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Existence of HSOLSSOMs of type $4^n u^1$

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Abstract

This paper investigates the existence of holey self-orthogonal Latin squares with a symmetric orthogonal mate of type $4^n u^1$ (briefly HSOLSSOM($4^n u^1$)). For $u > 0$, the necessary conditions for existence of such an HSOLSSOM are (1) u must be even, and (2) $u \leq (4n-4)/3$, and either $(n, u) = (4, 4)$ or $n \geq 5$. We show that these conditions are sufficient except possibly (1) for 36 cases with $n \leq 37$, (2) for $n \geq 38$, n odd and $n < u \leq (4n-4)/3$, and (3) for $n \geq 38$, n even and $n + 14 < u \leq (4n-4)/3$.

As an application of the main result, we are able to construct various types of new idempotent incomplete self-orthogonal Latin squares with a symmetric orthogonal mate (briefly ISOLSSOM).

1 Introduction

We first present a formal description of the terms HSOLSSOM and ISOLSSOM, which will be used quite extensively. Let S be a finite set and $\mathcal{H} = \{S_1, S_2, ..., S_n\}$ be a set of disjoint subsets of S. A *holey Latin square* having *hole set* $\mathcal{H}$ is an $|S| \times |S|$ array L, indexed by S, satisfying the following properties:

(1) every cell of L either contains an element of S or is empty,

(2) every element of S occurs at most once in any row or column of L,

(3) the subarrays indexed by $S_i \times S_i$ are empty for $1 \leq i \leq n$ (these subarrays are referred to as *holes*),

(4) each element $s \in S$ occurs in row or column t if and only if the pair $(s,t) \in (S \times S) \setminus \bigcup_{1 \leq i \leq n}(S_i \times S_i)$.

The *order* of the array L is $|S|$. We refer to two holey Latin squares on symbol set S and hole set $\mathcal{H}$, say L_1 and L_2, as being *orthogonal*, if their superposition yields every ordered pair in $(S \times S) \setminus \bigcup_{1 \leq i \leq n}(S_i \times S_i)$. We denote by IMOLS$(s; s_1, ..., s_n)$ a pair of orthogonal holey Latin squares on symbol set S and hole set $\mathcal{H} = \{S_1, S_2, ..., S_n\}$, where $s = |S|$ and $s_i = |S_i|$ for $1 \leq i \leq n$. If $\mathcal{H} = \emptyset$, we obtain a MOLS(s). If $\mathcal{H} = \{S_1\}$, we simply write IMOLS(s, s_1) for the orthogonal pair of holey Latin squares.

If $\mathcal{H} = \{S_1, S_2, ..., S_n\}$ is a partition of S, then a holey Latin square is called a *partitioned incomplete Latin square*, denoted by PILS. The *type* of the PILS is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. We shall use an "exponential" notation to describe types: type $t_1^{u_1} ... t_k^{u_k}$ denotes u_i occurrences of $t_i, 1 \leq i \leq k$, in the multiset. Two orthogonal PILS of type T will be denoted by HMOLS(T).

We say that a holey Latin square is *self-orthogonal* if it is orthogonal to its transpose. For self-orthogonal holey Latin squares we use the notations SOLS(s), ISOLS(s, s_1) and HSOLS(T) for the cases of $\mathcal{H} = \emptyset, \mathcal{H} = \{S_1\}$ and a holey partition $\{S_1, S_2, ..., S_n\}$, respectively.

If any two PILS in a set of t PILS of type T are orthogonal, then we denote the set by t HMOLS(T). Similarly, we may define t MOLS(s) and t IMOLS(s, s_1).

A *holey* SOLSSOM having partition $\mathcal{P}$ is 3 HMOLS having partition $\mathcal{P}$, say A, B, C, where $B = A^T$ and $C = C^T$. Here a SOLSSOM stands for a *self-orthogonal Latin square* (SOLS) with a *symmetric orthogonal mate* (SOM). A holey SOLSSOM of type T will be denoted by HSOLSSOM(T). From 3 IMOLS(s, s_1) we can similarly define an incomplete SOLSSOM which is denoted by ISOLSSOM(s, s_1). Also, an ISOLSSOM(s, s_1) is called *idempotent* if every non-holey point appears once on the main diagonal of each square.

With regards to the existence of HSOLSSOMs, the following basic result is known:

Theorem 1. [20] *In an HSOLSSOM(T), if one hole has an odd size, then every hole must have an odd size and the number of holes must also be odd.*

It is perhaps worth mentioning that HSOLSSOMs have been useful in the construction of various types of combinatorial configurations, including resolvable orthogonal arrays invariant under the Klein 4-group [16], Steiner pentagon systems [17], [3], three-fold BIBDs with block size seven [24] and authentication perpendicular arrays [14]. HSOLSSOMs of both the uniform type h^n and also the nonuniform type $h^n u^1$ have proved to be quite useful. For the existence of an HSOLSSOM(h^n), which has been investigated by several researchers (see, for example, [5], [12], [19], [23], [20], [11], [10], [9]), the known results can be summarized in the following theorem.

Theorem 2. (1) *[6] A SOLSSOM(v) exists if and only if $v \geq 4$, except for $v = 6$ and possibly for $v \in \{10, 14\}$.*

(2) *([5], [12]) An HSOLSSOM(h^n) can exist only if $n \geq 5$; further, n must be odd whenever h is odd. These necessary conditions are also sufficient except possibly for $h = 6$ and $n \in \{12, 18\}$.*

We also have the following result relating to the existence of HSOLSSOMs of type $2^n u^1$:

Theorem 3. *[4, 22] Necessary conditions for existence of an HSOLSSOM of type $2^n u^1$ with $u \geq 2$ are that u is even and $n \geq \frac{3u}{2} + 1$. These conditions are sufficient in the following cases:*

(1) *When $4 \leq n \leq 30$ and $n \geq \frac{3u}{2} + 1$.*

(2) *When $n > 30$ and $n \geq 2(u-2)$, except possibly for 19 pairs (n, u) in Table 1.*

Table 1: Unknown HSOLSSOMs of type $2^n u^1$ with $n \geq 4$ and $n \geq 2(u-2)$.

n	31	32	33,34	37,38	41	46	53
u	4,6,8,10	6,8	4,6,8,18	20	22	24	28

Our main objective in this paper is to investigate the existence of HSOLSSOMs of type $4^n u^1$ with $u > 0$. The known necessary conditions for the existence of such an HSOLSSOM are (1) u is even, and (2) $u \leq \frac{4n-4}{3}$, and either $(n, u) = (4, 4)$ or $n \geq 5$. The first of these conditions follows from Theorem 1, while condition (2) follows from the following more general result which can be obtained by simple counting: If s orthogonal PILS of order v with 2 holes of sizes t_1 and t_2 exist, then $v \geq (s+1)t_1 + t_2$.

In this paper, we prove the following theorem.

Theorem 4. *Necessary conditions for existence of an HSOLSSOM($4^n u^1$) with $u > 0$ are (1) u is even, and (2) $u \leq (4n-4)/3$, and either $(n, u) = (4, 4)$ or $n \geq 5$. These conditons are sufficient, except possibly in the following cases:*

1. *For 36 pairs (n, u) with $n \leq 37$ listed in Table 2.*
2. *When $n > 37$, n is odd, and $n < u \leq (4n-4)/3$.*
3. *When $n > 37$, n is even, and $n + 14 < u \leq (4n-4)/3$.*

2 Preliminaries

Our main results will be established with a combination of both direct and recursive constructions. Difference methods will be instrumental in our main direct construction. In particular, the following useful construction is contained in Lemma 2.1 of [9].

Lemma 5. *Let* $G = \mathbb{Z}_g$ *with* g *even and* $H = \{0, g/n, 2g/n, \ldots, (h-1)g/n\}$*, the subgroup of* G *of order* h*. Let* X *be any set disjoint from* G*. Suppose there exists a set of* 5*-tuples* $\mathcal{B} \subseteq (G \cup X)^5$ *which satisfies the following properties:*

1. *for each* i*,* $1 \le i \le 5$*, and each* $x \in X$*, there is a unique* $B \in \mathcal{B}$ *with* $b_i = x$ *(*b_i *denotes the* i*'th co-ordinate of* B*);*
2. *no* $B \in \mathcal{B}$ *has two co-ordinates in* X*, or two coordinates whose difference is an element of* H*;*
3. *for each* i, j*,* $(1 \le i < j \le 5)$ *and for each* $d \in G \backslash H$*, there exists a unique* $B \in \mathcal{B}$ *with* $b_i, b_j \in G$ *and* $b_i - b_j = d$*;*
4. *for* $b_5 \in G$*, if* $(b_1, b_2, b_3, b_4, b_5) \in \mathcal{B}$ *then* $(b_2, b_1, b_4, b_3, b_5) \in \mathcal{B}$*;*
5. *for* $x \in X$*, if* $(b_1, b_2, b_3, b_4, x) \in \mathcal{B}$*, then* $(b_2, b_1, b_4, b_3, y) \in \mathcal{B}$ *for a unique* $y \in X$*,* $x \ne y$*, and the differences* $b_1 - b_2$*,* $b_3 - b_4$ *are both odd.*

Then there exists an HSOLSSOM($h^{g/h}|X|^1$*), where* $h = |H|$*.*

There is an important connection between HSOLSSOMs and the notion of holey Steiner pentagon systems (HSPSs), which we briefly describe below (see, for example, [3]).

