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A new parameter on resolving sets with a realizable triple

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Abstract

A set S of vertices in a graph G is a *resolving set* if for every pair of vertices $u, v \in V(G)$ there is a vertex $x \in S$ such that the distances $d(x, v) \neq d(x, u)$. We define a new parameter $\underline{\text{res}}(G)$, the size of the smallest subset S of $V(G)$ that is not a resolving set but every superset of S resolves G. We also demonstrate that for every triple (a, b, c), $a \leq (b+1) \leq c$, there is a graph G in which a is the metric dimension of G, $b = \underline{\text{res}}(G)$, and c is the resolving number.

1 Introduction

Throughout this paper we will consider all graphs to be simple, undirected, and connected unless otherwise noted. For a graph G we denote its set of vertices by $V(G)$. The distance between two vertices $u, v \in V(G)$ is the length of the shortest path between them, denoted $d(u, v)$. The order of a graph $|V(G)|$ is shortened to $n(G)$. The complement of a set S is denoted $\bar{S}$. Given a set $S \subseteq V(G)$ and vertex $u \in \bar{S}$, the set $S \cup \{u\}$ is shortened to S^u. We will also use standard notation from [14].

Recently the resolving number has been a topic of interest ([4], [5]). This extremal parameter can be compared to the *extremal number* of a graph G, introduced by Mantel [9] and again by Turán [13]. A graph H is *G-saturated* if it contains no subgraph isomorphic to G but the addition of any edge to H results in such a subgraph. The extremal number of G is the least number of edges on a fixed number of vertices that ensures a subgraph isomorphic to G . Therefore, we can think of the extremal number as one greater than the largest G-saturated graph. Subsequently, Erdős, Hajnal, and Moon [3] defined the *saturation number* of a graph to be the size of the smallest G-saturated graph on a particular set of vertices. We extend the comparison with the following definition.

On a path graph the distance of a vertex v to one specific endpoint u uniquely determines v. However, it is not the case in general that distance from single vertex is sufficient to locate a vertex, and often multiple vertices are necessary. Although introduced in [10] and [7], we use the following definitions from [2].

Definition 1.1. *Given a graph G, a subset of vertices $W = \{w_1, w_2, \ldots, w_k\}$, and a vertex u, the* distance vector *is $r(u|W) = (d(u, w_1), d(u, w_2), \ldots, d(u, w_k))$. If $r(u_i|W)$ is unique for all u_i in a set U, then the set W* resolves *the set U. The set W is a* resolving set *for, and* resolves*, the graph G if W resolves $V(G)$.*

They further define a *metric basis* (respectively *upper basis*) as a smallest (largest) minimal resolving set for G, with cardinality called *metric dimension* (*upper dimension*) and denoted $\dim(G)$ ($\dim^+(G)$). Beyond these two extremal parameters, a third has been studied.

Definition 1.2. *The* resolving number *$res(G)$ of a connected graph G is the smallest integer k such that every subset $S \subseteq V(G)$ of size k is a resolving set for G.*

As an example, the path graph P_n of order $n > 3$ has upper dimension and resolving number 2, and the family of path graphs comprises all graphs with metric dimension 1 ([10]).

Definition 1.3. *A set $S \subseteq V(G)$ that does not resolve G is a* nearly resolving set *if S^u resolves G for every vertex $u \in \bar{S}$. The size of a smallest nearly resolving set is the* nearly resolving number*, denoted $\underline{res}(G)$.*

Other work on extremal properties of resolving sets can be found in [6], in which the authors determine the maximum order of a graph with fixed metric dimension and diameter. We end this section with some obvious bounds on the nearly resolving number.

Remark 1. *If G is a connected graph, then $dim(G) \leq (\underline{res}(G) + 1) \leq res(G)$.*

Proof. If $\underline{\mathrm{res}}(G) = k$, then there is a resolving set S of G of size $k+1$. Since $\dim(G)$ is the smallest resolving set of G, $\dim(G) \leq k+1$. Similarly, since every set of $\mathrm{res}(G)$ vertices is a resolving set, k must be strictly smaller than $\mathrm{res}(G)$. □

We outline some properties of nearly resolving sets and calculate some values in Section 2. As a particular case, grid graphs are addressed in Section 3. Finally, a realizable triple is established in Section 4 and the nearly resolving number is compared to the upper dimension.

2 Properties of nearly resolving sets

2.1 Relation to the resolving number

Garijo, Gonzalez, and Márquez ([4]) prove that no integer $a \geq 4$ is realizable as the resolving number of an infinite family of graphs. Their argument uses the order and diameter of graphs with a resolving number a. We begin this section by addressing whether this holds for all nearly resolving sets.

Theorem 2.1. *For any integer $a \geq 1$ there is an infinite family of graphs $\mathcal{G}$ with $\underline{\text{res}}(G) = a$ for all $G \in \mathcal{G}$.*

Proof. Let $Br(x, y), x > 1, y > 0$ be the *broom graph* of length x with y bristles. That is, $Br(x, y)$ consists of a path $\{v_1, \dots, v_x\}$ with v_1 adjacent to the vertices $\{u_1, \dots, u_y\}$. Given $a \geq 1$, we claim that $\underline{\text{res}}(Br(x, a)) = a$ for all $x > 1$.

