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A study on H-line graphs

K. M. Kathiresan

Center for Research and Post Graduate Studies in Mathematics
Ayya Nadar Janaki Ammal College
Sivakasi – 626 124, Tamil Nadu
India
kathir2esan@yahoo.com

S. Akila Devi

Department of Mathematics
Sethu Institute of Technology
Kariapatti, Arupukkottai – 626115, Tamil Nadu
India
akis.akila@gmail.com

Abstract

For a connected graph H of order at least 3, the *H-line graph* of a graph is defined as that graph whose vertices are the edges of G and where two vertices are adjacent if and only if the corresponding edges of G are adjacent and belong to a common copy of H. In particular, when $H = P_3$, the H-line graph $HL(G)$ is the standard line graph $L(G)$ and for any connected graph G on at least three vertices, $GL(G) = L(G)$. For $k \geq 2$, the k^{th} iterated H-line graph $HL^k(G)$ is defined as $HL(HL^{k-1}(G))$, where $HL^1(G) = HL(G)$ and $HL^{k-1}(G)$ is assumed to be non-empty. Chartrand et al. characterized those graphs for which the sequence $\{HL^k(G)\}$ converges, when H is P_4, P_5 or $K_{1,n}$, $n \geq 3$. In this paper we characterize those graphs G for which the sequence $\{HL^k(G)\}$ converges, when H is P_6.

1 Introduction and Definitions

The *line graph* $L(G)$ of a nonempty graph G is that graph whose vertices are the edges of G and where two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. A sequence $\{G_k\}$ of graphs is said to *converge* to a graph G if there exists a positive integer N such that $G_k \cong G$ for every integer $k \geq N$. If the sequence $\{G_k\}$ is finite, it is said to terminate. If the sequence $\{G_k\}$ neither converges nor terminates, then the sequence diverges. For a connected graph H of order at least 3, the *H-line graph* of a graph G is defined as that graph whose vertices are the edges of G and where two vertices are adjacent if and only if the corresponding edges of G are adjacent and belong to a common copy of H. In particular, when $H = P_3$, the H-line graph $HL(G)$ is the standard line graph $L(G)$ and for any connected graph G on at least three vertices, $GL(G) = L(G)$. For $k \geq 2$, the k^{th} iterated H-line graph $HL^k(G)$ is defined as $HL(HL^{k-1}(G))$, where $HL^1(G) = HL(G)$ and $HL^{k-1}(G)$ is assumed to be non-empty. Necessary conditions for $\{HL^k(G)\}$ to converge to a connected limit graph and sufficient conditions for the sequence $\{HL^k(G)\}$ to diverge are discussed in [1].

Chartrand et al. [1] discussed the behaviour of the sequence $\{HL^k(G)\}$, when $H \cong P_4$ and $H \cong P_5$. Manjula [4] discussed the behavior of the sequence $\{HL^k(G)\}$ for a unicyclic graph G, which consists of a cycle C_t and a path P_m originating from a vertex v_i on the cycle such that C_t and P_m have only one vertex v_i in common, when $H \cong P_n$.

In this paper, as an extension of the result of Chartrand et al., we prove a necessary and sufficient condition for the convergence of $\{HL^k(G)\}$, when H is isomorphic to P_6.

2 Main Theorem

For $n \geq 4$, define F_n to be the graph of order n and size $n+1$ consisting of an n-cycle together with an edge joining some pair of vertices on the cycle whose distance is two. We will use the following theorem in our work.

Theorem 2.1 [1] *Suppose $H \cong P_n$ for $n \geq 4$, and let m be an integer with $m \geq n$; then the sequence $\{HL^k(F_m)\}$ diverges.*

Theorem 2.2 *Let $H \cong P_6$. Then the sequence $\{HL^k(G)\}$ converges if and only if each component of G is isomorphic to C_n for some $n \geq 6$ or each component is isomorphic to one of the graphs given below, namely, G_1, G_2, A_j, B_j, $C_{i,j}$.*