In a cyclically ordered block $(a_1, a_2, a_3, a_4, a_5)$ of size 5, the *distance* between two points a_i, a_j is defined to be the minimum of $(i - j + 5)$ (mod 5) and $(j - i + 5)$ (mod 5). By definition, the distance between any 2 distinct points in such a block is either 1 or 2. A *holey Steiner pentagon system* (or HSPS) of type $t_1 t_2 \cdots t_n$ is a design on a set T of $t = \sum_{i=1}^{n} t_i$ points satisfying the following conditions:

1. T is partitionable into subsets $T_1, T_2, \ldots, T_n$ of sizes $t_1, t_2, \ldots, t_n$.
2. No two points in any T_i appear together in any block.
3. Any two points in different T_i's appear in exactly one block with a distance of 1 and one block with a distance of 2.

Some types of HSOLSSOMs can be obtained from HSPSs. For this, we formally state this result as a lemma:

Lemma 6. *[11] If an HSPS of any type exists, then so does an HSOLSSOM of that type.*

Lemma 7. *There exist HSPSs of types* $4^{15}18^1$*,* $4^{21}10^1$*,* $4^{23}26^1$*,* $4^{29}34^1$ *and* $4^{33}36^1$*.*

Table 2: Unknown HSOLSSOMs of type $4^n u^1$ with $5 \le n \le 37$ and $2 \le u \le u_{max}$ where $u_{max} = \lceil (4n-4)/3 \rceil$.

n	u_{max}	Exceptions for u	n	u_{max}	Exceptions for u
5	4	None	22	28	2
6	6	None	23	28	24, 28
7	8	None	24	30	2
8	8	None	25	32	30
9	10	None	26	32	2
10	12	None	27	34	30, 32, 34
11	12	None	28	36	None
12	14	None	29	36	30, 32, 36
13	16	None	30	38	None
14	16	2	31	40	34, 36, 38
15	18	None	32	40	None
16	20	2	33	42	34, 38, 40, 42
17	20	None	34	44	None
18	22	2	35	44	None
19	24	22	36	46	None
20	24	None	37	48	40, 42, 44, 46
21	26	2, 6, 12, 14, 16, 18, 22, 24, 26			

Proof. For each of these HSPSs of type $4^n u^1$, let $X = \{\infty_1, \infty_2, \ldots, \infty_u\}$ and take the point set as $\mathbb{Z}_{4n} \cup X$. X forms the hole of size u, and the holes of size 4 are $\{y, y+n, y+2n, y+3n\}$ for $0 \le y \le n-1$. In each case, develop the given base blocks (mod $4n$); for type $4^{15}18^1$, the last block, $(0, 12, 24, 36, 48)$ generates only $4n/5 = 12$ blocks. Some base blocks are of the form $(0, a, b, c, \infty_i)$ with $i \le u/2$. For each of these, if c and $b-a$ are both odd integers, replace ∞_i by $\infty_{i+u/2}$ when adding odd values to that base block. If instead, $c \equiv 2 \pmod 4$ and $b - a \equiv 2 \pmod 4$, replace ∞_i by $\infty_{i+u/2}$ when adding any values $\equiv 2$ or $3 \pmod 4$ to that block.

$4^{15}18^1$:

$(0, 8, 2, 18, \infty_1)$, $(0, 4, 14, 38, \infty_2)$, $(0, 1, 4, 21, \infty_3)$, $(0, 14, 55, 33, \infty_4)$, $(0, 7, 49, 54, \infty_5)$, $(0, 34, 23, 43, \infty_6)$, $(0, 13, 52, 25, \infty_7)$, $(0, 25, 16, 53, \infty_8)$, $(0, 32, 1, 3, \infty_9)$, $(0, 12, 24, 36, 48)$.

$4^{21}10^1$:

$(0, 17, 39, 74, \infty_1)$, $(0, 1, 14, 69, \infty_2)$, $(0, 14, 47, 67, \infty_3)$, $(0, 41, 4, 61, \infty_4)$, $(0, 44, 33, 9, \infty_5)$, $(0, 34, 8, 5, 10)$, $(0, 32, 25, 44, 6)$, $(0, 2, 11, 36, 8)$, $(0, 4, 52, 22, 45)$ $(0, 12, 30, 83, 68)$.

$4^{23}26^1$:

$(0, 2, 56, 22, \infty_1)$, $(0, 4, 26, 10, \infty_2)$, $(0, 36, 50, 90, \infty_3)$, $(0, 37, 67, 18, \infty_4)$,
$(0, 57, 31, 58, \infty_5)$, $(0, 18, 15, 7, \infty_6)$, $(0, 1, 14, 5, \infty_7)$, $(0, 6, 27, 39, \infty_8)$,
$(0, 15, 76, 43, \infty_9)$, $(0, 25, 8, 37, \infty_{10})$, $(0, 24, 17, 45, \infty_{11})$, $(0, 53, 48, 29, \infty_9)$,
$(0, 44, 89, 57, \infty_{13})$, $(0, 20, 62, 52, 11)$.

$4^{29}34^1$:

$(0, 66, 92, 22, \infty_1)$, $(0, 82, 52, 46, \infty_2)$, $(0, 68, 50, 26, \infty_3)$, $(0, 20, 82, 98, \infty_4)$,
$(0, 21, 23, 54, \infty_5)$, $(0, 61, 39, 14, \infty_6)$, $(0, 39, 1, 106, \infty_7)$, $(0, 47, 2, 95, \infty_8)$,
$(0, 4, 7, 15, \infty_9)$, $(0, 1, 6, 13, \infty_{10})$, $(0, 12, 3, 31, \infty_{11})$, $(0, 15, 28, 71, \infty_{12})$,
$(0, 33, 16, 53, \infty_{13})$, $(0, 32, 5, 57, \infty_{14})$, $(0, 63, 4, 55, \infty_{15})$, $(0, 56, 37, 73, \infty_{16})$,
$(0, 14, 81, 41, \infty_{17})$, $(0, 44, 86, 76, 35)$.

$4^{33}36^1$:

$(0, 86, 112, 22, \infty_1)$, $(0, 34, 72, 62, \infty_2)$, $(0, 68, 54, 26, \infty_3)$, $(0, 84, 34, 74, \infty_4)$,
$(0, 91, 13, 114, \infty_5)$, $(0, 59, 29, 6, \infty_6)$, $(0, 61, 39, 14, \infty_7)$, $(0, 45, 115, 2, \infty_8)$,
$(0, 67, 38, 31, \infty_9)$, $(0, 12, 7, 15, \infty_{10})$, $(0, 9, 12, 1, \infty_{11})$, $(0, 24, 51, 83, \infty_{12})$,
$(0, 93, 16, 69, \infty_{13})$, $(0, 17, 4, 61, \infty_{14})$, $(0, 16, 37, 35, \infty_{15})$, $(0, 43, 80, 11, \infty_{16})$,
$(0, 72, 25, 45, \infty_{17})$, $(0, 56, 41, 77, \infty_{18})$, $(0, 128, 76, 82, 1)$, $(0, 88, 30, 48, 97)$.

□

For some of our recursive techniques, we shall utilize what might be described as generalized product type of construction. Let us denote by ILS(s, s_1) a holey Latin square of order s when it contains only one hole of size s_1. An element in the hole of an ILS is said to be *evenly distributed* if it does not appear on the main diagonal and if when it appears in one cell, then it must appear also in its symmetric cell. If each element in the hole is evenly distributed, then we say that the ILS is *balanced.* Given 3 IMOLS, if one of the three ILS is balanced and each element in the hole determines $s - s_1$ distinct entries above the main diagonal in the other two squares, then we say that the 3 IMOLS are *compatible.*

The following constructions are fairly well known and are essentially Lemmas 2.2.1 and 2.2.3 in [11].