First, note that when $y = 1$ the result is simply an infinite family of paths, each with nearly resolving number 1. This is clear since any vertex that is not an endpoint fails to resolve its neighbors, but any pair of vertices resolves the path. When $y > 1$ the set $S = \{u_i\}_{i=1}^{a-1} \cup \{v_1\}$, representing all bristles but one along with their shared neighbor, does not resolve the set $\{v_0, v_2\}$ and is therefore not a resolving set of G. However, all other pairs of vertices are resolved by S. Consider the sets S^w for any $w \in \{v_0, v_2, \dots, v_x\}$. Regardless of the choice of w, the set S^w resolves the set $\{v_0, v_2\}$, and therefore S^w is a resolving set of G. So, S nearly resolves G. Note that any resolving set of the graph must include at least $(a-1)$ bristles, and if it does not include all a of them, then it must also include at least one vertex from $\{v_2, \dots, v_x\}$. Therefore, any set smaller than S has a proper superset that is not a resolving set. Hence S is a smallest nearly resolving set for the graph.

Since $|S| = a$ and x is arbitrary, we have found an infinite family of graphs with nearly resolving number a. □

Since one less than the resolving number of a graph G indicates the largest set of vertices that is not a resolving set of G, and is therefore the largest nearly resolving set, it is natural to consider graphs for which $\underline{\text{res}}(G) = \text{res}(G) - 1$. The *randomly k-dimensional graphs*, those graphs for which $\dim(G) = \text{res}(G) = k$ for some integer k, also have this property, (see [8] for more on these graphs). This is because any graph G with $\underline{\text{res}}(G) < \text{res}(G) - 1$ contains a set of vertices S of size $\underline{\text{res}}(G)$ such that any superset S^v resolves G. Since $|S^v| = \underline{\text{res}}(G) + 1$, we cannot have $\underline{\text{res}}(G) < \dim(G) - 1 = \text{res}(G) - 1$. So, all randomly k-dimensional graphs have resolving number precisely one greater than their nearly resolving number. In [4] this family was shown to consist entirely of the complete graphs K_n and odd cycles C_{2k+1}.

There are, in fact, graphs that are not randomly k-dimensional with equal resolving number and nearly resolving number. This family include even cycles, which have metric dimension 2 but resolving number 3. In order to determine more members of this family of graphs, we require a theorem regarding blocks and cut vertices.

Definition 2.2. *A vertex v in a connected graph G is a* cut vertex *if its removal results in at least two connected components. A graph is* biconnected *if it contains no cut vertex, and a* block *in G is a maximal biconnected subgraph of G.*

Theorem 2.3. *If the graph G contains a cut vertex v and at least one non-path block (that is, a block not isomorphic to a path) containing v, then the set of vertices in all* *connected components of $G \setminus \{v\}$ other than that containing this block, along with v itself, does not resolve the graph G.*

Proof. Consider such a graph G. Let S be v along with all vertices in the components of $G \setminus \{v\}$ other than the component containing the block. Since this component contains only one vertex from S, and its metric dimension is at least 2, there must be a pair of vertices in G that is not resolved by S. So, S is not a resolving set of G. □

This leads us to the following corollary.

Corollary 2.4. *Let G be a graph with cut vertices and at least 2 non-path blocks. Denote by B_0 a smallest non-path block, and B_1 a largest. Then, $res(G) > n(G) - n(B_0) + 1$, and $\underline{res}(G) \geq n(G) - n(B_1) + 1$.*

Proof. From Theorem 2.3 any resolving set includes non-cut vertices from all non-path blocks. The set consisting of all vertices in G except for those in B_0, but including a cut vertex of G in B_0, is not sufficient to resolve all vertices in G, and hence $\mathrm{res}(G) > n(G) - n(B_0) + 1$. Similarly, any nearly resolving set of G must not only contain non-cut vertices from all non-path blocks, but the addition of any new vertex is sufficient to resolve G. Therefore, at most one block can be excluded from any nearly resolving set. Hence, $\underline{\mathrm{res}}(G) \geq n(G) - n(B_1) + 1$. □

We continue by examining graphs with cut vertices that join a larger number of connected components.

Theorem 2.5. *If there is a vertex v in a connected graph G such that the graph $G \setminus \{v\}$ has at least 3 connected components $\{G_1, G_2, G_3, \ldots\}$, then $res(G) - 1 \geq n(G) - n(G_i \cup G_j)$, where G_i and G_j are the two smallest connected components in $G \setminus \{v\}$. Also, $\underline{res}(G) \geq n(G) - n(G_k \cup G_m)$, where G_k and G_m are the largest two components.*

Proof. Let v be a cut vertex of G such that $G \setminus \{v\}$ has at least 3 connected components. Let S be the set of vertices from all but precisely two of these components, say G_a and G_b, along with the vertex v. If the subgraph of G induced by v and the vertices in G_a and G_b is a path, then it is not resolved by S since only one vertex in the path is in S and it is neither terminal vertex. If this induced subgraph is not a path, then it has metric dimension greater than 1 and is therefore not resolved by the set S. Because $\underline{\mathrm{res}}(G)$ and $\mathrm{res}(G) - 1$ are the sizes of the least and greatest nearly resolving sets of G, respectively, we can choose which connected components contribute vertices to S based on their orders. □

2.2 The nearly resolving numbers of some graph families

As mentioned in Section 2.1, the randomly k-dimensional graphs K_n and C_{2k+1} have nearly resolving numbers $n-2$ and 1, respectively. In this section we determine the nearly resolving numbers of some other families and characterize all graphs with nearly resolving number 1.

