The Raney generalization of Catalan numbers and the enumeration of planar embeddings

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Abstract

The Raney numbers $R_{p,r}(k)$ are a two-parameter generalization of the Catalan numbers that were introduced by Raney in his investigation of functional composition patterns. We give a new combinatorial interpretation for the Raney numbers in terms of planar embeddings of certain collections of trees, a construction that recovers the usual interpretation of the p-Catalan numbers in terms of p-ary trees via the specialization $R_{p,1}(k) = {}_pc_k$. Our technique leads to several combinatorial identities involving the Raney numbers and ordered partitions. We then give additional combinatorial interpretations of specific Raney numbers, including an identification of $R_{p^2,p}(k)$ with oriented trees whose vertices satisfy the "source-sink property." We close with comments applying these results to the enumeration of connected (non-elliptic) A_2 webs that lack an internal cycle.

1   Introduction

We investigate a two-parameter generalization of the Catalan numbers known as the *Raney numbers*, as first studied by Raney [12]. These Raney numbers are defined as $R_{p,r}(k) = \dfrac{r}{kp+r}\dbinom{kp+r}{k}$ for all positive integers k, p, r, and specialize to the p-Catalan numbers as $R_{p,1}(k) = {}_pc_k$, which further specialize to the usual Catalan numbers as $R_{2,1}(k) = c_k$. Raney numbers have previously seen applications to compositional patterns [12] and probability theory [9]. In this paper we give a new set of combinatorial interpretations for $R_{p,r}$ that directly generalize the well-known application of p-Catalan numbers to the enumeration of p-ary trees [5, 13]. Our work can also be interpreted as a generalization of the planted plane tree enumeration techniques developed by Harary, Prins and Tutte [4] and Klarner [7], and our results specialize to all of those tree enumeration results via specific choices of p and r.

We begin in Section 2 with a careful description of our "generalized p-ary trees," which are in one-to-one correspondence with planar embeddings of trees with specific vertex structures. Section 2 goes on to provide two independent methods for counting these generalized trees: Proposition 2.3 is a "tiered approach" that generalizes a more specialized result of Klarner [7], whereas Theorem 2.5 is a modification of a construction by Hilton and Pedersen [5] that directly relates our enumerations to the Raney numbers. Our two techniques are brought together by the combinatorial identity of Theorem 2.6, which is summarized below:

Theorem 1.1. *Let k be a positive integer. Then for all positive integers p, r we have:*

$$R_{p,r}(k) = \frac{r}{kp+r}\binom{kp+r}{k} = \sum_{\lambda}\binom{r}{\lambda_1}\binom{p\lambda_1}{\lambda_2}\binom{p\lambda_2}{\lambda_3}\cdots\binom{p\lambda_{j-1}}{\lambda_j}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ ranges over all ordered partitions of k.

In Section 3 we apply our primary results to give combinatorial interpretations of Raney numbers for specific values of p and r. Our result of most interest in this section is our identification of $R_{p^2,p}(k)$ with edge-oriented trees whose $(p + 1)$-valent vertices coherently obey the "source-sink" property. In the case of $p = 2$ this gives an enumeration of connected, non-elliptic A_2 webs with no internal cycles: a significant subclass of the non-elliptic A_2 webs introduced by Kuperberg [8] to graphically encode the representation theory of the quantum enveloping algebra $U_q(sl_3)$. In Corollary 3.5 we eventually prove the following, which characterizes a certain subset of $Hom_{sl_3}(V^{\otimes 3(k+1)}, \mathbb{C})$ for the three-dimensional irreducible sl_3-module V:

Theorem 1.2. *For all positive integers k, $R_{4,2}(k)$ equals the number of connected, non-elliptic A_2 webs that lack an internal face and have a constant boundary string with $3(k + 1)$ pluses.*

After giving another application of $R_{4,1}$ to a different class of non-elliptic A_2 webs, we close the paper with a series of conjectures that hope to generalize our results to sl_n webs, which similarly encode the representation theory of $U_q(sl_n)$. In particular, we assert a correlation between the Raney number $R_{n+1,n-1}(k)$ and linearly-independent connected sl_n webs that lack an internal cycle and have a boundary string corresponding to $n(k + 1)$ total 1's.