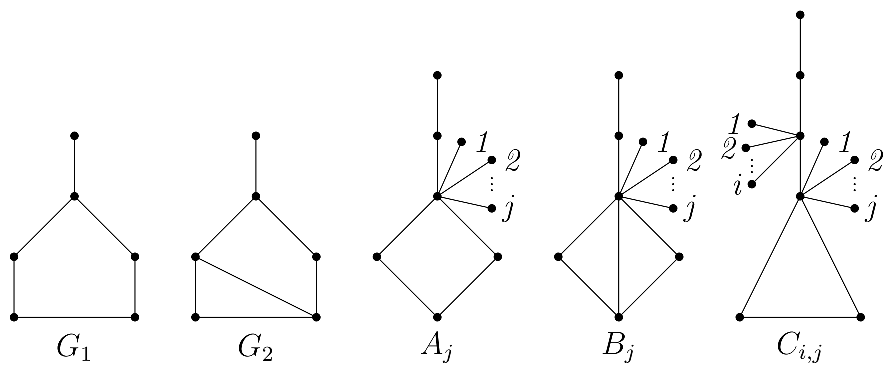

Figure 2.1

In particular, if G is connected and $G \cong C_n, n \geq 6$, then the sequence $\{HL^k(G)\}$ converges to C_n, and if $G \cong G_i$ for $i = 1, 2$ or $G \cong A_j$ or B_j or $C_{i,j}$ $(i, j \geq 0)$, then $\{HL^k(G)\}$ converges to C_6.

Proof. Let $H \cong P_6$. Without loss of generality, let us assume that G is connected. Suppose $G \cong C_n i$, $n \geq 6$. Then $HL(G) \cong G$. Hence $\{HL^k(G)\}$ converges. Suppose $G \cong G_1$. Then $HL(G) \cong C_6$. Suppose $G \cong G_2$. Then $HL(G) \cong C_6$. Hence $\{HL^k(G)\}$ converges. Hence if $G \cong G_1$ or $G \cong G_2$, then $HL^k(G)$ converges to C_6.

Suppose $G \cong A_j$ or $G \cong B_j$, $j \geq 0$. Then $HL(A_j)$ contains exactly one non trivial component which is isomorphic to G_1 and hence $\{HL^k(G)\}$ converges to C_6. Similarly $HL(B_j)$ contains exactly one non trivial component which is isomorphic to G_1 and hence $\{HL^k(G)\}$ converges to C_6.

Suppose $G \cong C_{i,j}$, $(i, j \geq 0)$. Then $HL(G)$ contains exactly one non trivial component which is isomorphic to G_1 and hence $\{HL^k(G)\}$ converges to C_6. Thus if G is isomorphic to $G_i, i = 1, 2$ (or) $G \cong A_j$ (or) $G \cong B_j$ or $G \cong C_{i,j}$, $(i, j \geq 0)$, then the sequence $\{HL^k(G)\}$ converges to C_6.

Now consider the graph G in which each component is isomorphic to G_i, $i = 1, 2$ (or) $G \cong A_j$ (or) $G \cong B_j$ (or) $G \cong C_{i,j}$. Then by the above case, each component of G converges and hence $\{HL^k(G)\}$ converges.

Conversely, assume that $\{HL^k(G)\}$ converges.

Let us show that each component G' of G is isomorphic to G_1 or G_2 or A_j or B_j or $C_{i,j}$. It is enough to show that if each component G' of G is not isomorphic to G_1 or G_2 or A_j or B_j or $C_{i,j}$, then the sequence $\{HL^k(G)\}$ diverges. Thus in what follows, we assume that G is connected and G contains a subgraph isomorphic to P_6, for otherwise the sequence $\{HL^k(G)\}$ terminates. If G contains an n-cycle for some $n \geq 6$ but G is not isomorphic to C_n, then $HL(G)$ contains a subgraph isomorphic to the graph F_{n+1}, and hence by Theorem 2.1 the sequence $\{HL^k(G)\}$ diverges. Thus we may assume that G contains no cycles of length 6 or more.

We consider four cases:

(i) G has a 5-cycle

(ii) G has a 4-cycle but no 5-cycle

(iii) G has a triangle but no 5-cycle

(iv) G is a tree

Case 1. Suppose that G has a 5-cycle and G is not isomorphic to either G_1 or G_2. Suppose first that G has no triangle. Since G must contain P_6 as a subgraph, it follows that G contains G_1 as a proper subgraph. Then there exists an edge e of G not in G_1 such that the edge e is incident to at least one of the vertices of G_1. It follows that G contains one of the graphs of Figure 2.2 as a subgraph.

