. We use pairs of matchings to give combinatorial proofs for the enumeration of alternating permutations with given peak set. This structure seems to be more direct and can lead naturally to some results on the pattern avoidance of alternating permutations.

2 Combinatorial Proofs of Theorem 2

A matching π on $[2k]$ is a partition of the set $[2k]$ where each block has exactly two elements, and it can be represented by a graph with vertices $1, 2, \ldots, 2k$ drawn on a horizontal line in increasing order, where two vertices i and j are connected by an edge if and only if $\{i, j\}$ (with $i < j$) is a block, and we say that i is the opener and j is the closer of this edge. Since the graph is undirected, there is no difference to denote by (i, j) or (j, i) for the edge. The set of all the closers (respectively, openers) of a matching is called its closer set (respectively, opener set).

It is known that the number of matchings on $[2k]$ equals $(2k-1)!!$, and we also have the following lemma if the k closers or the k openers are given among these $2k$ vertices.

Lemma 3. *The number of matchings on* $[2k]$ *with closer set* $\{i_1 < i_2 < \cdots < i_k\}$ *or opener set* $\{2k+1-i_1 > 2k+1-i_2 > \cdots > 2k+1-i_k\}$ *is equal to*

$$\prod_{1\le j\le k} (i_j - 2j + 1).$$

Proof. We will construct a matching on $[2k]$ with closer set $\{i_1, i_2, \ldots, i_k\}$ by determining the k edges step by step from left to right as follows. For the closer i_1, there are $i_1 - 1$ openers before it with labels $1, 2, \ldots, i_1 - 1$, so there are $i_1 - 1$ choices for the first closer to form an edge. Generally, for $2 \le j \le k$, there are $j - 1$ closers before the closer i_j, thus $j - 1$ openers have been chosen so far by these closers, and the j-th closer has $i_j - 1 - 2(j - 1) = i_j - 2j + 1$ openers to be chosen to generate a new edge.

Similarly, if the opener set $\{2k+1-i_1 > 2k+1-i_2 > \cdots > 2k+1-i_k\}$ is given, the matching on $[2k]$ can be constructed by determining the k edges from right to left step by step. For the opener $2k + 1 - i_1$, there are $2k - (2k + 1 - i_1) = i_1 - 1$ closers after it, so there are $i_1 - 1$ choices for the last opener to form an edge. Generally, for $2 \le j \le k$, there are $j - 1$ openers after the opener $2k + 1 - i_j$, thus $j - 1$ closers have been chosen so far by these openers, and the j-th opener has $2k - (2k + 1 - i_j) - 2(j - 1) = i_j - 2j + 1$ closers to be chosen to generate a new edge.

By considering all the k closers or the k openers, we see that there are $\prod_{1\le j\le k}(i_j - 2j + 1)$ possibilities to form a matching with this given closer or opener set. ∎

As in [5], we can represent a permutation of $[n]$ as a graph on n vertices labeled $1, 2, \ldots, n$, with an edge from i to j if and only if $\sigma_i = j$. Explicitly, put the n vertices on a horizontal line ordered from left to right in increasing label, then we draw an edge from i to σ_i above the line if $i \le \sigma_i$ and under the line otherwise. Using this drawing, cycle up-down permutations with only even cycles correspond precisely to a pair of independent matchings whose vertices agree on openers and closers. For convenience, we refer the matching with edges above the line and below the line as above matching and below matching, respectively. With this representation of permutations in mind, we are now in the position to give a combinatorial proof of Theorem 2.

Combinatorial Proof of (1.1). Given an alternating permutation $\sigma \in \mathcal{E}_{2k}$ with peak set $\{i_1, i_2, \ldots, i_k\}$, the reverse permutation σ^r defined by $\sigma^r_i = \sigma_{n+1-i}$ is an up-down permutation on $[2k]$, and $\sigma' := \tau(\sigma^r)$ is a cycle up-down permutation with only even cycles. Let $\mathcal{G}(\sigma')$ be the corresponding graph of the permutation σ'. The elements in the peak set of σ are the values of the even positions of σ^r, which correspond to the closers of the pair of matchings in the graph $\mathcal{G}(\sigma')$. On the other hand, given a pair of matchings with closer set $\{i_1, i_2, \ldots, i_k\}$, we can recover an alternating permutation on $[2k]$ with peak set $\{i_1, i_2, \ldots, i_k\}$ by reversing the above procedure. By Lemma 3, the number of above matchings and the number of below matchings with closer set $\{i_1, i_2, \ldots, i_k\}$ are both $\prod_{1\le j\le k}(i_j - 2j + 1)$, thus identity (1.1) follows. ∎

For example, if $\sigma = 5\,3\,8\,1\,4\,2\,7\,6 \in \mathcal{E}_8$, then $\sigma^r = 6\,7\,2\,4\,1\,8\,3\,5$ and $\tau(\sigma^r) = (6,7)\,(2,4)\,(1,8,3,5)$, the graph $\mathcal{G}(\tau(\sigma^r))$ with closer set $\{4,5,7,8\}$ is depicted in Figure 1.