Proposition 2.6. *The nearly resolving numbers of paths and even cycles are* $\underline{\mathrm{res}}(P_n) = 1$*,* $\underline{\mathrm{res}}(C_{2k}) = 2$ *for* $n \geq 3$*,* $k \geq 3$*.*

Proof. Although $\dim(P_n) = 1$ ([10]), not every singleton subset of the vertices of P_n resolves the graph. However, every non-terminal vertex of the path is a nearly resolving set, since the addition of any vertex to a singleton set is a set of size 2, and $\mathrm{res}(P_n) = 2$. On an even cycle $(v_0, v_1, \dots v_{2k-1})$, let $\{v_i\}$ be a set consisting of any one vertex. Since the set $\{v_i, v_{i+k}\}$, with addition modulo $(2k)$, does not resolve the graph, we know that $\underline{\mathrm{res}}(C_{2k}) > 1$. Let $S = \{v_0, v_k\}$. Although S does not resolve the graph, any superset does. Therefore, $\underline{\mathrm{res}}(P_n) = 1$ and $\underline{\mathrm{res}}(C_{2k}) = 2$. □

Next we turn our attention to the wheel graph on n vertices, W_n. As a family of arbitrarily large graphs with diameter 2 they make for an interesting subject in the study of resolving sets. First we identify and reprove certain required and forbidden structures of resolving sets on wheel graphs, first identified in [1].

Theorem 2.7 (Buczkowski, Chartrand, Poisson, Zhang, 2003)**.** *Let* W_n *be the wheel graph with* $(n-1)$*-cycle* $(v_0, v_1, \dots v_{n-2})$ *and center vertex* u*. If* S *is a resolving set for* W_n *and* $T = V(W_n) \setminus S$*, then the following structures are not present. All arithmetic is modulo* $(n-1)$*.*

1. *Two disjoint sets of the form* $\{v_i, v_{i+1}, v_{i+2}\} \subseteq T$*.*
2. *A set of the form* $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \subseteq T$*.*
3. *A set of the form* $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\} \subseteq T$ *with* $v_i \in S$*.*

Proof. Let us consider these structures in turn. Firstly, if $\{v_i, v_{i+1}, v_{i+2}\}$, $\{v_j, v_{j+1}, v_{j+2}\} \subseteq T$ are disjoint sets, then the vertices v_{i+1} and v_{j+1} are not disambiguated by S. This is because every vertex in S is distance 2 from both vertices, except u if it is in S, and it is adjacent to both vertices. So, S does not disambiguate this pair. Similarly, if $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \subseteq T$, then no vertex in S, except for possibly u, has distance other than 2 to v_{i+1} and v_{i+2}. Finally, if $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\} \subseteq T$, $v_i \in S$, then the pair $\{v_{i-1}, v_{i+1}\}$ is not disambiguated by S. □

We restate the following theorem from [1] and [11].

Theorem 2.8 (Sooryanarayana, Shanmukha, 2002; Buczkowski, Chartrand, Poisson, Zhang, 2003)**.** *The wheel* W_n *has metric dimension* 3 *when* $n = 4, 7$ *and* $\dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ *otherwise.*

Theorem 2.7 is now used to determine the remaining relevant parameters.

Theorem 2.9. *For all $n \geq 7$, the resolving number of the wheel W_n is $(n-3)$. The nearly resolving number is $(n-4)$ when $7 \leq n \leq 10$, and is $(n-6)$ for $n \geq 11$.*

Proof. A largest set of vertices in $W_n, n \geq 6$ that excludes a forbidden structure from Theorem 2.7 is $V(W_n) \setminus \{v_0, v_1, v_2, v_3\}$. This gives us that $\text{res}(W_n) = (n-3)$. Any nearly resolving set must exclude exactly one forbidden structure from Theorem 2.7, and include every remaining vertex in the wheel. Otherwise, the addition of any unaccounted for vertex to the set may not obviate the forbidden structure. This includes the center vertex u which, although not especially useful in disambiguating pairs of vertices in W_n, must be included in any nearly resolving set lest it be included in a superset that does not resolve the graph. When $7 \leq n \leq 10$ only the forbidden structures that include 4 vertices from the set T are possible, and hence $\underline{\text{res}}(W_n) = (n-4)$ and is realized by the set $S = V(W_n) \setminus \{v_0, v_1, v_2, v_3\}$. When $n \geq 11$ the value $\underline{\text{res}}(W_n) = (n-6)$ is realized by $S = V(W_n) \setminus \{v_0, v_1, v_2, v_5, v_6, v_7\}$. Note that in either case, the addition of any vertex to the set eliminates the forbidden structure and yields a resolving set for W_n. □

For smaller wheels we simply employ case analysis.