Lemma 8. *Suppose $q \geq 5$, and either q is an odd prime power, or $q \equiv 1$ or $5 \pmod 6$. Suppose m is even and there exist 3 compatible IMOLS$(m + e_t, e_t)$ where $t = 1, 2, ..., (q-1)/2$, $k = \sum_{1 \leq t \leq (q-1)/2}(2e_t)$. Then there exists an HSOLSSOM of type $m^q k^1$.*

Lemma 9. *Suppose q is an odd prime power, $q \geq 7$. Suppose m is even and there exist 3 MOLS(m) and 3 compatible IMOLS$(m+e_t, e_t)$ where $t = 1, 2, ..., (q-5)/2$, $k = \sum_{1 \leq t \leq (q-5)/2}(2e_t)$. Then there exists an HSOLSSOM of type $m^{(q-1)}(m + k)^1$.*

Our next recursive construction is simple, but also quite useful.

Construction 10. (Filling in Holes) *Suppose there exists an HSOLSSOM of type $\{s_i : 1 \leq i \leq n\}$. Let $w \geq 0$ be an integer. For each $i, 1 \leq i \leq n-1$, if there exists* *an HSOLSSOM of type* $\{s_{ij} : 1 \le j \le k_i\} \cup \{w\}$*, where* $s_i = \sum_{1\le j\le k_i} s_{ij}$*, then there is an HSOLSSOM of type* $\{s_{ij} : 1 \le j \le k_i, 1 \le i \le n-1\} \cup \{s_n + w\}$*.*

We wish to remark that the existence of an idempotent ISOLSSOM$(n+u, u)$ is equivalent to that of an HSOLSSOM$(1^n u^1)$; further, by Theorem 1, existence requires n to be even and u to be odd.

Theorem 11. *There exists an HSOLSSOM of type* $1^n u^1$ *(and hence also an idempotent ISOLSSOM*$(n+u,u)$*) whenever* u *is odd,* $u \le 15$*,* n *is even and* $n \ge 3u+1$*, except possibly for* $(n,u) = (10,3)$*.*

Proof. A solution for $(n,u) = (54,3)$ can be found in the Appendix. The case $(n,u) = (48,3)$ can be found in [4], while the case $(n,u) = (58,7)$ can be obtained by applying Construction 10 with $w = 1$ to an HSOLSSOM of type $6^9 10^1$, also in [4]. See Theorem 4.3 in [12] for $(n,u) \in \{(54,11),(58,13),(58,15)\}$, and Theorem 4.12 of [5] for other cases with $u \ge 3$, $(n,u) \ne (10,3)$. Finally, when $u = 1$, see Theorem 2(2) with $h = 1$. □

It should be noted that from Lemma 3.2 in [11], there exist three compatible IMOLS(v,u) for $(v,u) = (10,2)$. Also, since existence of an idempotent ISOLSSOM $(n+u,u)$ implies existence of three compatible IMOLS$(n+u,u)$, we have the following result.

Corollary 12. *There exist* 3 *compatible IMOLS*$(n+u,u)$ *whenever* u *is odd,* $u \le 15$*,* n *is even and* $n \ge 3u+1$*, except possibly for* $(n,u) = (10,3)$*.*

We also need some other recursive constructions for HSOLSSOMs, utilizing the notion of group divisible designs. A *group divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

1. $\mathcal{G}$ is a partition of a set X of *points* into subsets (called *groups*);

2. $\mathcal{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point;

3. every pair of points from distinct groups occurs in a unique block.

The *group type* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. A GDD$(X,\mathcal{G},\mathcal{B})$ will be referred to as a K-GDD if $|B| \in$ K for every block B in $\mathcal{B}$. A TD(k,n) is a GDD for which all blocks have size k, and the group type is n^k. An RTD(k,n) is a resolvable TD(k,n), that is one whose blocks can be partitioned into parallel classes. Also, a *pairwise balanced design* or (v,K)-PBD, is a K-GDD on v points in which all groups have size 1. A $(v, K \cup \{w^*\})$-PBD is a $(v, K \cup \{w\})$-PBD with one block of size w and all other block sizes from K.

Wilson's fundamental construction for GDDs [21] can also be applied to obtain a similar construction for HSOLSSOMs. The following is Construction 2.3.3 in [11].

Construction 13. (Weighting) *Suppose* $(X,\mathcal{G},\mathcal{B})$ *is a GDD and let* $w : X \to \mathbf{Z}^+ \cup \{0\}$*. Suppose there exists an HSOLSSOM of type* $\{w(x) : x \in B\}$ *for every* $B \in \mathcal{B}$*. Then there exists an HSOLSSOM of type* $\{\sum_{x\in G} w(x) : G \in \mathcal{G}\}$*.*

Lemma 14. *Suppose there exists a K-GDD of type T where K is a set of odd integers all ≥ 5. Then there exists an HSOLSSOM of type T.*

Proof. Apply Construction 13, giving weight one to each point in the GDD. □

The following product construction is similar to that given in Lemma 3.4 of [20].

Construction 15. *Suppose an HSOLSSOM of type $\{t_j : 1 \leq j \leq n\}$ exists. Let $m \geq 4$ and $m \notin \{6, 10\}$. Then an HSOLSSOM of type $\{mt_j : 1 \leq j \leq n\}$ also exists.*

The next lemma is our main tool for constructing HSOLSSOMs of type $4^n u^1$ when n is even and $u \geq n$:

Lemma 16. *If n is even, $n \geq 8$ and $n \neq 10$, then there exists an HSOLSSOM of types $4^n u^1$ whenever u is even and $n \leq u \leq min(n + 14, (4n - 4)/3)$.*

Proof. Since $n \geq 8$ and $n \neq 10$, a TD$(5, n)$ exists; add an extra point x and form a block of size $n+1$ on each group of the TD plus x. Delete a point (other than x), and take the blocks containing this deleted point as groups. This gives a $\{5, n+1\}$-GDD of type $4^n n^1$. Give the point x weight w where w is odd and $1 \leq w \leq min(15, (n-1)/3)$; also, give all other points weight 1. Now apply Construction 13 using HSOLSSOMs of types 1^5 and $1^n w^1$; since $n \neq 10$, an HSOLSSOM of type $1^n w^1$ exists by Lemma 11. This gives an HSOLSSOM of type $4^n u^1$ for $u = n+w-1$. Since $1 \leq w \leq min(15, (n-1)/3)$ and $u = n + w - 1$, we have $n \leq u \leq min(n + 14, (4n - 4)/3)$, as required. □

The next lemma gives a similar construction for n odd, although this one works only for one value of u:

Lemma 17. *If n is odd and $n \geq 5$, then there exists an HSOLSSOM of type $4^n(n - 1)^1$.*

Proof. Start with a TD$(5, n)$ and delete any point in 5th group plus all blocks containing this point. Then form a block of size n on each of the first 4 groups. This gives a $\{5, n\}$-GDD of type $4^n(n - 1)^1$. Applying Lemma 14, using HSOLSSOMs of types 1^5 and 1^n, we obtain the desired HSOLSSOM of types $4^n(n - 1)^1$. □

3 HSOLSSOM$(4^n u^1)$ for $u = 2$

First we consider the case $u = 2$. Existence of an HSOLSSOM of type $4^n 2^1$ requires $n \geq 5$. We prove:

Lemma 18. *There exist HSOLSSOMs of type $4^n 2^1$ for $n \geq 5$, except possibly for $n \in \{14, 16, 18, 21, 22, 24, 26\}$.*

Proof. Solutions for $n \in \{6, 8, 12\}$ are obtained by direct constructions given in the Appendix.

Types $4^n 2^1$ with $n \in \{5, 10, 15, 20\}$ can be obtained by Lemma 6, using HSPSs of these types. These HSPSs are given in Lemma 5.12 of [3] for $n = 5$, in Lemma 3.4 of [7] for $n \in \{10, 15\}$, and in Lemma 3.6 of [1] for $n = 20$.

1. If n is an odd prime power ≥ 5, or $n = 6t+1$ or $6t+5$, then Lemma 8 guarantees existence of an HSOLSSOM of type $4^n 2^1$.

2. If $n \equiv 3 \pmod 6$ and $n \geq 45$, write $n = 6t + 1 + 8$ where $t \geq 6$. By Lemma 8, we have an HSOLSSOM of type $4^{6t+1}34^1$. Now apply Construction 10, filling in the hole of size 34 with an HSOLSSOM of type $4^8 2^1$ to obtain an HSOLSSOM of type $4^n 2^1$.

3. If $n \equiv 0 \pmod 6$ and $n \geq 30$, write $n = 6t + 1 + 5$ where $t \geq 4$. By Lemma 8, we have an HSOLSSOM of type $4^{6t+1}22^1$. Applying Construction 10, filling in the hole of size 22 with an HSOLSSOM of type $4^5 2^1$ now gives an HSOLSSOM of type $4^n 2^1$.