In any of these cases, either $HL(T_i)$ or $HL^2(Ti)$ contains F_6 or F_7 as a subgraph. Hence by Theorem 2.1, $\{HL^k(G)\}$ diverges. We now show that if G is not isomorphic to G_2 and G contains a triangle, then $\{HL^k(G)\}$ diverges. Suppose that G is such a graph and let T denote a triangle in G and C denote a 5-cycle in G. Then $E(T) \cap E(C)$ has zero or two edges since G does not contain a 6-cycle.

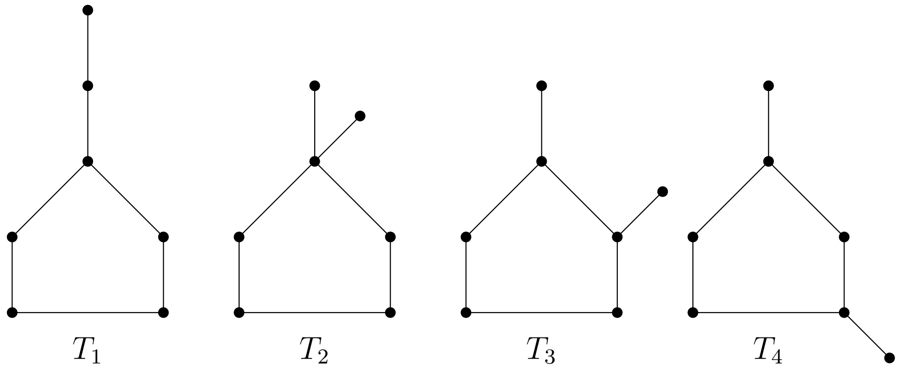

Figure 2.2

If $E(T) \cap E(C) = \emptyset$, then G contains T_1 as a subgraph, and hence $\{HL^k(G)\}$ diverges. Suppose $|E(T) \cap E(C)| = 2$. Since G must contains P_6 as a subgraph, it follows that G contains one of the graphs $J_i (1 \leq i \leq 4)$ of Figure 2.3 as a subgraph.

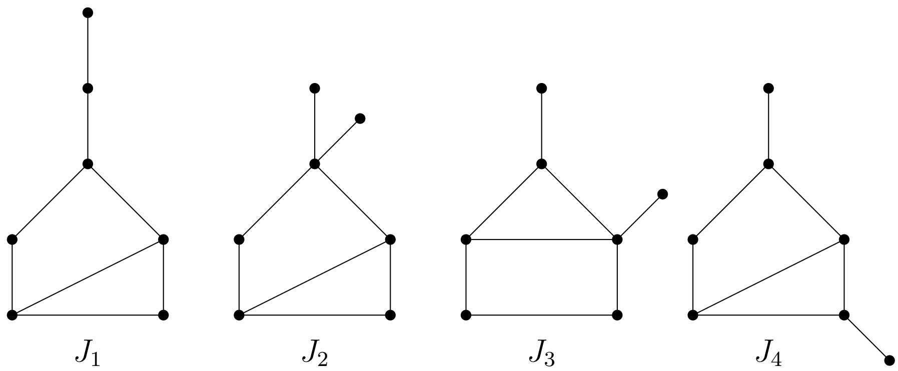

Figure 2.3

Now for each $i = 1, 2, \ldots, 4$, the graph T_i is a subgraph of J_i, and so by Theorem 2.1, the sequence $\{HL^k(J_i)\}$ diverges. For J_2, $HL^4(J_3)$ contains F_7 as a subgraph and hence by Theorem 2.1, the sequence $HL^k(J_3)$ diverges. Thus if G is any graph having a 5-cycle but no cycle of length 6 or more, and G is isomorphic to neither G_1 nor G_2, then $\{HL^k(G)\}$ diverges.

Case 2. Suppose that G has a 4-cycle but no 5-cycle or more and G is isomorphic to neither A_j nor B_j $(j \geq 0)$. Consider the graph $S_{m,n}$.