Since $W_4 = K_4$ we know from Section 2.1 that $\underline{\text{res}}(W_4) = 2$. The set S consisting of the center vertex and one vertex from the outer cycle of W_6 does not resolve the pair of vertices at distance two from the latter, and since any set consisting of the center vertex and two vertices on the outer cycle resolves the graph it is evident that the set S is a nearly resolving set. Similarly, the center vertex and one vertex v on the outer cycle of W_7 fail to resolve any pair of vertices at distance 2 from v, and the singleton set $\{v\}$ has a proper superset consisting of a pair of opposite vertices on the outer cycle. This set of size 2 is also not a resolving set of W_7, and so a smallest resolving set consists of a pair of vertices adjacent on the outer cycle.

Consider now any set S of 2 vertices in W_5. If S contains the center vertex, then no superset of size 2 resolves the pair of remaining shared neighbors along the cycle. If instead S contains two opposite vertices on the outer cycle, then the same superset arises. Finally, any pair of adjacent vertices along the outer cycle is itself a resolving set. So, $\underline{\text{res}}(W_5) > 2$, and it is easy to see that any size 3 set of vertices that does not resolve W_5 is a nearly resolving set.

Hence, the nearly resolving number of small wheels are as follows: $\underline{\text{res}}(W_4) = \underline{\text{res}}(W_6) = \underline{\text{res}}(W_7) = 2, \underline{\text{res}}(W_5) = 3$.

We now proceed to characterize all connected graphs with nearly resolving number 1. We begin by demonstrating a restriction on the degree of a single vertex that alone resolves a graph G. We then set forth a construction for a family of graphs that have such a vertex. Finally, we demonstrate that this family, along with all path graphs, completely characterizes all such graphs.

Consider a graph G with $|S| = \underline{\text{res}}(G) = 1$ that is neither a path nor an odd cycle, and let $\{v\}$ be a nearly resolving set. Since this implies that $\dim(G) = 2$ we need the following theorem from [12].

Theorem 2.10 (Sudhakara, Kumar, 2009)**.** *If $\{v, u\}$ is a resolving set in G, then neither u nor v has degree greater than* 3*.*

We continue by examining the possible degrees of the vertex v in G. Let u be any other vertex in G.

Lemma 2.11. *The vertex v has degree at most* 2*.*

Proof. First note that by Theorem 2.10 the degree of v is at most 3. Say that v has exactly three neighbors: $\{u_1, u_2, u_3\}$. If these vertices are mutually non-adjacent, then the set $\{v, u_1\}$ does not disambiguate the set $\{u_2, u_3\}$, since v is adjacent to both and u_1 is at distance 2 from each of them. So, without loss of generality we may assume that u_1 is adjacent to u_2. If u_1 is also adjacent to u_3, then we have the same problem, since both v and u_1 are adjacent to both of the other vertices. Therefore, there must be exactly one edge among the neighbors of v. If the vertices $\{v, u_1, u_2\}$ form a triangle and u_3 is not adjacent to either u_1 or u_2, then the set $\{v, u_3\}$ fails to disambiguate $\{u_1, u_2\}$, as both are adjacent to v and at distance 2 from u_3. We therefore have a contradiction and v has at most 2 neighbors. $\square$

The next lemma provides a lower bound for the degree of v.

Lemma 2.12. *The vertex v has degree at least* 2*.*

Proof. Since v has exactly one neighbor u, and G is not a path, then the set $\{v, u\}$ fails to resolve any pair of vertices in G equidistant from u. $\square$

By Lemmas 2.11 and 2.12 the vertex v has degree exactly 2. We now proceed to completely characterize the graphs with nearly resolving number 1. We begin with a construction.

Construction 2.13. *By a* monument graph *we will mean an odd cycle*

$$\{x', x_k, x_{k-1}, \ldots, x_1, v, y_1, \ldots, y_k, y'\}$$

with possible edges of the form $x_i y_i$, and a path that includes the edge $x'y'$, (*Fig. 1*)*.*

Theorem 2.14. *A graph G has a nearly resolving number of* 1 *if and only if G is either a path or a monument graph.*

Proof. As above we let $S = \{v\}$ be a nearly resolving set of a graph G. If G is a tree, then either G is a path or by Theorem 2.5 S cannot be a nearly resolving set of G. So, we may assume that G contains a cycle. If v is not on a cycle and u is a cut vertex adjacent to a non-path block of G, then S^u is not a resolving set of G.