4. If $n \equiv 2 \pmod 6$ and $n \geq 38$, write $n = 6t + 1 + 7$, where $t \geq 5$. By Lemma 8, we have an HSOLSSOM of type $4^{6t+1}30^1$. Applying Construction 10, filling in the hole of size 30 with an HSOLSSOM of type $4^7 2^1$ now gives an HSOLSSOM of type $4^n 2^1$.

5. If $n \equiv 4 \pmod 6$ and $n \geq 28$, write $n = 6t + 5 + 5$ where $t \geq 3$. By Lemma 8, we have an HSOLSSOM of type $4^{6t+5}22^1$. Applying Construction 10, filling in the hole of size 22 with an HSOLSSOM of type $4^5 2^1$ now gives an HSOLSSOM of type $4^n 2^1$.

Finally, for $n = 32$, 33 and 39, apply Lemma 16 to obtain HSOLSSOMs of types $4^{26}26^1$, $4^{26}30^1$ and $4^{30}38^1$ respectively. Now apply Construction 10 to these HSOLSSOMs, filling in the holes of sizes 26, 30 and 38 with HSOLSSOMs of types $4^6 2^1$, $4^7 2^1$ and $4^9 2^1$ respectively. $\square$

4 HSOLSSOM($4^n u^1$) with $5 \leq n \leq 30$ and $u \geq 4$

For $u \geq 4$, existence of an HSOLSSOM of type $4^n u^1$ requires (1) u to be even, (2) either $(n, u) = (4, 4)$, or $n \geq 5$ and $u \leq (4n - 4)/3$. When $u = 4$, HSOLSSOMs of type $4^n u^1$ are already known to exist for all $n \geq 4$ by Theorem 2.

We now investigate existence of HSOLSSOMs of type $4^n u^1$ with $u \geq 4$. We start by looking at the smallest values of n.

Lemma 19. *Suppose either* (1) $(n, u) = (4, 4)$ *or* (2) $5 \leq n \leq 19$, *u is even*, $u \geq 4$ *and* $u \leq (4n - 4)/3$. *Then an HSOLSSOM of type* $4^n u^1$ *exists, except possibly for* $(n, u) = (19, 22)$.

Proof. For $u = 4$ and $n \geq 4$, the result follows from Theorem 2(2). For $n \in \{5, 7, 9, 11, 13, 17, 19\}$ and $u < n$, apply Lemma 8 with $m = 4$, $q = n$, $e_t \in \{0, 1\}$ and $\sum_{t=1}^{(q-1)/2}(2e_t) = u$. For $n \in \{6, 8, 10, 12, 16, 18\}$, $n + 1$ is an odd prime power; here, when $u \leq n$, we can apply Lemma 9 with $m = 4$, $q = n + 1$, $e_t \in \{0, 1\}$ and $\sum_{t=1}^{(q-5)/2}(2e_t) = u - 4$. For n even, $n \geq 12$, and $u \geq n$, apply Lemma 16. For types $4^9 10^1$, $4^{11}12^1$, $4^{13}14^1$, $4^{14}u^1$ for $u \in \{6, 8, 10\}$, $4^{15}10^1$, $4^{17}20^1$, and $4^{19}20^1$, direct constructions using Lemma 5 are given in the Appendix. For types $4^7 8^1$, $4^{10}12^1$, $4^{13}16^1$ and $4^{19}24^1$, we can apply Lemma 14, since $(5,1)$-GDDs of these types are obtainable by adding 8, 12, 16 or 24 points to the parallel classes in 4-RGDDs of types 4^7, 4^{10}, 4^{13} and 4^{19}. For types $4^{15}8^1$ and $4^{15}12^1$, we can also apply Lemma 14, since 5-GDDs of these types are obtainable by deleting one point from the large block (and the blocks containing this point) in a $(69, \{5, 9^*\})$-PBD or $(73, \{5, 13^*\})$-PBD. These PBDs both exist [8].

This leaves the following cases to be handled: $4^{14}12^1$, $4^{15}u^1$ with $u \in \{6, 14, 16, 18\}$ and $4^{17}18^1$. For type $4^{14}12^1$, apply Construction 15 with $m = 4$ to an HSOLSSOM of type $1^{14}3^1$, which exists by Lemma 11. For type $4^{15}14^1$, apply Lemma 17. For types $4^{15}u^1$ with $u \in \{6, 16, 18\}$ and $4^{17}18^1$, we can apply Lemma 6 since HSPSs of these types can be found in Lemma 3.4 of [7] for $4^{15}6^1$ and $4^{15}16^1$, in Lemma 7 for $4^{15}18^1$, and in Lemma 3.6 of [2] for $4^{17}18^1$. □

Lemma 20. *Suppose* $20 \le n \le 30$*, u is even, $u \ge 4$ and* $u \le (4n-4)/3$*. Then an HSOLSSOM of type $4^n u^1$ exists, except possibly for* (1) $n = 21$*,* $u \in \{6, 12, 14, 16, 18, 22, 24, 26\}$*,* (2) $n = 23$*,* $u \in \{24, 28\}$*,* (3) $n = 25$*,* $u = 30$*,* (4) $n = 27$*,* $u \in \{30, 32, 34\}$*, and* (5) $n = 29$*,* $u \in \{30, 32, 36\}$*.*

Proof. For $n \in \{23, 25, 27, 29\}$ and $u < n$, apply Lemma 8 with $m = 4$, $q = n$, $e_t \in \{0, 1\}$ and $\sum_{t=1}^{(q-1)/2}(2e_t) = u$. For $n \in \{22, 24, 26, 28, 30\}$, and $u \le n$, use Lemma 9 with $q = n + 1$, $m = 4$, $e_t \in \{0, 1\}$ and $\sum_{t=1}^{(q-5)/2}(2e_t) = u - 4$. For n even, and $u \ge n$, use Lemma 16.

For types $4^{20}u^1$ with $u \le 20$, we start with a TD$(6, 8)$, and apply Construction 13. Here we give weight 2 to all points in first 5 groups; in the 6th group give all points weight 0 or 2 so that total weight is $u - 4$. This gives an HSOLSSOM of type $16^5(u - 4)^1$. Now apply Construction 10, filling in the first 5 groups with 4 extra points, using an HSOLSSOM of type 4^5.

For type $4^{21}20^1$, apply Lemma 17, and for type $4^{21}4^1$, see Theorem 2(2). For type $4^{21}10^1$, apply Lemma 6, using the HSPS of this type in Lemma 7. For type $4^{21}8^1$, add 21 points to the parallel classes of a 4-RGDD of type 9^8 (which exists, see [13]); then form a $(21, 5, 1)$ BIBD on the extra points and a block of size 9 on each group. This gives a $(93, \{5, 9\}, 1)$-PBD. Deleting a point from a block of size 9 (and all its blocks) now gives a $\{5, 9\}$-GDD of type $4^{21}8^1$, so we can apply Lemma 14.

For $(n, u) \in \{(23, 26), (29, 34)\}$, we can apply Lemma 6, since HSPSs of types $4^{23}26^1$ and $4^{29}34^1$ exist by Lemma 7. For $(n, u) \in \{(25, 28), (25, 32), (27, 28)\}$, we have a $(5, 1)$-GDD of type $4^n u^1$ by deleting one point from the block of size $u + 1$ in a $(4n + u + 1, \{5, (u + 1)^*\})$-PBD. These PBDs all exist [8]. Now apply Lemma 14 to these GDDs.

For type $4^{25}26^1$, delete 5 points in one block of a TD$(6, 11)$ to obtain a $\{5, 6\}$-GDD of type $10^5 11^1$. Applying Lemma 14, using HSOLSSOMs of types 2^5 and 2^6 gives an HSOLSSOM of type $20^5 22^1$. Applying Construction 10, filling in the holes of size 20 with 4 extra points and an HSOLSSOM of type 4^6, we now obtain the desired HSOLSSOM of type $4^{25}26^1$. □

5 HSOLSSOM($4^n u^1$) with $31 \le n \le 37$ and $u \ge 4$

Lemma 21. *There exist HSOLSSOMs of types $4^n u^1$ for $n \in \{31, 37\}$ and either $4 \le u \le n$ or $(n, u) \in \{(31, 40), (37, 48)\}$. Also there exist HSOLSSOMs of types $4^n u^1$ for $n \in \{35, 36\}$ and $4 \le u \le (4n-4)/3$.*

Proof. For $n = 36$, and $4 \le u \le 36$, we can apply Lemma 8 with $m = 4$, $q = 37$, $e_t \in \{0, 1\}$ and $\sum_{t=1}^{(q-1)/2}(2e_t) = u-4$. For $n = 36$ and $36 \le u \le 46$, apply Lemma 16.

