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On co-regular signed graphs

Shahul Hameed K.

Department of Mathematics
K.M.M. Government Women's College, Kannur-670 004
India
shabrennen@gmail.com

Viji Paul

Department of Mathematics
WMO Arts & Science College, Muttil, Kerala
India
vijipaul10@gmail.com

K.A. Germina

Department of Mathematics, Mary Matha Arts and Science College
Wayanad, Kerala
India
srgerminaka@gmail.com

Abstract

A graph whose edges are labeled either as positive or negative is called a signed graph. A signed graph is said to be net-regular if every vertex has constant net-degree k, namely, the difference between the number of positive and negative edges incident with a vertex. In this paper, we analyze some properties of co-regular signed graphs which are net-regular signed graphs with the underlying graphs being regular. An r-regular graph G is said to be co-regularizable with the co-regularity pair (r, k) if we can find a signed graph Σ of net-degree k with the underlying graph G. $\mathrm{Net}(G)$ is the set of all possible values of net-degree k, such that G is co-regularizable with the co-regularity pair (r, k); we find $\mathrm{Net}(G)$ for some special classes of graphs. Also we establish a connection between co-regularizability of G and its graph factors. An algorithm for producing a co-regular signed Harary graph, of which the co-regular signed graph on the complete graph K_n is a particular example, is included. We also establish the fact that every signed graph can be embedded as an induced subgraph of a net-regular or co-regular signed graph.

1 Introduction

All graphs in this article are simple. For all definitions in (unsigned) graph theory used here, unless otherwise mentioned, reader may refer to [3]. Signed graphs (also called sigraphs) are graphs with positive or negative labels on the edges. Formally, a sigraph is an ordered pair $\Sigma = (G, \sigma)$ where $G = (V, E)$ is a graph called the *underlying graph* of Σ and $\sigma : E \to \{+1, -1\}$ called a *signing*, is a function (also called a *signature*) from the edge set E of G into the set $\{+1, -1\}$.

A signed graph is *all-positive* (respectively, *all-negative*) if all of its edges are positive (negative); further, it is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise. The notation $+G$ denotes an all-positive graph, and $+K_n$ an all-positive complete graph. Similarly, $-K_n$ represents an all-negative complete graph. Note that a graph can be considered to be a homogeneous signed graph. A signed graph Σ is said to be *balanced* or *cycle balanced* if all of its cycles are positive, where the sign of a cycle in a signed graph is the product of the signs of its edges.

We denote by $P_n^{(r)}xp$, where $0 \leq r \leq n-1$, signed paths of order n and size $n-1$ with r negative edges where the underlying graph is the path P_n. Also $C_n^{(r)}$, for $0 \leq r \leq n$, denotes signed cycles with r negative edges. The *net-degree* $d_\Sigma^\pm(v)$ of a vertex v of a signed graph Σ is defined as $d_\Sigma^\pm(v) = d_\Sigma^+(v) - d_\Sigma^-(v)$, where $d_\Sigma^+(v)$ and $d_\Sigma^-(v)$ denote, respectively, the number of positive edges and the number of negative edges incident with v. If no confusion arises, we may omit the suffix and write these as $d^+(v)$ and $d^-(v)$. Also, as usual, $d(v)$ denotes the total number of edges incident at v and of course $d(v) = d^+(v) + d^-(v)$. A signed graph Σ is called *net-regular* if every vertex has the same net-degree, and in that case we write the common value of the net-degree as $d^\pm(\Sigma)$. We define a signed graph $\Sigma = (G, \sigma)$ to be *co-regular* if the underlying graph G is r-regular for some positive integer r and Σ is net-regular with net-degree k for some integer k. In this case we also define the *co-regularity pair* to be the ordered pair (r, k). For example, the alternately signed cycle $C_{2n}^{(n)}$ is a co-regular signed graph with co-regularity pair $(2, 0)$.

The objective of this paper is to deal with some classes of co-regular signed graphs and their properties. We also establish the fact that every signed graph can be embedded as an induced subgraph of a net-regular or co-regular signed graph.

2 Basic results on co-regular signed graphs

Lemma 2.1. *If $\Sigma = (G, \sigma)$ is a co-regular signed graph with co-regularity pair (r, k), then the following assertions hold:*

(i) $d^+(v) = \frac{1}{2}(r+k)$ and $d^-(v) = \frac{1}{2}(r-k)$ for every vertex v of Σ.