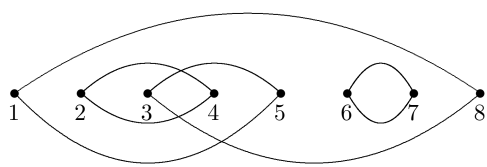

Figure 1: The graph of the cycle up-down permutation (6,7) (2,4) (1,8,3,5).

Combinatorial Proof of (1.2). Given an alternating permutation $\sigma \in \mathcal{E}_{2k+1}$ with peak set $\{i_1, i_2, \ldots, i_{k+1}\}$, let $\overline{\sigma} := \sigma\, 0$ be a permutation on the set $\{0, 1, 2, \ldots, 2k, 2k+1\}$ obtained by appending a "0" after the last element of σ. Since $\overline{\sigma}_{2k} = \sigma_{2k} < \overline{\sigma}_{2k+1} = \sigma_{2k+1} > 0 = \overline{\sigma}_{2k+2}$, and $\overline{\sigma}_i = \sigma_i$ for $i \le 2k+1$, we can view $\overline{\sigma}$ as an alternating permutation in $\mathcal{E}_{2k+2}$ and the reverse permutation $\overline{\sigma}^r$ is an up-down permutation on $2k+2$ elements with the first element being 0. From the position of 0, we see that it is the unique left to right minimum of $\overline{\sigma}^r$, thus $\overline{\sigma}' := \tau(\overline{\sigma}^r)$ is a cycle up-down permutation with only one even cycle.

Let $\mathcal{G}(\overline{\sigma}')$ be the corresponding graph of the permutation $\overline{\sigma}'$, then the closer set of the pair of matchings is $\{i_1, i_2, \ldots, i_{k+1}\}$. Since there is only one cycle in $\overline{\sigma}'$, the above matching and the below matching are dependent now. It requires that in the graph $\mathcal{G}(\overline{\sigma}')$, the edges which not contain the last closer i_{k+1} cannot form a closed loop. That is to say, for any closer i_j $(j \le k)$, there does not exist an opener a such that $(i_j, a), (a, b_1), (b_1, b_2), \ldots, (b_m, i_j)$ are edges of $\mathcal{G}(\overline{\sigma}')$. On the other hand, given a pair of matchings with closer set $\{i_1, i_2, \ldots, i_{k+1}\}$ satisfying the above condition, we can also reverse the above procedure to get an alternating permutation on $[2k+1]$ with peak set $\{i_1, i_2, \ldots, i_{k+1}\}$.

Similarly to Lemma 3, we can obtain that the number of above matchings on $\{0, 1, 2, \ldots, 2k, 2k+1\}$ with closer set $\{i_1, i_2, \ldots, i_{k+1}\}$ equals $\prod_{1\le j\le k+1}(i_j - 2j + 2)$ since there is an extra opener 0. It also equals $\prod_{1\le j\le k}(i_j - 2j + 2)$ from $i_{k+1} - 2(k+1) + 2 = 1$. Once the above matching is determined, we can construct the edges of the below matching. For $1 \le j \le k$, there are $i_j - 1 - 2(j-1) + 1$

openers before the j-th closer i_j. Among these openers, the opener a such that $(i_j, a), (a, b_1), (b_1, b_2), \ldots, (b_m, i_j)$ are edges of the graph have been constructed so far cannot be chosen, otherwise there will exist at least two cycles in $\overline{\sigma}'$. Hence the j-th closer of the below matching has $i_j - 2j + 1$ choices for its opener to generate an edge. For the last closer $i_{k+1} = 2k + 1$, there is just one opener left, the below matching is completely determined by joining these two left vertices. Combining the number of possible ways of constructing this pair of matchings with the same given closer set leads to identity (1.2) immediately. ∎

Let us illustrate the argument with an example. If $\sigma = 8\,6\,7\,3\,4\,1\,9\,2\,5 \in \mathcal{E}_9$, then $\overline{\sigma} = 8\,6\,7\,3\,4\,1\,9\,2\,5\,0 \in \mathcal{E}_{10}$, $\overline{\sigma}^r = 0\,5\,2\,9\,1\,4\,3\,7\,6\,8$, $\tau(\overline{\sigma}^r) = (0, 5, 2, 9, 1, 4, 3, 7, 6, 8)$ and its corresponding graph is shown in Figure 2.