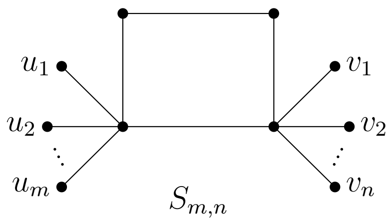

Figure 2.4

Then $HL(S_{m,n})$ contains no subgraph isomorphic to P_6 and thus, if G is a subgraph of $S_{m,n}$ for $m, n \geq 1$, then $\{HL^k(G)\}$ terminates. Consider the graph $A_{m,n}$.

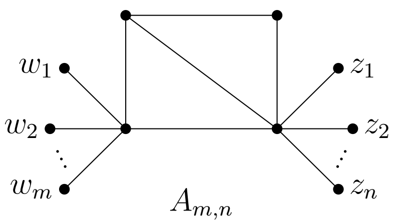

Figure 2.5

Then $HL(A_{m,n})$ contains no subgraph isomorphic to P_6 and thus, if G is a subgraph of $A_{m,n}$ for $m, n \geq 1$, then $\{HL^k(G)\}$ terminates. So we assume that G is not a subgraph of $S_{m,n}$ or $A_{m,n}$ for all $m, n \geq 1$.

This implies that G must contain one of the graphs $M_i(1 \leq i \leq 16)$ as shown below:

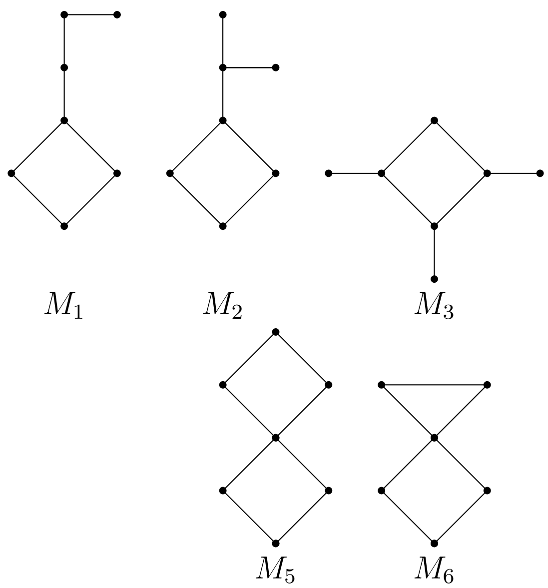

Figure 2.6

For each $i = 1, 2, 3$, the H-line graph of M_i is isomorphic to one of the graphs given in Figure 2.2 and for M_4, $HL(M_4)$ has F_6 as a subgraph, and hence by Theorem 2.1, the sequence $\{HL^k(G)\}$ does not converge. For M_5, $HL^3(M_5)$ contains F_7 as a subgraph and for M_6, $HL^3(M_6)$ contains F_7 as a subgraph and hence by Theorem 2.1, the sequence $HL^k(G)$ diverges.

Case 3. Suppose G has a triangle but no cycles of length 5 and $G \ncong C_{i,j}, (i, j \geq 0)$.

Since G must contain P_6 as a subgraph, it follows that G contains the above G_0 as a proper subgraph. Then there exists an edge e of G not belonging to G_0 such that e is incident to at least one of the vertices of G_0. Hence G must contain one of the graphs given in Figure 2.8.

For the graphs S_1 and S_2, $HL^4(S_1)$ and $HL^4(S_2)$ contain F_7 as a subgraph. For the graph S_3, $HL(S_3)$ contains F_5 as a subgraph. Thus by Theorem 2.1, $\{HL^k(G)\}$ diverges.