(ii) If $r = k$, then Σ is all-positive and if $k = -r$, then Σ is all-negative.

(iii) For a fixed r, k satisfies the inequality $-r \leq k \leq r$.

(iv) The degree r and the net-degree k must be of the same parity.

Proof. (i) From the definitions, we have $d^+(v)+d^-(v) = r$ and $d^+(v)-d^-(v) = k$; solving these, we get $d^+(v) = \frac{1}{2}(r+k)$ and $d^-(v) = \frac{1}{2}(r-k)$.

(ii) This follows from (i).

(iii) We have $d^+(v) \geq 0$ and $d^-(v) \geq 0$. These imply, using (i), that $-r \leq k \leq r$.

(iv) Again, from (i), we have $r+k \equiv 0 \pmod 2$ and $r-k \equiv 0 \pmod 2$, which imply that r and k must be of the same parity.

□

Corollary 2.2. *The signed cycle $C_n^{(r_1)}$ for a fixed r_1 is co-regular if and only if*

(i) $r_1 = 0$, that is, when it is an all-positive cycle $+C_n$;

(ii) $r_1 = n$, which makes it an all-negative cycle $-C_n$; and

(iii) $n = 2m$, $r_1 = m$, or $d^{\pm}(C_{2m}^{(m)}) = 0$.

Proof. Since the underlying cycle is 2-regular, the net-degree of each vertex must be either -2, 0 or 2 in view of Lemma 2.1. Thus the only possible co-regularity pairs are $(2,-2)$, in which case we get an all-negative cycle, the pair $(2,2)$ leading to an all-positive cycle and $(2,0)$ leading to the case (iii). □

A forest has trees as its components, and if it is not totally disconnected, since each tree component definitely has two vertices of degree 1, we have the following immediate result where $n(+K_2)$ denotes a forest with each component being $+K_2$. The case with notation $n(-K_2)$ is similar.

Corollary 2.3. *A signed forest F is co-regular if and only if it is either a totally disconnected graph $F = n(+K_2)$ or $F = n(-K_2)$ for some n. Thus the only co-regular signed paths are K_1, $+K_2$ and $-K_2$ with respective co-regularity pairs $(0,0)$, $(1,1)$ and $(1,-1)$.*

3 Net-regularity index and Net(G) of a regular graph G

From the definition of the co-regularity of a signed graph $\Sigma = (G,\sigma)$, it is essential that the underlying graph G is regular of degree r. Given a positive integer r, the all-positive complete graph $+K_{r+1}$ is a co-regular signed graph with the co-regularity pair (r,r). For the same r, the all-negative complete graph $-K_{r+1}$ is another co-regular signed graph with the pair $(r,-r)$. In a similar way, given an integer k, the all-negative complete graph $-K_{|k|+1}$ is a co-regular graph with the co-regularity pair $(|k|,k)$ if k is negative and the all-positive complete graph $+K_{k+1}$ with co-regularity pair (k,k) is another co-regular signed graph when k is positive. Now we define a *co-regularizable graph* as an r-regular graph G such that there exists a net-regular signed graph $\Sigma = (G, \sigma)$ with net-degree k belonging to the closed interval $[-r, r]$. The set of all such possible values of $k \in [-r, r] \cap \mathbb{Z}$ is denoted by $\mathrm{Net}(G)$ and we call this the *net-regularity index set* of G. We also call a $k \in \mathrm{Net}(G)$ a *net-regularity index* of G. Indeed, $|\mathrm{Net}(G)|$ gives an idea about the possible number of net-regular signed graphs with the underlying regular graph G. From Corollary 2.2, if the cycle is alternately signed, $\mathrm{Net}(C_{2n}) = \{-2, 0, 2\}$ and $\mathrm{Net}(C_{2n+1}) = \{-2, 2\}$. We call an r-regular graph G a *completely co-regularizable graph* if $|\mathrm{Net}(G)| = r + 1$, that is, when G is co-regularizable for all net-regularity indices $k \in [-r, r]$.

We list in Fig. 1 some of the non-isomorphic co-regular or net-regular signed graphs for at most 5 vertices. An interesting question is whether there exist net-regular signed graphs whose underlying graphs are not regular. In an attempt to answer this question, in Fig. 1 we give disconnected signed graphs for $n = 5$ in column (8) and a connected signed graph in column (9) for $n = 5$.