When $n \in \{31, 35, 37\}$ and $u < n$, the required HSOLSSOMs are obtainable by applying Lemma 8 with $m = 4$, $q = n$, $e_t \in \{0, 1\}$ and $\sum_{t=1}^{(q-1)/2}(2e_t) = u$. In addition, HSOLSSOMs of types $4^{31}40^1$ and $4^{37}48^1$ are obtainable by Lemma 14, since $(5, 1)$-GDDs of these types are obtainable by adding 40 or 48 points to the parallel classes of a 4-RGDD of type 4^{31} or 4^{37}. For types $4^{35}u^1$ with u even and $36 \le u \le 44$, apply Lemma 8 with $m = 28$, $q = 5$ and $e_t = 9$ for $t = 1, 2$ to obtain an HSOLSSOM of type $28^5 36^1$. Now apply Construction 10, filling in the holes of size 28 with s extra points where $0 \le s \le 8$, using HSOLSSOMs of types $4^7 s^1$. □

The next lemma helps us obtain one more case for each $n \in \{31, 37\}$.

Lemma 22. *If p is a prime power, then a $\{p-1, p\}$-GDD of type $(p-2)^{p-1}(p-1)^1 p^1$ exists.*

Proof. Take the blocks in one parallel class of the RTD$(p-1, p)$ as groups, and the groups of this RTD as blocks to obtain a $\{p-1, p\}$-RGDD of type $(p-1)^p$. Delete all points in 1 block of size $p-1$ to obtain a $\{p-2, p-1\}$-RGDD of type $(p-2)^{p-1}(p-1)^1$. This design has p parallel classes; finally add p points, each to a parallel class of blocks to obtain a $\{p-1, p\}$-RGDD of type $(p-2)^{p-1}(p-1)^1 p^1$. □

Lemma 23. *There exist HSOLSSOMs of types $4^{31}32^1$ and $4^{37}38^1$.*

Proof. From the previous lemma, a $\{6, 7\}$-GDD of type $5^6 6^1 7^1$ and a $\{7, 8\}$-GDD of type $6^7 7^1 8^1$ both exist. Deleting one group of size 5 from the first GDD and 2 groups of size 6 from the second, we obtain a $\{5, 6, 7\}$-GDD of type $5^5 6^1 7^1$ and a $\{5, 6, 7, 8\}$-GDD of type $6^5 7^1 8^1$. Applying Construction 13 to these GDDs, giving all points weight 4, we obtain HSOLSSOMs of types $20^5 24^1 28^1$ and $24^5 28^1 32^1$. Now apply Construction 10 to these HSOLSSOMs. By filling in the holes of sizes 20, 24 in the first HSOLSSOM with 4 new points, using HSOLSSOMs of types 4^6 and 4^7, we obtain an HSOLSSOM($4^{31}32^1$). Similarly, filling the holes of sizes 24, 28 in the second HSOLSSOM with 6 extra points, using HSOLSSOMs of types $4^6 6^1$ and $4^7 6^1$ gives an HSOLSSOM of type $4^{37}38^1$. □

Now we consider the case $n \in \{32, 33, 34\}$. Here we require the following lemma for some values of u:

Lemma 24. *Suppose a TD$(6, m)$ exists and $0 \le x \le m$. If HSOLSSOMs of types $4^m t^1$ and $4^{m-x} t^1$ exist for t even, $0 \le t \le m-x$, then there also exists an HSOLSSOM of type $4^{5m-x} u^1$ for u even and $4m \le u \le 5m-x$.*

Proof. Start with a TD$(6, m)$, and truncate one of its groups to size $m - x$, giving a $\{5, 6\}$-GDD of type $m^5(m-x)^1$. Apply Construction 13, giving all points weight

4 to obtain an HSOLSSOM of type $(4m)^5(4(m-x))^1$. Now apply Construction 10, filling in all holes (except one of size $4m$) with t points where $0 \le t \le m-x$; here we require HSOLSSOMs of types $4^m t^1$ and $4^{m-x}t^1$. This gives an HSOLSSOM of type $4^{4m+(m-x)}(4m+t)^1$ for $0 \le t \le m-x$, i.e. an HSOLSSOM($4^{5m-x}u^1$) for $4m \le u \le 5m-x$, as required. □

Lemma 25. *There exists an HSOLSSOM of type $4^n u^1$ for $n \in \{32, 34\}$, u even and $4 \le u \le (4n-4)/3$. Also an HSOLSSOM of type $4^n u^1$ exists for $n = 33$, u even and either $4 \le u \le n$ or $n = 36$.*

Proof. First we deal with the case $4 \le u \le 22$. Take $s = 2$, 3 and 4 respectively, when $n = 32$, 33 and 34. Start with a TD(11, 11), let B be a specified block in this TD, and delete the 10 points in both B and one of the first 10 groups. Now truncate s of the first 10 groups to size 2, and $4-s$ of them to size 0. This gives a $\{6, 7, 8, 9\}$-GDD of type $10^6 2^s 11^1$. Apply Construction 13, giving weight 2 to all points in the groups of sizes 2 and 10; in the group of size 11, give points weight 0 or 2 so that the total weight is u. This gives an HSOLSSOM of type $20^6 4^s u^1$. Now apply Construction 10, with $w = 0$, filling in the holes of size 20 with an HSOLSSOM of type 4^5.

For $(n, u) = (33, 24)$, $(34, 24)$, $(33, 26)$ and $(34, 26)$ respectively, we can similarly truncate one block and one or two groups in a TD(8, 11) to obtain $\{5, 6, 7, 8\}$-GDDs of types $10^6 8^2$, $10^7 8^1$, $10^5 8^2 11^1$, and $10^6 8^1 11^1$. Applying Construction 13, giving weight 2 to all points gives HSOLSSOMs of types $20^6 16^2$, $20^7 16^1$, $20^5 16^2 22^1$ and $20^6 16^1 22^1$. Now apply Construction 10 with $w = 4$, filling all holes with 4 extra points, except one hole of size 20 (in the first two cases) or size 22 (in the last two cases). Here we use HSOLSSOMs of types 4^5 and 4^6. This gives HSOLSSOMs of types $4^{33}24^1$, $4^{34}24^1$, $4^{33}26^1$ and $4^{34}26^1$, respectively.

For $n \in \{33, 34\}$ and $28 \le u \le n$, use Lemma 24 with $m = 7$, and $x = 35 - n \in \{1, 2\}$. Here $m - x \in \{5, 6\}$. For the required HSOLSSOMs of types $4^5 t^1$ with t even $0 \le t \le 4$, and $4^6 t^1$, $4^7 t^1$ with t even, $0 \le t \le 6$, see Lemma 19.

For $n \in \{32, 34\}$ and $u \ge n$, apply Lemma 16. For $n = 33$ and $u = 36$, we can apply Lemma 6, since an HSPS of type $4^{33}36^1$ exists by Lemma 7.

For $n = 32$ and $20 \le u \le n$, we can use Lemma 9 with $m = 16$, $q = 9$, $e_t \in \{0, 1, 3, 5\}$ for $t \in \{1, 2\}$, and $\sum_{t=1}^{2}(2e_t) = u - 20$ to obtain an HSOLSSOM of type $16^8(u-4)^1$. Note that for these values of e_t, there exist an idempotent ISOLSSOM($16 + e_t, e_t$) and hence also 3 compatible IMOLS($16 + e_t, e_t$) by Theorem 11. Now apply Construction 10, filling in all holes of size 16 with 4 extra points, using an HSOLSSOM of type 4^5. □

6 HSOLSSOM($4^n u^1$) with $38 \le n \le 66$ and $u \ge 4$

For $n \ge 38$, we are able to prove the existence of HSOLSSOM($4^n u^1$) whenever $u \le n$ (if n is odd) or $u \le min(n + 14, (4n-4)/3)$ (if u is even). For n even and $n \le u \le min(n + 14, (4n-4)/3)$, this HSOLSSOM exists by Lemma 16. Below is one major lemma which will be used in establishing this result for $u \le n$.