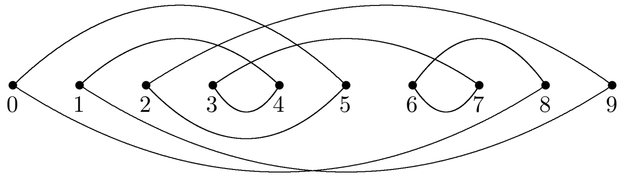

Figure 2: The graph of the cycle up-down permutation (0,5,2,9,1,4,3,7,6,8).

3 Refinements by the number of left to right maxima

In this section, we will give a refinement of Theorem 2 by considering the number of left to right maxima. Denote by $S_k(i_1, i_2, \ldots, i_k; m)$ the subset of $S_k(i_1, i_2, \ldots, i_k)$ whose elements have m left to right maxima, and let $s_k(i_1, i_2, \ldots, i_k; m) = |S_k(i_1, i_2, \ldots, i_k; m)|$. Similarly, let $T_k(i_1, i_2, \ldots, i_{k+1}; m)$ be the subset of $T_k(i_1, i_2, \ldots, i_{k+1})$ whose elements have m left to right maxima, and set $t_k(i_1, i_2, \ldots, i_{k+1}; m) = |T_k(i_1, i_2, \ldots, i_{k+1}; m)|$.

Theorem 4. *For $k \geq 1$ and $1 \leq m \leq k-1$, we have*

$$s_k(i_1, i_2, \ldots, i_k; k-m) = \prod_{1 \leq j \leq k} (i_j - 2j + 1) \left(\sum_{\substack{\{i_{a_1}, i_{a_2}, \ldots, i_{a_m}\} \\ \subset \{i_1, i_2, \ldots, i_{k-1}\}}} \prod_{1 \leq j \leq m} (i_{a_j} - 2a_j) \right), \quad (3.1)$$

and

$$s_k(i_1, i_2, \ldots, i_k; k) = \prod_{1 \leq j \leq k} (i_j - 2j + 1). \quad (3.2)$$

Proof. It is easy to see that the map $\sigma \mapsto \sigma^c$ sends a permutation with $k-m$ left to right maxima in $\mathcal{E}_{2k}$ to a permutation with $k-m$ left to right minima in $\mathcal{E}^c_{2k}$. Let

$\mathcal{G}(S_k) := \{\mathcal{G}(\tau(\sigma^c)) \mid \sigma \in S_k(i_1, i_2, \ldots, i_k)\}$. From the proof of (1.1), we see that the permutations in the set $S_k(i_1, i_2, \ldots, i_k; k-m)$ are in the correspondence to the graphs having $k-m$ cycles in the set $\mathcal{G}(S_k)$ with opener set $\{n+1-i_1, n+1-i_2, \ldots, n+1-i_k\}$ where $n = 2k$. Such a graph can be constructed by first establishing an above matching in $\prod_{1\le j\le k}(i_j - 2j + 1)$ ways, then the below matching can be constructed by the following manner. Since there are $k-m$ cycles in the graph, we can choose m openers such that there is no loop after the edge formed by this opener; while for the other openers, their corresponding closers are uniquely determined to form a cycle. For the opener $n+1-i_k = 1$, it always completes a cycle at last, this requires that the m chosen openers lie in the set $\{n+1-i_1, n+1-i_2, \ldots, n+1-i_{k-1}\}$. Suppose that $n+1-i_{a_1}, n+1-i_{a_2}, \ldots, n+1-i_{a_m}$ are the openers not in a loop in the process of the construction of the below matching, then for the opener $n+1-i_{a_j}$, it has $i_{a_j} - 2a_j$ choices for its closer. The identity (3.1) follows by summing up all the possible values of those m openers.

If $m = 0$, then the permutations in $S_k(i_1, i_2, \ldots, i_k; k)$ are in the correspondence to the graphs with k cycles in the set $\mathcal{G}(S_k)$. This implies that the edges in the above matching and the below matching are the same, and summing up the number of possible above matchings yields the identity (3.2). ∎

Theorem 5. *For $k \ge 1$ and $1 \le m \le k$, we have*

$$t_k(i_1, i_2, \ldots, i_{k+1}; k+1-m) = \prod_{1\le j\le k}(i_j - 2j + 1)\left(\sum_{\substack{\{i_{a_1}, i_{a_2}, \ldots, i_{a_m}\}\\ \subset\{i_1, i_2, \ldots, i_k\}}} \prod_{1\le j\le m}(i_{a_j} - 2a_j + 1)\right), \tag{3.3}$$

and

$$t_k(i_1, i_2, \ldots, i_{k+1}; k+1) = \prod_{1\le j\le k}(i_j - 2j + 1). \tag{3.4}$$

Proof. Given an alternating permutation $\sigma \in T_k(i_1, i_2, \ldots, i_{k+1})$ with $k+1-m$ left to right maxima, let $\overline{\sigma} := \sigma\, 0$, then $\overline{\sigma}^c$ is an up down permutation on $[2k+2]$ with $k+1-m$ left to right minima. It is easy to check that the graph in the set $\mathcal{G}(\tau(\overline{\sigma}^c))$ has $k+1-m$ cycles with opener set $\{n+1-i_1, n+1-i_2, \ldots, n+1-i_{k+1}\}$ where $n = 2k+1$.