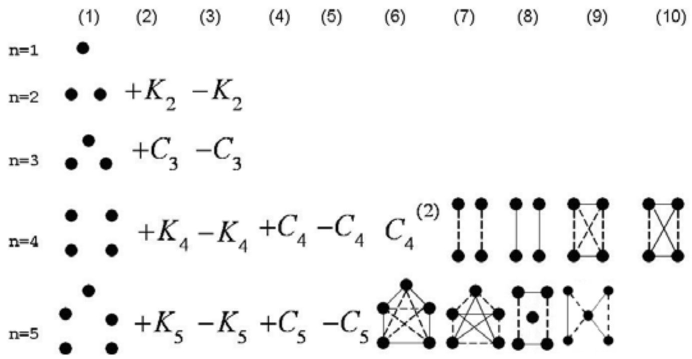

Figure 1: Net-regular signed graphs with at most five vertices some of which are co-regular

Now we establish that co-regularizability and graph factors have strong connections.

Lemma 3.1. *An $r-$regular graph G is co-regularizable with co-regularity pair (r, k) if and only if it has an $\frac{1}{2}(r-k)-$factor.*

Proof. If $\Sigma = (G, \sigma)$ is a co-regular signed graph with the co-regularity pair (r, k), by Lemma 2.1, $d^+(v) = \frac{1}{2}(r+k)$ and $d^-(v) = \frac{1}{2}(r-k)$ for every vertex v of Σ. Then, the subgraph of G induced by the positive edges form an $\frac{1}{2}(r+k)$-factor of G and the subgraph induced by the negative edges form an $\frac{1}{2}(r-k)$-factor of G. Conversely, let G has an $\frac{1}{2}(r-k)$-factor, say, F. Since G is r-regular, the edges of G, not belonging to (the $\frac{1}{2}(r-k)$-factor) F form an $\frac{1}{2}(r+k)$-factor of G. Define $\sigma : E(G) \to \{1, -1\}$ by

$$\sigma(e) = \begin{cases} -1, & \text{if } e \in F \\ 1, & \text{if } e \in E(G) \setminus F \end{cases}$$

Then $\Sigma = (G, \sigma)$ is a co-regular signed graph with the co-regularity pair (r, k). $\square$

Remark 3.2. It is evident from the proof of the above theorem that once an r-regular graph has an $\frac{1}{2}(r-k)$-factor, then it possesses an $\frac{1}{2}(r+k)$-factor as well, and vice-versa.

The following theorem found in [4] is quite useful for the discussion that follows.

Lemma 3.3 ([4]). *Let a, b, c be odd integers such that $1 \leq a \leq c \leq b$. If a graph G has both an a-factor and b-factor, then G has a c-factor. In particular, if an r-regular graph H has a 1-factor, then for every integer h, $1 \leq h \leq r$, H has an h-factor.*

Theorem 3.4. *An r-regular graph G is completely co-regularizable if and only if G has a 1-factor.*

Proof. Since G has a 1-factor, from Lemma 3.3 it has an h-factor for all $h = 0, 1, \ldots, r$. Therefore, using Lemma 3.1, $\mathrm{Net}(G)$ contains $k \in [-r, r] \cap \mathbb{Z}$ such that G has an $\frac{1}{2}(r-k)$-factor. Taking $h = \frac{1}{2}(r-k)$ we have $k = r - 2h$ for $h = 0, 1, \ldots, r$. Hence $\mathrm{Net}(G) = \{r - 2h : h = 0, 1, \ldots, r\}$ and $|\mathrm{Net}(G)| = r + 1xp$, i showing that G is completely co-regularizable.

Conversely, supposing that G has all admissible $\frac{1}{2}(r-k)$-factors, taking $k = r-2$, it has a 1-factor. □

Lemma 3.5 ([3]). *A graph G is 2-factorable if and only if G is r-regular for some positive even integer r.*

Theorem 3.6. *If G is an r-regular graph of odd order then* $\mathrm{Net}(G) = \{r - 4j : j = 0, 1, \ldots, r/2\}$ *and* $|\mathrm{Net}(G)| = \frac{r+2}{2}$.