2   Raney Numbers & the Enumeration of Planar Tree Embeddings

In this section we present the primary construction of this paper, which gives a geometric realization of the Raney numbers in terms of planar embeddings of certain types of trees. We begin by introducing our graph theoretic terminology:

Definition 2.1. *Let p be a positive integer. Then a **p-star** is a rooted tree with p terminal edges lying above a single base vertex.*

In this section we will directly use p-stars as building blocks for larger graphs. In Section 2 we will modify stars by allowing their edges to be directed, or by replacing the basic p-star with more complicated subgraphs that retain a single base vertex and p terminal edges. For a fixed p, p-stars are used to construct planar graphs that we refer to as coral diagrams:

Definition 2.2. *Let p, r be positive integers. A **coral diagram of type** (p, r) is a rooted tree that is constructed from a $(r + 1)$-valent base vertex via the repeated placement of p-stars atop terminal edges that are not the leftmost edge adjacent to the base vertex.*
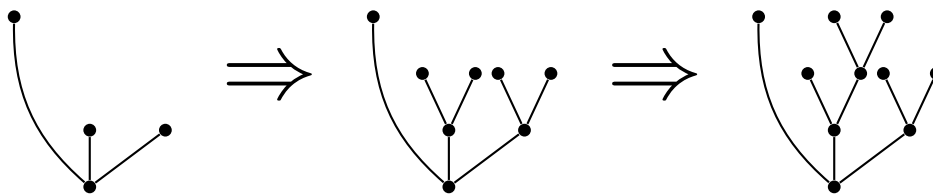

Figure 1: Construction of a (2,2)-coral diagram with three 2-stars

We refer to the vertices that serve as base points for our p-stars as the *attachment sites* of our coral diagram. In our upcoming constructions it will be useful to think of coral diagrams as a collection of $k+1$ trees (k p-stars and a single "base" $(r+1)$-star) where we have identified the vertices corresponding to the k attachment sites.

The condition from Definition 2.2 that we cannot add stars to the leftmost initial edge in a coral diagram is essential for our combinatorial interpretations. Our primary concern is planar embeddings of graphs, with equivalence given by homeomorphisms that fix a linear ordering of the terminal vertices. When associating planar embeddings with rooted trees, the principal difficulty is consistently dealing with the fact that a single embedding may be rooted at multiple distinct vertices. Not attaching stars to the leftmost edge of our coral diagram gives us a consistent way of selecting a base vertex for our planar embeddings and ensures that homeomorphic trees cannot correspond to distinct coral diagrams: each equivalence class of homeomorphic planar embeddings contains a unique member where there is a single edge connecting the leftmost boundary point with a "bottom" vertex. Although somewhat awkward from a diagrammatic perspective, this way of selecting root vertices will allow for a particularly elegant correspondence between coral diagrams and Raney numbers in Theorem 2.5.

In Figure 2 we show the (2,2)-coral diagram from Figure 1 and a homeomorphic embedding of the same graph. Assume in all of our figures that terminal edges have been extended to a fixed boundary line at the top of the diagram. In all constructions we assume a boundary that is homeomorphic to $\mathbb{R}$. Notice that the embedding on the right does not represent a valid coral diagram of any type.

We are now ready for our primary results enumerating coral diagrams. Henceforth denote the number of distinct (p, r)-coral diagrams with exactly k total p-stars added to the base star by $T_{p,r}(k)$. The proposition below is the more direct of our
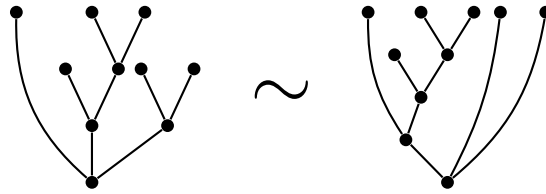
Figure 2: A (2,2)-coral diagram and an equivalent planar embedding

two methodologies, and will be used in Theorem 2.6 to introduce an entirely new combinatorial identity involving the Raney numbers. Notice that Proposition 2.3 closely resembles (20) from [7] after specializing to $r = 1$.