Once the opener set is given, the below matching on $[2k+2]$ can be constructed as done in Lemma 3. Since the element $2k+2$ is the last element of $\overline{\sigma}^c$, this requires that the arc with opener 1 and closer $2k+2$ must always be an edge in the below matching of the corresponding graph. For the opener $n+1-i_j$, there are $n+1-(n+1-i_j) = i_j$ vertices after it. Since j closers have been chosen so far by the opener 1 and those $j-1$ openers after the opener $n+1-i_j$, the j-th opener has $i_j-(j-1)-j = i_j-2j+1$ closers to be chosen to generate a new edge. By considering all the $k+1$ openers, we see that there are $\prod_{1\le j\le k}(i_j - 2j + 1)$ possibilities to form a below matching.

Now, it remains to construct the above matching. Since there are $k+1-m$ cycles in the graph, we can choose m openers such that there is no loop after the edge with this opener is formed; while for the other openers, their corresponding closers are uniquely determined to form a cycle. Suppose that $n+1-i_{a_1}, n+1-i_{a_2}, \ldots, n+1-i_{a_m}$ are the openers not in a loop in the process of the construction of the above matching. Then, for the opener $n+1-i_{a_j}$, it has $n+1-(n+1-i_{a_j})-2(a_j-1)-1 = i_{a_j}-2a_j+1$ choices for its closer. The identity (3.3) follows by summing up all the possible values of those m openers lie in the set $\{n+1-i_1, n+1-i_2, \ldots, n+1-i_k\}$.

The identity (3.4) follows by observing that the edges in the above matching and the below matching are the same, and the number of below matching with this given opener set is $\prod_{1\leq j\leq k}(i_j-2j+1)$. ∎

Classically, a pattern is a permutation $\pi \in \mathfrak{S}_m$, and a permutation $\sigma \in \mathfrak{S}_n$ avoids π if there is no subword in σ whose letters are in the same relative order as the letters of π. If one requires that two adjacent letters in a pattern must be adjacent in the permutation, the pattern is called a generalized pattern. The absence of a dash between two adjacent letters in a pattern indicates that the corresponding letters in the subword of a permutation must be adjacent. For example, a generalized pattern $13-24$ of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is a subword $\sigma_i\sigma_{i+1}\sigma_j\sigma_{j+1}$ $(i+1<j)$ such that $\sigma_i < \sigma_j < \sigma_{i+1} < \sigma_{j+1}$. For more information on generalized patterns, see [3, 4]. A matching is noncrossing (respectively, nonnesting) if there do not exist four distinct elements $a<b<c<d$ with a, c (respectively, a, d) both in one block and b, d (respectively, b, c) both in another.

From the proof of Theorem 4, we see that the above matching is noncrossing (respectively, nonnesting) if and only if σ_{2k} avoids both the patterns $31-42$ and $42-31$ (respectively, $32-41$ and $41-32$). Since the noncrossing or nonnesting matching is uniquely determined with given opener set, we have

Proposition 6. *The number of alternating permutations in $\mathcal{E}_{2k}$ with k left to right maxima equals the number of alternating permutations in $\mathcal{E}_{2k}$ which avoid the patterns $31-42$ and $42-31$ (or $32-41$ and $41-32$).*

Likewise, from the proof of Theorem 5, we see that the above matching is noncrossing (respectively, nonnesting) if and only if σ_{2k+1} avoids simultaneously the patterns $31-42$, $42-31$ and $31-\bar{2}$ (respectively, $32-41$, $41-32$ and $21-\bar{3}$). Here the letter with a bar means that it lies in the end of σ. Since the noncrossing or nonnesting matching is uniquely determined with given opener set, we have

Proposition 7. *The number of alternating permutations in $\mathcal{E}_{2k+1}$ with $k+1$ left to right maxima equals the number of alternating permutations in $\mathcal{E}_{2k+1}$ which avoid the patterns $31-42$, $42-31$ and $31-\bar{2}$ (or $32-41$, $41-32$ and $21-\bar{3}$).*

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