Proof. Since G is of odd order, r is definitely even and hence 2-factorable, by Lemma 3.5. Thus G has a l-factor for all $l = 0, 2, \ldots, r$. Therefore, using Lemma 3.1, $\mathrm{Net}(G)$ contains $k \in [-r, r] \cap \mathbb{Z}$ such that G has an $\frac{1}{2}(r-k)$-factor. Taking $l = 2j = \frac{1}{2}(r-k)$ where $j = 0, 1, 2, \ldots, r/2$, we have $k = r - 4j$ for $j = 0, 1, \ldots, r/2$. Moreover, the order of the graph being odd means that it has no odd factors. Hence $\mathrm{Net}(G) = \{r - 4j : j = 0, 1, \ldots, r/2\}$ and $\mathrm{Net}(G) = r/2 + 1 = \frac{r+2}{2}$. □

Lemma 3.7 ([3]). *The complete graph K_n is 1-factorable when n is even and K_n is 2-factorable when n is odd.*

Corollary 3.8. *K_n is completely co-regularizable when n is even.*

$$\mathrm{Net}(K_n) = \begin{cases} \{k = n - 1 - 2i : i = 0, 1, 2, \ldots, n-1\}, \textit{if } n \equiv 0 (\textit{mod } 2) \\ \{k = n - 1 - 4j : j = 0, 1, 2, \ldots, \frac{n-1}{2}\}, \textit{if } n \equiv 1 (\textit{mod } 2) \end{cases}$$

and

$$|\mathrm{Net}(K_n)| = \begin{cases} \frac{n+1}{2}, \textit{if } n \equiv 1 \pmod 2 \\ n, \textit{if } n \equiv 0 \pmod 2 \end{cases}$$

Proof. The proof follows from Lemma 3.7, Theorem 3.4 and Theorem 3.6. □

Although there are only finitely many theoretically possible net-regular signed graphs for a given a complete graph K_n, actually finding and constructing these net-regular signed graphs is difficult in practice. In this regard we deal with Harary graphs, of which K_n is only a particular case, and the discussion of which is postponed until Section 5.

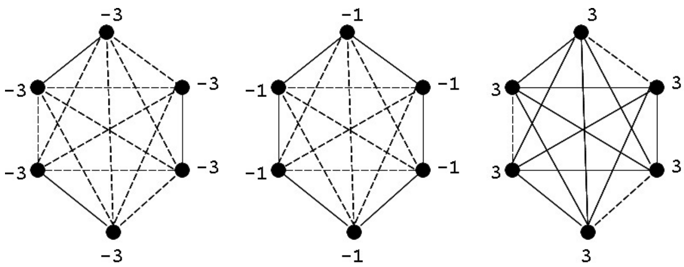

Figure 2: Co-regular signed graphs of various net-degrees on K_6

4 Embedding a Signed Graph in a Net-regular or a Co-regular Signed Graph

It is one of the fundamental results in graph theory that every graph G can be embedded in an r-regular graph as an induced subgraph when $r \geq \Delta(G)$. We extend this result to signed graphs in the following way. Theorem 4.1 and its proof contain new symbols with the following meanings attached to them.

For a signed graph Σ, $\Delta_\Sigma^+ = \max\{d^+(v) : v \in V(\Sigma)\}, \Delta_\Sigma^- = \max\{d^-(v) : v \in V(\Sigma)\}, \delta_\Sigma^+ = \min\{d^+(v) : v \in V(\Sigma)\}$ and $\delta_\Sigma^- = \min\{d^-(v) : v \in V(\Sigma)\}$

Theorem 4.1. *Every signed graph Σ can be embedded in a net-regular signed graph as an induced subgraph.*

Proof. In the following, we say attaching $+K_2$ to a vertex means adding a single new vertex to the graph and connecting it to a single vertex using a positively signed edge. A similar meaning is implied if we say we attach a negative edge or $-K_2$ to a vertex. If $d_\Sigma^{\pm}(v) <= 0$, for some vertices v, we attach as many $+K_2$ as required so as to make the new net-degree of that vertex be 1. If a vertex in the original signed graph has a positive net-degree, we add negative edges first to make it have net-degree 1 and then add two positive edges to each of these new vertices. After this process, we see that Σ is embedded in a net-regular signed graph of index 1. $\square$

Theorem 4.2. *Every signed graph Σ can be embedded in an (r,k)-co-regular signed graph for all $r \geq \Delta_\Sigma^+ + \Delta_\Sigma^-$ and for all k with the same parity as that of r satisfying $2\Delta_\Sigma^+ - r \leq k \leq r - 2\Delta_\Sigma^-$.*