Lemma 26. *Suppose a TD$(5{+}z,m)$ exists and $4 \le a_i \le m$ for $1 \le i \le z$. Then there exists an HSOLSSOM of type $(4m)^5$ $(4a_1)^1$ $(4a_2)^1 \ldots$, $(4a_z)^1$. Also, for $t = \sum_{i=1}^{z-1} a_i$, there exists an HSOLSSOM of type and $4^{5m+t}(4(a_z+1))^1$.*

Proof. Start with a TD$(5+z,m)$, and apply Construction 13, giving weight 4 to all points in groups 1 to 5; in group $5+i$ for $1 \le i \le z$, give weight 4 to a_i points and zero weight to the rest. This gives an HSOLSSOM of type $(4m)^5$ $(4a_1)^1$ $(4a_2)^1 \ldots$, $(4a_z)^1$. By applying Construction 10 with $w = 4$ to this HSOLSSOM, forming an HSOLSSOM of type 4^{m+1} or 4^{a_i+1} on each hole (except the one of size $4a_z$) plus 4 infinite points, we now obtain the required HSOLSSOM of type $4^{5m+t}(4(a_z{+}1))^1$. □

Lemma 27. *There exists an HSOLSSOM of type $4^n u^1$ for $n \in \{38, 39\}$, u even and $4 \le u \le 38$.*

Proof. For $4 \le u \le 26$, start with a TD$(9, 11)$. Delete the 8 points in both a specified block and one of the first 8 groups; also delete two more points from both the first and second groups (when $n = 38$) or from just the first group (when $n = 39$). This gives $\{6, 7, 8, 9\}$-GDDs of types $10^6 8^2 11^1$ and $10^7 8^1 11^1$. Now we apply Construction 13: here we give weight 2 to all points in the groups of sizes 8 and 10, and in the group of size 11, we give points weight 0 or 2 so that the total weight is $u - 4$. This gives HSOLSSOMs of types $20^6 16^2 (u-4)^1$ and $20^7 16^1 (u-4)^1$. We now apply Construction 10, filling in the holes of sizes 16 and 20 with 4 extra points, using HSOLSSOMs of types 4^5 and 4^6 to give the desired result.

For $u \in \{28, 30\}$, apply Lemma 26 with $m = 7$ and $z = 2$ to obtain an HSOLSSOM of type $28^5 20^2$ (when $n = 38$) or $28^5 20^1 24^1$ (when $n = 39$). For $u = 28, 30$ respectively, take $t = 0, 2$, and apply Construction 10, filling in all holes except one of size 28 with t extra points, using HSOLSSOMs of types $4^5 t^1$, $4^6 t^1$ and $4^7 t^1$.

Finally, for $32 \le u \le 38$, use Lemma 24 with $m = 8$ and $x = 40 - n$. □

Lemma 28. *There exists an HSOLSSOM of type $4^n u^1$ for $n \in \{44, 45\}$, u even and $4 \le u \le n$.*

Proof. When $4 \le u \le 36$, apply Lemma 26 with $m = 8$, $z = 2$, to obtain HSOLSSOMs of types $32^5 16^1 (u{-}4)^1$ and $32^5 20^1 (u{-}4)^1$. Now, apply Construction 10, filling in all holes of sizes 16, 20 and 32 with 4 extra points, using HSOLSSOMs of types 4^5, 4^6 and 4^9.

When $36 \le u \le 44$, apply Lemma 24 with $m = 9$ and $x = 45 - n$. □

Lemma 29. *There exists an HSOLSSOM of type $4^n u^1$ for $n \in \{50, 51, 54, 56, 57\}$, u even and $4 \le u \le n$.*

Proof. For $4 \le u \le 36$, apply Lemma 26 with $m = 9$ to obtain HSOLSSOMs of types $36^5 20^1 (u-4)^1$, $36^5 24^1 (u-4)^1$, $36^6 (u-4)^1$, $36^5 20^1 24^1 (u-4)^1$ and $36^5 24^2 (u-4)^1$ respectively, when $n = 50, 51, 54, 56, 57$. Filling in the holes of sizes 20, 24, 32 and 36 with 4 extra points, using HSOLSSOMs of type 4^y with $y \in \{6, 7, 9, 10\}$ now gives the desired result.

When $36 \le u \le 42$, we similarly apply Lemma 26 with $m = 9$ to obtain HSOLSSOMs of types $36^5 32^1 24^1$, $36^6 24^1$, 36^7, $36^5 24^1 28^2$ and $36^5 28^3$ respectively, for $n = 50$, 51, 54, 56 and 57. Now let $w = u - 36 \in \{0, 2, 4, 6\}$, and apply Construction 10, filling in all holes except one of size 36 with w extra points, using HSOLSSOMs of types $4^y w^1$ for $y \in \{6, 7, 8, 9\}$.

When $u \in \{44, 46\}$, and $n \in \{56, 57\}$, we apply Lemma 26 with $m = 11$ to obtain an HSOLSSOM of type $44^5 24^2$ (when $n = 56$) or $44^5 24^1 28^1$ (when $n = 57$). Now let $w = u - 44 \in \{0, 2\}$, and apply Construction 10, filling in all holes except one of size 44 with w extra points, using HSOLSSOMs of types $4^y w^1$ for $y \in \{6, 7, 11\}$.

When $44 \le u \le n$ and $n \in \{50, 51, 54\}$, use Lemma 24 with $m = 11$ and $x = 55 - n$. When $48 \le u \le n$ and $n \in \{56, 57\}$, use Lemma 24 with $m = 12$ and $x = 60 - n$. □

Lemma 30. *There exists an HSOLSSOM of type $4^n u^1$ for $n \in \{62, 63, 64\}$, u even and $4 \le u \le n$.*

Proof. For $u \le 44$, apply Lemma 26 with $m = 11$ to obtain HSOLSSOMs of types $44^5 28^1 (u-4)^1$, $44^5 32^1 (u-4)^1$ and $44^5 36^1 (u-4)^1$ for $n = 62, 63, 64$ respectively. Now apply Construction 10, filling in the holes of sizes 28, 32, 36 and 44 with 4 extra points, using HSOLSSOMs of type 4^t with $t \in \{8, 9, 10, 12\}$.

When $44 \le u \le 50$, we can similarly apply Lemma 26 with $m = 11$ to obtain HSOLSSOMs of types $44^5 36^2$, $44^5 36^1 40^1$ and $44^5 40^2$. Now let $w = u - 44 \in \{0, 2, 4, 6\}$, and apply Construction 10, filling in all groups except one of size 44 with w extra points, using HSOLSSOMs of type $4^y w^1$ with $y \in \{9, 10, 11\}$.

When $52 \le u \le n$, use Lemma 24 with $m = 13$ and $x = 65 - n$. □

Lemma 31. *There exists an HSOLSSOM of type $4^n u^1$ for $38 \le n \le 66$, u even, and $4 \le u \le n$.*

Proof. The cases $n \in \{38, 39, 44, 45, 50, 51, 54, 56, 57, 62, 63, 64\}$ were handled in Lemmas 27, 28, 29 and 30. For other even values of n, $n + 1$ is a prime power, and we can use Lemma 9 with $m = 4$, $q = n + 1$, $e_t \in \{0, 1\}$. For other odd values of n, $n \equiv 1$ or 5 (mod 6), and we can use Lemma 8 with $m = 4$, $q = n$, $e_t \in \{0, 1\}$. □

7 HSOLSSOM($4^n u^1$) with $n > 66$ and $u \ge 4$

We now consider HSOLSSOMs of type $4^n u^1$ for all values of $n > 66$. When n is even and $n \le u \le min(n + 14, (4n - 4)/3)$ these HSOLSSOMs can be obtained by Lemma 16. The following two lemmas will be used to obtain the majority of these HSOLSSOMs when $4 \le u \le n$:

Lemma 32. *If a TD$(13, m)$ exists, then an HSOLSSOM$(4^n u^1)$ exists for $n \in [10m + 4, 12m]$, u even and $4 \le u \le 12m + 4$. In particular, if $n \in [10m + 4, 12m]$, this HSOLSSOM exists whenever u is even and $4 \le u \le n$.*

Proof. Let $n = 10m + s + t$ where s, t are either zero or in the range $[4, m]$. Start with a TD$(13, m)$, and give all points in the first 10 groups weight 4; in groups 11 and 12, give weight 4 to s points in group 11, t points in group 12 and zero weight to the rest. In group 13, give points weight 0, 2, 4, 6, 8, 10 or 12 so that the total weight is $u-4$. Now apply Construction 13 with these weightings, using HSOLSSOMs of types $4^y a^1$ with $y \in \{10, 11, 12\}$ and $a \in \{0, 2, 4, 6, 8, 10, 12\}$; this gives an HSOLSSOM($(4m)^{10}(4s)^1(4t)^1(u-4)^1$). We now add four infinite points and apply Construction 10, forming HSOLSSOMs of types 4^x with $x \in \{m+1, s+1, t+1\}$ on each of the first 12 holes plus the infinite points, giving an HSOLSSOM of type $4^{10m+s+t}u^1$ as required. □

Lemma 33. *If a TD*$(10, m)$ *exists, then an HSOLSSOM*$(4^n u^1)$ *exists for* $n \in [7m+4, 9m]$, u *even and* $4 \le u \le 8m+4$. *In particular, if* $n \in [7m+4, 8m+4]$, *this HSOLSSOM exists whenever* u *is even and* $4 \le u \le n$.