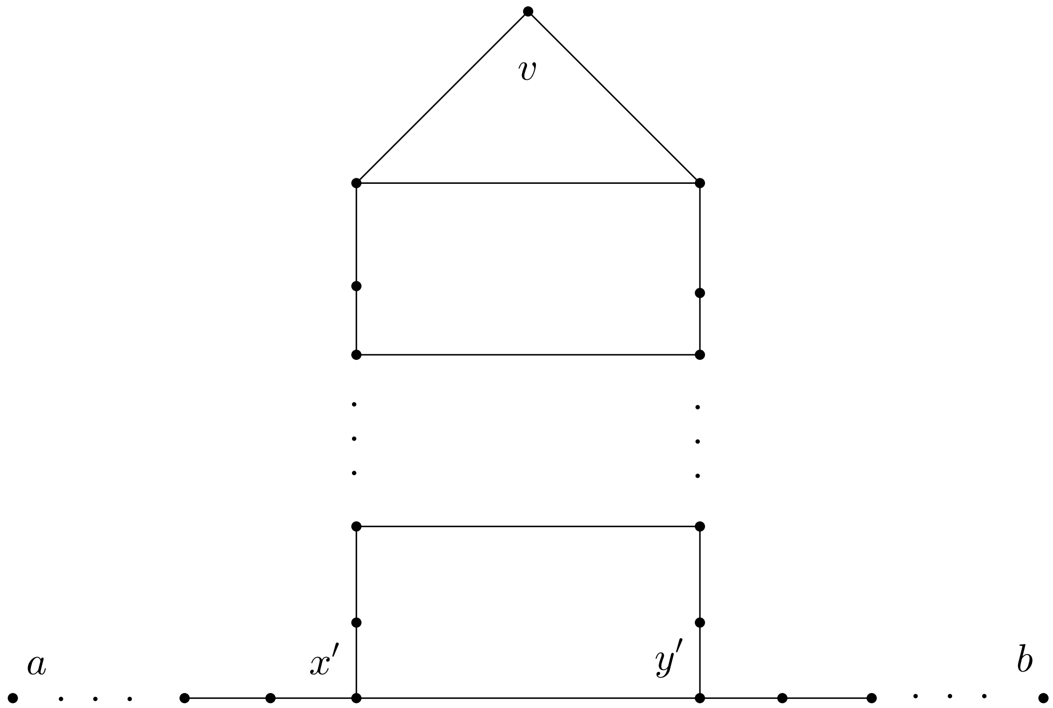

Figure 1: A monument graph from Construction 2.13

Therefore, since $deg(v) = 2$, v is on exactly one chordless cycle, say C. If $n(C)$ is even, then the set S^u, where u is the vertex opposite v in C, is not a resolving set for G. Therefore, v is on an odd chordless cycle. Now consider the vertices x, y at maximum distance along C from v. If any vertex other than x, y has a neighbor in $G \setminus C$, then let w be such a vertex on C closest to v. Let b be the neighbor of w not on C. If a is the vertex on C adjacent to w with distance $d(v, a) = d(v, w) + 1$, then the set S^w fails to resolve $\{a, b\}$. Therefore, no vertex on $C \setminus \{x, y\}$ has a neighbor in G outside of C.

If x (or y) has degree greater than 3, then S^x (S^y) does not disambiguate its neighbors at maximum distance from v. Therefore, the vertices x and y each have at most one neighbor in G not in C. If either vertex is a cut vertex terminating in a non-path tree, then, similarly, any leaf vertex added to S results in a non-resolving set of G. If instead the edge $\{x, y\}$ is on another chordless cycle C', then this cycle must be of even length. Otherwise, the addition of the vertex on the even cycle $C \cup C'$ opposite v does not generate a resolving set. This process of adding even cycles can be continued. Let x', y' be the vertices on the resulting cycle at maximum distance from v. Each of these vertices can terminate a path of arbitrary length, with terminal vertices a and b, respectively.

Let G be a monument graph from Construction 2.13, and let v be as labeled in Fig. 1. Let P_a be the shortest path from v to a, and P_b be the shortest path from

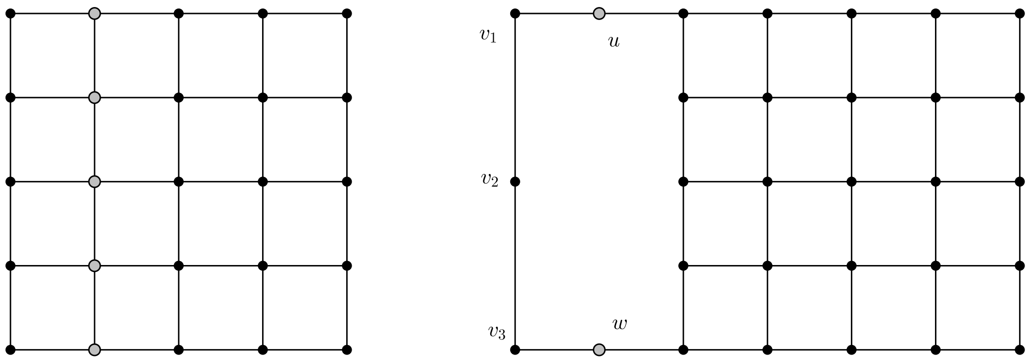

Figure 2: (a) The grid graph G_5 and (b) the graph G'_5 from Theorem 3.2, with highlighted nearly resolving sets

v to b. Since G is not a path the set $S = \{v\}$ is not a resolving set. Let u, w be distinct vertices in $V(G) \setminus \{v\}$. Without loss of generality we may assume that u is on P_a. We claim that the ordered pair $(d(v, w), d(u, w)) = (\alpha, \beta)$ uniquely identifies any vertex w in G. We may assume that $\alpha \neq 0 \neq \beta$. There are at most two vertices in G at distance α from v, at most one on P_a and at most on P_b. First consider the case in which $\alpha < d(v, u)$. If $\beta + \alpha = d(v, u)$, then w is on P_a. Otherwise, w is on P_b. Next, assume that $\alpha > d(v, u)$. If $\alpha - \beta = d(v, u)$, then again w is on the path P_a. Otherwise, w is on P_b. Therefore, for any vertex $u \in V(G) \setminus \{v\}$ the set $S^u = \{v, u\}$ is a resolving set for G. Hence, $\underline{\text{res}}(G) = 1$ if and only if G is a path or a monument graph. □