Proof. A signed graph Σ is (r,k)-co-regular if and only if $\Delta_\Sigma^+ = \delta_\Sigma^+ = \frac{1}{2}(r+k)$ and $\Delta_\Sigma^- = \delta_\Sigma^- = \frac{1}{2}(r-k)$. Suppose $r \geq \Delta_\Sigma^+ + \Delta_\Sigma^-$ and $2\Delta_\Sigma^+ - r \leq k \leq r - 2\Delta_\Sigma^-$ where r and

k are of the same parity. Then, $\Delta^+_\Sigma \leq \frac{1}{2}(r+k)$ and $\Delta^-_\Sigma \leq \frac{1}{2}(r-k)$. We construct an (r,k)-co-regular signed graph Σ^2 containing Σ as an induced subgraph in two steps.

In step 1, we construct a signed graph Σ^1 with $\Delta^+_{\Sigma^1} = \delta^+_{\Sigma^1} = \frac{1}{2}(r+k)$ containing Σ as an induced subgraph. Suppose that Σ has order n and $V(\Sigma) = \{v_1, v_2, \ldots, v_n\}$. If $\Delta^+_\Sigma = \delta^+_\Sigma = \frac{1}{2}(r+k)$, then we let $\Sigma^1 = \Sigma$. Otherwise, let Σ' be another copy of Σ with $V(\Sigma') = \{v'_1, v'_2, \ldots, v'_n\}$, where each vertex v'_i in Σ' corresponds to v_i in Σ for $1 \leq i \leq n$. We now construct a signed graph Σ_1 from Σ and Σ' by adding the edges $v_iv'_i$, for all v_i of Σ having $d^+(v_i) < \frac{1}{2}(r+k)$ and by letting $\sigma(v_iv'_i) = +1$. Then Σ is an induced subgraph of Σ_1 and $\delta^+_{\Sigma_1} = \delta^+_\Sigma + 1$. If $\Delta^+_{\Sigma_1} = \delta^+_{\Sigma_1}$, we let $\Sigma^1 = \Sigma_1$. If not, we continue the process until we get a signed graph Σ_s with $\Delta^+_{\Sigma_s} = \delta^+_{\Sigma_s}$, where $s = \frac{1}{2}(r+k) - \delta^+(\Sigma)$. Then we let $\Sigma^1 = \Sigma_s$.

In step 2, we construct Σ^2 from Σ^1 with $\Delta^-_{\Sigma^2} = \delta^-_{\Sigma^2} = \frac{1}{2}(r-k)$ containing Σ as an induced subgraph by proceeding as described above but by assigning a negative sign to each of the new edges joining the identical vertices of the two copies obtained in the process of construction. The graph Σ^2 is the desired (r,k)-co-regular signed graph. □

In Fig. 3, we provide an illustration of the embedding of a signed graph in a co-regular signed graph.

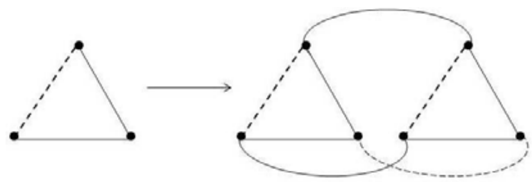

Figure 3: Embedding of a Signed graph in a Co-regular signed graph

5 Co-regular signed Harary graphs

In [2], Chartrand and Ping Zang give a method to construct an $r-$regular graph of order n when one of r or n is even and they call the graph thus obtained as Harary graph, denoted by $H_{r,n}$. The construction of $H_{r,n}$ with the vertex set $\{v_1, v_2, \ldots, v_n\}$ is briefly desribed as follows. If $r = 2s$ for some non-negative integer s, join v_i to $v_{i+1}, v_{i+2}, \ldots v_{i+s}$ and to $v_{i-1}, v_{i-2}, \ldots, v_{i-s}$ where the indices are taken modulo n. If $r = 2s+1$, so that n is even, say $n = 2t$, then we join v_i to the $2s$ vertices described above and then to the vertex v_{i+t}. Let us denote the co-regular signed Harary graph by $H_{r,k,n}$ with the co-regularity pair (r,k) and of order n. For an illustration see figure Fig. 4. Note that $H_{n-1,n} = K_n$. Now we shall find $\mathrm{Net}(H_{r,n})$.