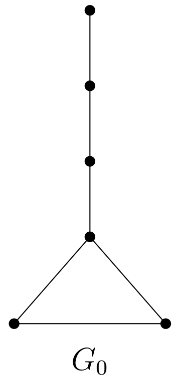

Figure 2.7

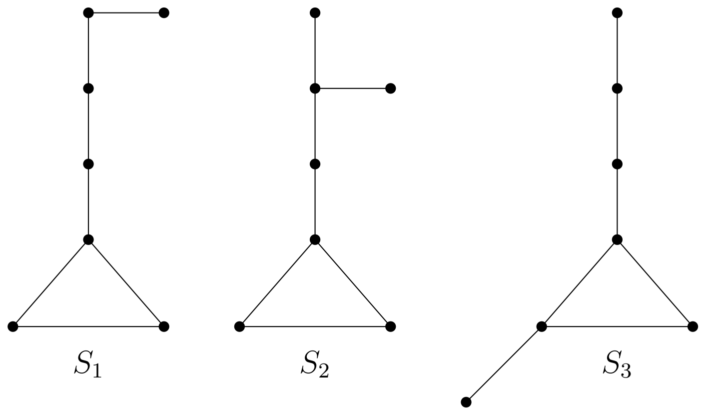

Figure 2.8

Case 4. G is a tree. Suppose that G is a tree containing P_6 as a subgraph. For $m \geq 3$, let T_m denote the tree having the property that the removal of its end vertices leaves a path of order m and such that if $v \in V(T_m)$ with deg $v \geq 3$, then v is either an end vertex of P_m or v is adjacent to an end vertex of P_m.

Some of the examples are given below:

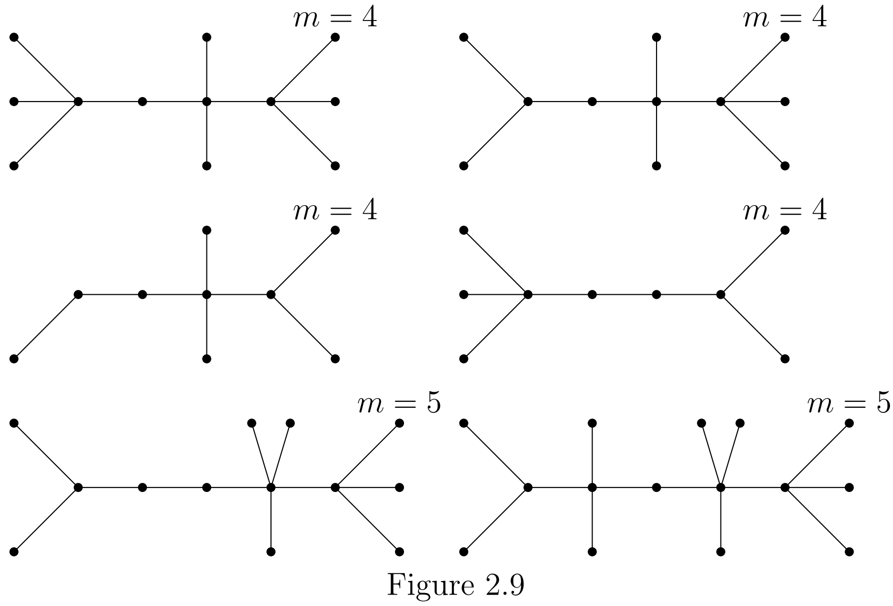

Figure 2.9

Observe that $HL^{m-1}(T_m)$ consists no subgraph isomorphic to P_6, and thus if G is a subtree of T_m for some $m \geq 4$ then $\{HL^k(G)\}$ terminates. Hence we may assume that G is not a subtree of any such T_m. Thus G must contain a subgraph isomorphic to the graph Y_1 or Y_2 of Figure 2.10.

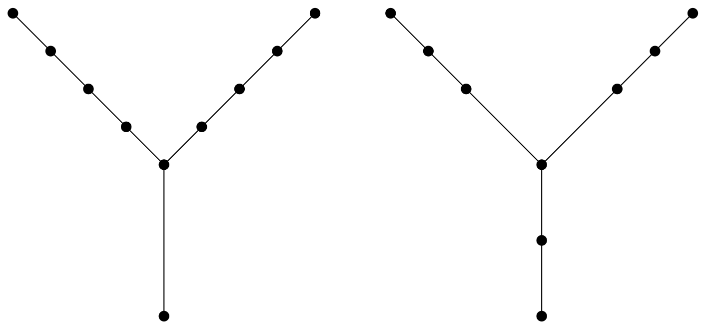

Figure 2.10

Since $HL^k(Y_i)$ for $k > 1$, $i = 1, 2$, contains F_n subgraphs, by Theorem 2.1 the sequence $\{HL^k(G)\}$ diverges.

Acknowledgements

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