3 Resolving parameters of grid graphs

Definition 3.1. *By the* grid graph $G(m, n)$ *we mean the Cartesian product* $P_m \square P_n$, (*see Fig. 2*). *This graph is sometimes referred to as a* 2-dimensional grid graph. *Since* $G(m, n)$ *and* $G(n, m)$ *are isomorphic, we will assume without any loss of generality that in the graph* $G(m, n)$, $m \leq n$. *We label the vertices in* $G(m, n)$ *as* $\{(x, y) : 1 \leq x \leq m, 1 \leq y \leq n\}$. *If* $m = n$, *then we denote the graph* G_m.

In [4] the authors determine $\dim(G)$ and $\dim^+(G)$ where G is a grid graph. In particular, $\dim(G(m, n)) = 2$ for $m, n \geq 2$.

Theorem 3.2. *If* $G(m, n)$ *is a grid graph with* $3 \leq m \leq n$, *then* $\underline{\text{res}}(G(m, n)) = m$. *In addition, let* $G'(m, n) = G(m, n)$ *with the addition of vertices* u, v_1, v_2, v_3, w *adjoined via the path* $\{(1, 1), u, v_1, v_2, v_3, w, (m, 1)\}$, (*see Fig. 2*). *Then* $\underline{\text{res}}(G') = 2$.

Proof. Consider $G = G(m, n)$ and let S be a non-resolving set of vertices in G with $|S| < m$. There is some pair of vertices $\{(x, y), (x', y')\}$ in G that S does not resolve.

We may assume that $y' \geq y$ with no loss of generality. Consider first the case for which $|x' - x| \leq |y' - y|$. Note that since G is bipartite every vertex disambiguates any pair of vertices in different partite sets. Therefore $d((x, y), (x', y')) = 2k$ for some integer k. Let $a = (y + k)$ and $b = (y' - k)$. The vertices (x, a) and (x', b) are equal distance from both (x, y) and (x', y'). If $x \leq x'$, then let $U = \{(i, a), (j, b) : 1 \leq i \leq x, x' \leq j \leq m\}$, otherwise let $U = \{(i, a), (j, b) : x \leq i \leq m, 1 \leq j \leq x'\}$. Thus, U is the set of vertices from (x, a) to one edge of the grid and from (x', b) to the opposite edge. Note that every vertex in U has a shortest path to (x, y) and (x', y') through (x, a) or (x', b), and hence no vertex in U resolves the pair. Also, note that $|b-a| = |x'-x|$, and hence we can consider the vertices (x, a) and (x', b) to be corners of a $G_{|b-a|}$ subgraph of $G(m, n)$. Let D be the vertices in this subgraph along the diagonal between (x, a) and (x', b). Since each vertex in D is distance k from (x, y) and (x', y'), and since $|U \cup D| = m$, there is at least one vertex $u \in (U \cup D)$ not in S. Since S^u is not a resolving set of G, the set S is not a nearly resolving set of G.

Next we consider the case where $|y' - y| < |x' - x|$. We proceed the same as above with the following changes. We let $a = (x + k), b = (x' - k)$ if $x \leq x'$, and $a = (x - k), b = (x' + k)$ otherwise. Consider the vertices (a, y) and (b, y'), each at distance k from (x, y) and (x', y'). We also let $U = \{(a, i), (b, j) : 1 \leq i \leq y, y' \leq j \leq n\}$. This yields the set $U \cup D$ of order $n \geq m$ that does not disambiguate the pair $\{(x, y), (x', y')\}$. Again, there is at least one vertex $u \in (U \cup D)$ not in S, and S^u is not a resolving set of G.

Now we let $S = \{(x, 2)\}_{x=1}^{m}$. Note that S does not disambiguate any pair in $\{(x, 1), (x, 3)\}_{x=1}^{m}$. In fact, it is easy to see that no other pairs of vertices fail to be resolved by S. However, S^u for any vertex $u \notin S$ clearly disambiguates every set of vertices in G since $d(u, (x, 1)) \neq d(u, (x, 3))$ for all $1 \leq x \leq m$. Therefore, S is a nearly resolving set of size m for G.

The argument easily extends to the graph $G'(m, n)$. It is not a monument graph, hence $\underline{\text{res}}(G'(m, n)) \geq 2$, and $\{u, w\}$ is a nearly resolving set. □

4 Realizability

In [4] the authors mention that it would be interesting to study the realization of triples (a, b, c) of integers as the metric dimension, the upper dimension, and the resolving number. We now study a similar triple involving the nearly resolving number.

4.1 Metric dimension, nearly resolving number, and resolving number

In this section we characterize the triples (a, b, c) such that there is a graph G with $\dim(G) = a$, $\underline{\text{res}}(G) = b$, and $\text{res}(G) = c$.