Theorem 5.1. *The Harary graph $H_{r,n}$ is completely co-regularizable if and only if n*

is even.

$$\mathrm{Net}(H_{r,n}) = \begin{cases} \{k = r - 2i : i = 0, 1, 2, \ldots, r\}, \textit{if } n \equiv 0 \pmod 2 \\ \{k = r - 4j : j = 0, 1, 2, \ldots, \frac{r}{2}\}, \textit{if } n \equiv 1 \pmod 2 \end{cases}$$

and

$$|\,\mathrm{Net}(H_{r,n})| = \begin{cases} \frac{r+2}{2}, \textit{if } n \equiv 1 \pmod 2 \\ r + 1, \textit{if } n \equiv 0 \pmod 2 \end{cases}$$

Proof. Suppose first that n is even. If $r = 1$, $H_{1,n} = \frac{n}{2}K_2$ and hence completely co-regularizable. If $r \geq 2$, by the construction of $H_{r,n}$, it is Hamiltonian and the alternative edges of a Hamiltonian cycle constitute a $1-$factor of $H_{r,n}$ and thus by Theorem 3.4, it is completely co-regularizable.
Now suppose that n is odd. Proof of this case follows from Theorem 3.6. $\square$

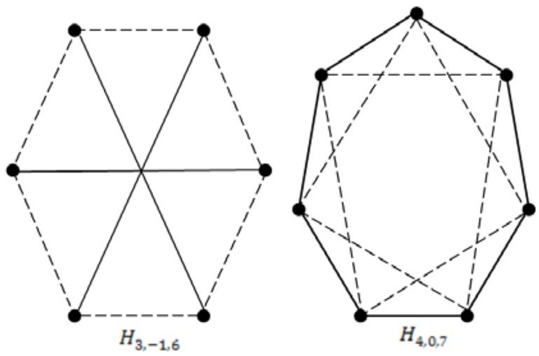

Figure 4: Co-regular signed Harary graphs

The following is an algorithm that produces $H_{r,k,n}$ for a given Harary graph $H_{r,n}$. The style and syntax adopted in the algorithm can be found in [1].

Algorithm 5.2. The following procedure finds out the adjacency matrix of $H_{r,k,n}$ for an admissible k when the input is the adjacency matrix $A = (a_{ij})$ of the Harary graph $H_{r,n}$.
Procedure HARARYCO-REGULAR(A, r, k, n)
(1) $dp \leftarrow \frac{1}{2}(r + k)$ and $dm \leftarrow \frac{1}{2}(r - k)$
(2) **if** $dm \equiv 2 \pmod 0$ **then**
(3) $s \leftarrow dm/2$
(4) **for** $i = 1$ to n **do**
(5) **for** $j = i + 1$ to $i + s$ **do**
(6) $a(i, j) \leftarrow -a(i, j)$
(7) $a(j, i) \leftarrow a(i, j)$
(8) **od**
(9) **od**
(10) **for** $p = 1$ to s **do**

```
    for q = n - s + p to n do
        a(q, p) <- -a(q, p)
        a(p, q) <- a(q, p)
        od
od
   fi
if dm ≡ 2(mod 1) then
s <- (dm - 1)/2
for i = 1 to n - 1 in steps of 2 do
    for j = i + 1 to n in steps of 2 do
        a(i, j) <- -a(i, j)
        a(j, i) <- a(i, j)
        od
od
for i = 1 to n do
    for j = i + 2 to i + s + 1 do
        a(i, j) <- -a(i, j)
        a(j, i) <- a(i, j)
        od
od
for i = 1 to s + 1 do
    for j = i + n - s to i + n - 1 do
if i ≠ s + 1 then
        a(i, j) <- -a(i, j)
        a(j, i) <- a(i, j)
        fi
if i = s + 1 and  j = n then
        a(i, j) <- -a(i, j)
        a(j, i) <- a(i, j)
      exit the loop
        fi
      od
od
```

6 Further direction

Although we have dealt with the net-regularity indices of a regular graph G in some detail, it is natural to investigate further how many non-isomorphic co-regular signed graphs are possible with a particular admissible net-regularity index k, and to enumerate such isomorphic and non-isomorphic co-regular signed graphs built on G. Also it would be worthwhile to find how many of them are balanced or unbalanced.

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