Proof. Let $n = 7m+s+t$ where s, t are either zero or in the range $[4, m]$. Start with a TD$(10, m)$, and give all points in the first 7 groups weight 4; in groups 8 and 9, give weight 4 to s points in group 8, t points in group 9 and zero weight to the rest. In group 10, give points weight 0, 2, 4, 6 or 8 so that the total weight is $u-4$. Now apply Construction 13, with these weightings, using HSOLSSOMs of types $4^y a^1$ with $y \in \{7, 8, 9\}$ and $a \in \{0, 2, 4, 6, 8\}$; this gives an HSOLSSOM($(4m)^7(4s)^1(4t)^1(u-4)^1$). We now add four infinite points and apply Construction 10, forming an HSOLSSOM of type 4^x with $x \in \{m+1, s+1, t+1\}$ on each of the first 9 holes plus the infinite points, giving an HSOLSSOM of type $4^{7m+s+t}u^1$ as required. □

Lemma 34. *Suppose* $67 \le n \le 140$. *Then an HSOLSSOM of type* $4^n u^1$ *exists for* u *even and* $4 \le u \le n$.

Proof. For $67 \le n \le 140$, and n not in one of the ranges $[77, 80]$, $[93, 94]$ or $[109, 115]$, these HSOLSSOMs can all be obtained by Lemma 33. Table 3 gives the values of m used in this lemma; for each m, it also gives the ranges $[7m+4, 8m+4]$ for n for which this lemma works for all even u, $4 \le u \le n$.

Table 3: Values of m used when applying Lemma 33 for $67 \le n \le 140$.

m	Range for n	m	Range for n
9	$[67, 76]$	16	$[116, 132]$
11	$[81, 92]$	17	$[121, 140]$
13	$[95, 108]$		

Now we have to handle the case where n is in one of the ranges $[77, 80]$, $[93, 94]$ or $[109, 115]$. For $n \in [77, 80]$, apply Lemma 33 with $m = 9$ when $4 \le u \le 76$, and Lemma 24 with $m = 17$, $5 \le x \le 8$ when $68 \le u \le n$. For $n \in [93, 94]$, apply Lemma 33 with $m = 11$ when $4 \le u \le 92$, and Lemma 16 when $n = u = 94$. For $n \in [109, 115]$, apply Lemma 33 with $m = 13$ for $4 \le u \le 104$; for $104 \le u \le n$, apply Lemma 24 with $m = 25$, $13 \le x \le 16$ if $109 \le n \le 112$, or with $m = 27$, $20 \le x \le 22$ if $113 \le n \le 115$. □

Lemma 35. *Suppose $n \geq 134$. Then an HSOLSSOM of type $4^n u^1$ exists for u even, and $4 \leq u \leq n$.*

Proof. For $157 \leq n \leq 163$, we can apply Lemma 33 with $m = 19$ when $4 \leq u \leq 152$, and Lemma 24 with $m = 38$, $27 \leq x \leq 33$ when $152 \leq u \leq n$.

For $229 \leq n \leq 233$, we can apply Lemma 33 with $m = 31$ when $4 \leq u \leq n$.

For $134 \leq n \leq 1164$, $n \notin [157, 163]$ or $[229, 233]$, apply Lemma 32 using the values of m in Table 4. We indicate the range for n covered by each value of m, that is, $[10m + 4, 12m]$. In all cases, the given construction works for $4 \leq u \leq 12m + 4$ and hence for $4 \leq u \leq n$.

Table 4: Values of m used when applying Lemma 32 for $134 \leq n \leq 1164$.

m	Range for n	m	Range for n	m	Range for n
13	$[134, 156]$	27	$[274, 324]$	53	$[534, 636]$
16	$[164, 192]$	32	$[324, 384]$	61	$[614, 732]$
17	$[174, 204]$	37	$[374, 444]$	71	$[714, 852]$
19	$[194, 228]$	43	$[434, 516]$	83	$[834, 996]$
23	$[234, 276]$	47	$[474, 564]$	97	$[974, 1164]$

For $n \geq 1164$, we first note that for any set of 9 consecutive odd integers m, at most three are divisible by 3, two by 5, two by 7 and one by 11. Therefore there is always at least one integer m in the set not divisible by 3, 5, 7 or 11, and a TD$(13, m)$ exists for this value of m. Also, for $m \geq 97$, the intervals $[10m + 4, 12m]$ and $[10(m + 18) + 4, 12(m + 18)]$ overlap; hence for any $n \geq 12 \cdot 97 = 1164$, there is always at least one value of m for which $n = 10m + s + t$, with m, s, t satisfying the conditions in Lemma 32. Thus for any $n \geq 1164$, Lemma 32 can be used to obtain all HSOLSSOMs of type $4^n u^1$ with u even and $4 \leq u \leq n$.

Summarizing the results of the last three sections, we have proved our main result, Theorem 4, which we restate for convenience. The proof of this theorem follows from Lemmas 18, 19, 20, 21, 23, 25, 31, 34 and 35.

Theorem 36. *Necessary conditions for existence of an HSOLSSOM of type $4^n u^1$ with $u > 0$ are (1) u even, and (2) $u \leq (4n - 4)/3$, and either $(n, u) = (4, 4)$ or $n \geq 5$. These conditions are sufficient, except possibly in the following cases:*

1. *For 36 pairs (n, u) with $n \leq 37$ listed in Table 2.*
2. *When $n > 37$, n is odd, and $n < u \leq (4n - 4)/3$.*
3. *When $n > 37$, n is even, and $n + 14 < u \leq (4n - 4)/3$.*

Concluding Remark 37. *As indicated earlier in Theorem 11, the necessary conditions for existence of an idempotent ISOLSSOM$(n + u, u)$, or equivalently, an HSOLSSOM$(1^n u^1)$ (i.e. n is even, u is odd and $n \geq 3u + 1$) are sufficient when*

$u \le 15$, *except for* $(n, u) = (10, 3)$. *For larger* u, *if* $n = 4t$, *then several idempotent ISOLSSOM*$(n+u, u)$*s can be obtained from the results in Theorem 36, since by filling in the holes of an HSOLSSOM*$(4^t(u-1)^1)$ *with one extra point, an idempotent ISOLSSOM*$(4t+u, u)$ *is obtained. However, a more complete investigation of the existence of ISOLSSOM*$(n+u, u)$*s with* $u > 15$ *is yet to be carried out.*

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Appendix: Direct constructions for HSOLSSOM$(4^n u^1)$

To ease the notation for these HSOLSSOMs of type $h^n u^1$ (obtained by Lemma 5), we shall always take $G = Z_{hn}$ having a subgroup H of order h (here, except for one example of type $1^{54}3^1$, we have $h = 4$). If u is even, let $u = 2t$ and $X = \{x_1, ..., x_t, y_1, ..., y_t\}$, and if u is odd, let $u = 2t+1$ and $X = \{x, x_1, ..., x_t, y_1, ..., y_t\}$. In both cases, we choose x_i and y_i so that they form a pair satisfying condition 5 in Lemma 5 for $1 \le i \le t$. According to the conditions 4 and 5 in Lemma 5, to construct an HSOLSSOM$(h^n u^1)$ we may record half the 5-tuples instead of listing all of them. That is, only one of the two 5-tuples (b_1, b_2, b_3, b_4, x) and (b_2, b_1, b_4, b_3, y) is recorded if either (1) $x = y$ and $x \in G$ or (2) x and y belong to X. In order to save space, we write half of 5-tuples of $\mathcal{B}$ vertically as columns of an array.

For the last two examples, $4^{17}20^1$ and $4^{19}20^1$, only one quarter of the 5-tuples are given as columns; here each given 5-tuple $(b_1, b_2, b_3, b_4, z_1)$ generates four 5-tuples of the form $(b_1, b_2, b_3, b_4, z_1)$, $(b_2, b_1, b_4, b_3, z_2)$, $(b_3, b_4, b_1, b_2, z_3)$ and $(b_4, b_3, b_2, b_1, z_4)$. Here either $z_1 \in G$ and $z_1 = z_2 = z_3 = z_4$, or z_1, z_2, z_3, z_4 are distinct elements of X.