Given the bounds in Remark 1 we will only consider triples (a, b, c) such that $a \leq (b + 1) \leq c$. Also, in [10] it is shown that the only graphs with metric dimension

1 are paths. So $(1,1,2)$ is the only realizable triple in which $a = 1$. For triples in which $a > 1$ we require a new construction.

Construction 4.1. *For integers $2 \le a < b < c$ define the graph $P_{(a,b,c)}$ in the following way. Let $V = \{v_0, \dots, v_a\}$ be the vertices of a complete graph on $(a+1)$ vertices. Let $V' = \{v_{a+1}, v_{a+2}, \dots, v_{b-1}\}$ be a path, with v_a adjacent to v_{a+1}, if $a < b+1$, otherwise let V' be empty. Define $U = \{u_0, \dots, u_{c-b+1}\}$ to be a path in which $u_{\lfloor (c-b+1)/2 \rfloor} = v_{b-1}$. Thus, $P_{(a,b,c)}$ is a clique attached to a tree as in Fig. 3.*

If $2 \le a = (b+1) \le c$, then we let $V' = \{v_a, \dots, v_c\}$ and exclude U.

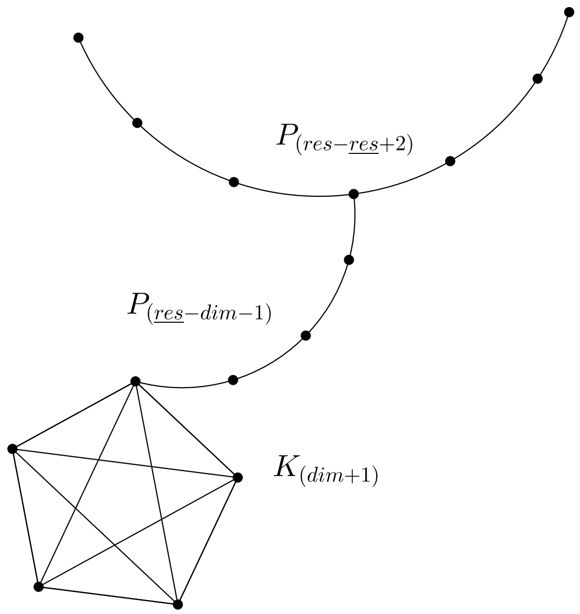

Figure 3: The graph $P_{(4,9,14)}$ from Construction 4.1

We now demonstrate that this graph in Construction 4.1 realizes nearly all triples.

Lemma 4.2. *For $2 \le a < b < c$ or $2 \le a = (b+1) < c$ the graph $P_{(a,b,c)}$ has the properties $dim(P_{(a,b,c)}) = a$, $\underline{res}(P_{(a,b,c)}) = b$, $res(P_{(a,b,c)}) = c$.*

Proof. First consider $a < b$ and note that the set $S = \{v_0, \dots, v_{a-2}, u_0\}$ is a resolving set for $P_{(a,b,c)}$. Any resolving set S' must contain at least one vertex in U and at least $(a-1)$ vertices in V since $V \setminus \{v_a\}$ share the same closed neighborhood. Therefore, S realizes $\dim(P_{(a,b,c)}) = a$. It is easy to see that the set $V(P_{(a,b,c)}) \setminus \{v_0, v_1\}$ is not a resolving set, and therefore $\mathrm{res}(P_{(a,b,c)}) = c$.

First notice that the set $V \cup V'$ is a nearly resolving set in $P_{(a,b,c)}$. Now, suppose there is another nearly resolving set S. The set S must include at least one vertex in U and at least $(a-1)$ vertices in V. Consider a vertex $w \in V'$ not in S. Since S^w must be a resolving set for the graph, S must already contain $(a-1)$ vertices from V and a vertex $u \in U$. If u is the center vertex of the path U, then S^w does not disambiguate the neighbors of u in U unless S is already a resolving set. If the center vertex of this path is not already in S, then its addition does not create a resolving set. Thus $V' \subseteq S$. In addition, if $v \in V$ is not in S, then in order for S^v to be a resolving set S contains $(a-2)$ vertices from V, all of V', and therefore all of U. So, $|S| \geq |P_{(a,b,c)}| - 3$. However, $V \cup V'$, which contains b vertices, is a smaller nearly resolving set. Hence S contains V' and V and the graph $P_{(a,b,c)}$ realizes (a, b, c).

Now, we assume that $a = (b+1)$. Again, it is clear that $\mathrm{res}(P_{(a,b,c)}) = c$ is one less than the order of the graph. Since any resolving set must contain either $(a-1)$ vertices in V and one other in V', or a vertices in V, we see that $\dim(P_{(a,b,c)}) = a$. The set $\{v_0, v_1, \ldots, v_{a-1}\}$ is a smallest nearly resolving set, with size $a - 1 = b$. □

We next turn our attention to the exigent case.

Lemma 4.3. *The graph $Br(c - b + 1, a)$ realizes the triple (a, b, c) when $b = a$.*

Proof. In Theorem 2.1 it was demonstrated that $\underline{\mathrm{res}}(Br(c - b + 1, a)) = a = b$. Since no set that excludes more than one bristle is a resolving set, we see that $\mathrm{res}(Br(c - b + 1, a) = (c - b + 1) + (a) - 1 = c$. Also, since a smallest resolving set of the graph consists of the a bristles $\dim(Br(c - b + 1, a)) = a$. □

Lemmas 4.2 and 4.3 bring us to the main theorem of this section.