$1^{54}3^1$:

0	0	x	x_1	y_1	52	11	48	7	47	18	34	6	53	25
27	19	29	24	26	5	42	30	13	10	51	44	9	12	1
1	29	37	21	25	43	9	35	39	13	49	40	16	47	14
28	22	8	31	51	x	x_1	y_1	36	26	1	3	18	33	15
x	x_1	0	0	0	0	0	0	0	0	0	0	0	0	0

32	46	15	37	19	23	2	33	8	14	40	36	21	20	38
3	50	43	49	27	45	17	22	28	16	31	41	35	4	39
44	45	52	19	41	34	53	28	32	48	42	12	29	5	2
20	11	17	30	46	38	22	10	24	6	4	27	7	50	23
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$4^6 2^1$:

0	x_1	y_1	9	16	15	4	2	1	10	8	13
11	20	17	19	23	7	5	21	3	14	11	22
3	11	19	20	13	5	14	1	16	17	4	21
20	22	15	x_1	y_1	2	9	10	8	7	3	23
x_1	0	0	0	0	0	0	0	0	0	0	0

$4^8 2^1$:

0	x_1	y_1	31	10	9	15	19	26	1	18	20	28	14	2	12
11	29	4	13	17	11	21	22	7	5	30	25	6	23	3	27
9	4	6	22	27	30	25	21	19	20	1	14	26	13	29	23
14	3	18	x_1	y_1	15	28	10	12	2	11	5	7	9	31	17
x_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$4^9 10^1$:

0	0	0	0	0	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3
13	7	3	15	17	34	26	25	3	11	14	4	20
1	20	5	32	29	32	30	31	11	25	33	35	10
16	33	10	3	32	8	13	17	19	23	3	15	6
x_1	x_2	x_3	x_4	x_5	0	0	0	0	0	0	0	0

y_4	y_5	29	16	31	2	22	19	10	35	13	24	33
15	1	7	21	5	6	28	17	30	23	12	32	8
4	24	28	22	12	21	20	16	5	7	34	26	1
29	14	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	2
0	0	0	0	0	0	0	0	0	0	0	0	0

$4^{11} 12^1$:

0	0	0	0	0	0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4
19	13	5	21	15	7	3	2	40	19	31	14	4	25	12	9
31	23	29	26	40	2	43	5	10	23	39	6	20	2	13	35
34	32	30	19	35	17	27	41	24	36	19	18	16	8	15	1
x_1	x_2	x_3	x_4	x_5	x_6	0	0	0	0	0	0	0	0	0	0

y_5	y_6	28	1	23	20	27	10	36	26	6	7	21	24	35	17
5	34	32	13	37	29	43	30	42	16	8	15	39	41	38	18
32	3	34	42	30	38	40	9	26	31	7	21	37	17	12	25
14	28	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	y_5	y_6	29	4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$4^{12}2^{1}$:

0	x_1	y_1	30	20	34	1	40	35	23	26	5
33	16	25	38	39	45	2	47	44	41	28	27
31	38	27	3	26	11	29	19	5	8	34	45
38	20	18	x_1	y_1	42	32	21	31	9	47	16
x_1	0	0	0	0	0	0	0	0	0	0	0

17	15	10	32	21	4	13	14	46	29	6	3
37	11	33	42	8	7	18	31	19	43	22	9
40	28	1	41	2	30	15	13	44	23	17	35
6	7	43	37	22	14	4	46	39	33	25	10
0	0	0	0	0	0	0	0	0	0	0	0

$4^{13}14^{1}$:

0	0	0	0	0	0	0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	y_1	y_2	y_3	y_4	y_5
19	17	5	37	3	9	1	25	36	5	7	35	9	32	29	30	49	47	42
5	21	37	44	24	14	30	7	51	22	47	27	15	50	5	14	42	48	36
48	42	14	25	9	11	37	11	1	23	19	33	4	28	37	9	17	38	24
x_1	x_2	x_3	x_4	x_5	x_6	x_7	0	0	0	0	0	0	0	0	0	0	0	0

y_6	y_7	51	40	6	21	19	17	34	16	14	3	11	38	45	12	33	4	2
18	15	1	50	27	37	23	24	46	48	43	28	41	10	31	20	44	22	8
41	34	44	10	29	18	2	21	45	31	46	6	30	32	3	40	8	20	35
25	16	x_1	x_2	x_3	x_4	x_5	x_6	x_7	y_1	y_2	y_3	y_4	y_5	y_6	y_7	43	12	49
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$4^{14}6^{1}$:

0	0	0	x_1	x_2	x_3	y_1	y_2	y_3	39	26	43	40	41	15	49
3	15	7	19	4	45	34	3	8	23	44	37	18	9	17	16
33	39	22	15	39	50	52	53	37	31	21	2	30	29	40	38
38	16	31	45	17	43	25	55	5	x_1	x_2	x_3	y_1	y_2	y_3	46
x_1	x_2	x_3	0	0	0	0	0	0	0	0	0	0	0	0	0

32	6	27	55	22	1	2	5	20	52	7	38	54	33	11	13
36	51	46	24	35	21	29	31	30	53	12	47	10	25	50	48
49	22	48	19	16	13	20	24	47	4	10	23	9	35	34	51
12	32	33	8	41	26	36	6	3	7	11	27	44	18	54	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$4^{14}8^1$:

0	0	0	0	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	29	25	44	5	38
51	13	25	23	33	52	41	10	11	43	53	37	45	51	54	13	4
33	30	24	25	21	4	9	23	16	41	50	48	52	20	24	46	35
40	3	19	16	43	36	29	15	12	51	34	7	x_1	x_2	x_3	x_4	y_1
x_1	x_2	x_3	x_4	0	0	0	0	0	0	0	0	0	0	0	0	0

50	39	31	49	55	40	26	8	3	2	1	17	48	36	23	22	15
18	27	34	30	6	19	35	46	9	47	16	21	12	7	24	20	32
54	37	30	8	10	6	18	11	32	40	22	13	1	27	25	38	47
y_2	y_3	y_4	5	49	31	55	17	44	53	33	39	19	26	2	3	45
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$4^{14}10^1$:

0	0	0	0	0	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	32	26	44
3	15	7	25	13	49	23	21	39	5	52	43	40	45	20	36	31	4
29	25	39	30	24	11	25	19	47	29	39	32	52	30	45	27	2	1
36	16	6	3	19	51	10	41	23	49	31	36	6	48	24	x_1	x_2	x_3
x_1	x_2	x_3	x_4	x_5	0	0	0	0	0	0	0	0	0	0	0	0	0

15	12	25	51	1	34	27	50	55	10	53	30	3	6	2	24	17	22
35	18	7	29	33	46	38	13	8	37	54	9	11	16	41	47	19	48
12	8	46	4	55	50	16	5	9	18	38	22	20	33	40	43	7	13
x_4	x_5	y_1	y_2	y_3	y_4	y_5	44	34	17	35	3	26	21	53	54	37	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$4^{15}10^1$:

0	0	0	0	0	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	49	43	53	14
29	17	19	51	27	47	26	54	9	27	25	12	42	41	20	3	22	35	34
5	21	27	34	22	19	28	4	55	33	44	37	54	16	49	40	39	56	23
18	14	38	1	53	7	51	26	21	13	42	5	36	59	58	x_1	x_2	x_3	x_4
x_1	x_2	x_3	x_4	x_5	0	0	0	0	0	0	0	0	0	0	0	0	0	0

44	51	52	4	31	7	6	59	58	32	1	57	33	5	11	2	37	46	24
38	39	16	36	18	8	29	10	23	28	56	19	17	13	21	55	40	48	50
20	2	47	52	6	48	10	32	22	24	43	3	41	18	34	46	8	14	1
x_5	y_1	y_2	y_3	y_4	y_5	9	53	25	29	35	17	31	12	50	11	27	38	57
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$4^{17}20^1$:

0	0	0	0	0	14	x_1	x_2	x_3	x_4	x_5	y_1	y_2
19	45	29	31	15	35	67	20	61	48	13	55	10
1	14	22	37	58	21	63	64	49	28	41	46	31
42	15	35	46	65	24	27	1	5	50	39	16	57
x_1	x_3	x_5	x_7	x_9	0	0	0	0	0	0	0	0

y_3	y_4	y_5	42	33	58	60	32	12	22	7	43	59
44	45	15	56	66	62	2	38	40	30	19	54	9
52	4	26	37	65	23	47	8	53	25	3	18	11
36	29	6	x_6	x_7	x_8	x_9	x_{10}	y_6	y_7	y_8	y_9	y_{10}
0	0	0	0	0	0	0	0	0	0	0	0	0

$4^{19}20^1$:

0	0	0	0	0	75	39	58	x_1	x_2	x_3	x_4	x_5	y_1
35	49	29	15	21	35	69	13	22	60	25	64	8	53
9	6	26	54	3	31	6	11	12	28	73	48	30	41
62	7	31	61	6	45	34	24	14	37	21	4	72	2
x_1	x_3	x_5	x_7	x_9	0	0	0	0	0	0	0	0	0

y_2	y_3	y_4	y_5	49	10	52	36	71	46	16	3	26	42
56	63	65	47	61	43	62	40	1	68	32	20	51	50
67	23	18	74	44	66	7	15	9	59	33	55	27	70
17	5	29	54	x_6	x_7	x_8	x_9	x_{10}	y_6	y_7	y_8	y_9	y_{10}
0	0	0	0	0	0	0	0	0	0	0	0	0	0

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