Theorem 4.4. *Any triple (a, b, c) in which $2 \leq a \leq (b+1) \leq c$ is realizable as the metric dimension, nearly resolving number, and resolving number of a graph.*

4.2 Upper dimension and the nearly resolving number

It is obvious for any connected graph G that $\dim(G) \leq \dim^+(G) < \mathrm{res}(G)$ and that $\dim(G) \leq \underline{\mathrm{res}}(G) + 1 \leq \mathrm{res}(G)$. However, it is not clear how the upper dimension and the nearly resolving number relate. In this section we demonstrate that the two parameters can differ by an arbitrary number and neither is necessarily greater than the other.

First, we recount a theorem from [4].

Theorem 4.5. *There is a graph H_m with upper dimension $\dim^+(H_m) = 2m - 2$ and $\dim(H_m) = 2$.*

This graph H_m is a grid graph G_m modified by the addition of a triangle attached to a single vertex of degree 2, (Fig. 4). We modify the graph G'_m from Def. 3.1 in the same way, with a triangle attached at vertex v_3.

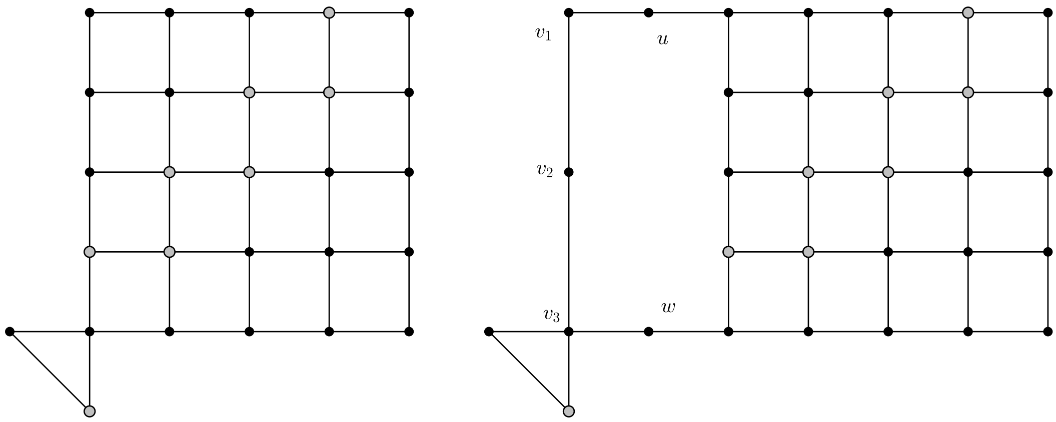

Figure 4: The modified grid H_5 with highlighted resolving set and H_5' with the same set of vertices highlighted

First note that $\dim(H_m') = \dim(H_m) = 2$ and $\dim^+(H_m') \geq \dim^+(H_m)$. This is because the largest minimal resolving set of H_m outlined in [4] and highlighted in Fig. 4 is also a minimal resolving set of H_m'. As demonstrated in Section 3 the nearly resolving number of G_m' is 2, and it is easy to see that $\underline{\mathrm{res}}(H_m') \leq 4$ by possibly including one or both vertices of the new triangle in any nearly resolving set. Hence, we have the following theorem.

Theorem 4.6. *For any $k \geq 0$ there is a graph G with $\dim^+(G) - \underline{res}(G) \geq k$.*

Proof. For any integer $j \geq 2$ there is a modified grid graph H_m with $\dim^+(H_m) = j$. Thus $\dim^+(H_m') \geq j, \underline{\mathrm{res}}(H_m') = 2$. Given k we let $j = (k+4)$ and choose G to be the appropriate modified grid graph. $\square$

The construction showing the inverse is a simple one.

Theorem 4.7. *For any $k \geq 0$ there is a graph G with $\underline{res}(G) - \dim^+(G) = k$.*

Proof. Let $k \geq 0$ be given, and let G be the graph consisting of three paths of length k, a vertex v, and one endpoint of each path adjoined to v. By Theorem 2.5 we know that $\underline{\mathrm{res}}(G) \geq (k+2)$, and since the vertices in one of the three paths along with v form a nearly resolving set we get that $\underline{\mathrm{res}}(G) = (k+2)$. Since any resolving set with more than two vertices is not minimal, $\dim^+(G) = 2$ and $\underline{\mathrm{res}}(G) - \dim^+(G) = k$. $\square$

5 Conclusion

In this manuscript we have examined a new graph parameter related to resolving sets, the *nearly resolving number*. This parameter is related to the *resolving number* of a graph in a way similar to the relationship between saturation and extremal numbers of graphs. We determined a number of properties of the new parameter, and related it to existing parameters. All graphs with nearly resolving number 1 were characterized.

A number of questions related to the nearly resolving number remain. It would be interesting to determine for which triples (a, b, c) there is a graph G in which $\dim(G) = a$, $\dim^+(G) = b$, $\underline{\mathrm{res}}(G) = c$. Other potential directions for research include characterizing all graphs with nearly resolving number 2, and relating this parameter to the girth and circumference.

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