Proposition 2.3. *Let p, r be positive integers. Then the number of coral diagrams of type (p, r) with precisely k p-stars is:*

$$T_{p,r}(k) = \sum_\lambda \binom{r}{\lambda_1} \binom{p\lambda_1}{\lambda_2} \binom{p\lambda_2}{\lambda_3} \cdots \binom{p\lambda_{j-1}}{\lambda_j}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ varies over all ordered partitions of k (of any length $j \geq 1$).

Proof. As in Figure 1, we will construct our coral diagram by attaching our k stars one "tier" at a time, beginning with the base tree and working upward. We assume that all stars are attached as low as possible, so that if a vertex is not used as an attachment point for a given tier that vertex cannot serve as an attachment point for later tiers. Hence the only valid attachment points at each tier are the terminal vertices of stars added in the previous tier.

So let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ be an ordered partition of k, and consider the number of (p, r)-coral diagrams with λ_1 stars attached in the first tier, λ_2 stars attached in the second tier, etc. For the first tier there are r available attachment sites, corresponding to the r terminal edges adjacent to the base that are not the leftmost edge. Hence there are $\binom{r}{\lambda_1}$ distinct ways to attach stars of this tier. For the jth tier $(j > 1)$ there are $p\lambda_{j-1}$ valid attachment sites, corresponding to the top vertices of the λ_{j-1} p-stars from the previous tier. Thus there are $\binom{p\lambda_{j-1}}{\lambda_j}$ distinct ways to attach stars of this tier.

As the resulting diagrams are rooted, and since we are not allowing star attachment to the leftmost initial edge, all trees produced in this manner are non-equivalent embeddings. This leaves $\binom{r}{\lambda_1}\binom{p\lambda_1}{\lambda_2}\binom{p\lambda_2}{\lambda_3}\cdots\binom{p\lambda_{j-1}}{\lambda_j}$ distinct trees corresponding to our partition λ. As coral diagrams produced from distinct partitions are clearly non-equivalent, this proves the result. □

Our second construction directly relates the number of (p, r)-coral diagrams to the Raney numbers $R_{p,r}(k)$. We preface our result with a characterization of the Raney numbers that is proven by Hilton and Pedersen in [5]. Our Raney number $R_{p,r}(n)$ corresponds to $_pd_{qk} = d_{qk}$ in [5] via $q = p - r, k = n + 1$.

Lemma 2.4. *[5, Theorem 2.6] Let $_pc_k = \frac{1}{k}\binom{pk}{k-1}$ be the kth p-Catalan number, and let $R_{p,r}(k)$ denote the Raney number. Then:*

$$R_{p,r}(k) = \sum_{i_1+\ldots+i_r=k} {}_pc_{i_1} \ {}_pc_{i_2} \ \cdots \ {}_pc_{i_r}.$$

Theorem 2.5. *Let p, r be positive integers. Then the number of (p, r)-coral diagrams with precisely k p-stars equals the kth evaluation of the Raney number $R_{p,r}(k)$:*

$$T_{p,r}(k) = R_{p,r}(k) = \binom{pk+r-1}{k-1}\frac{r}{k}.$$

Proof. It is well established (see [5]) that ${}_pc_j$ equals the number of connected trees constructed from j total p-stars. With this interpretation, Lemma 2.4 states that the Raney number $R_{p,r}(k)$ counts the number of distinct ordered r-tuples of trees constructed from p-stars such that a total of k total p-stars are utilized across the entire tuple.

Now consider a (p, r)-coral diagram. We may subdivide the coral diagram into $r+1$ subgraphs (some of which may be empty): one corresponding to the base r-star and one corresponding to everything added atop each of the r attachment sites of the base $(r + 1)$-star. Any coral diagram may then be described by an r-tuple of trees constructed from p-stars. As the leftmost edge of the base star still isn't a valid attachment site, it isn't possible to achieve equivalent coral diagrams from distinct r-tuples of trees. If we fix the total number of p-stars to be used at $k \geq 0$, it follows that $T_{p,r} = \displaystyle\sum_{i_1+\ldots+i_r=k} {}_pc_{i_1} \ {}_pc_{i_2} \ \cdots \ {}_pc_{i_r}$. Lemma 2.4 then gives the desired result. $\square$

As a quick corollary of Proposition 2.3 and Theorem 2.5, we have the following combinatorial identity that relates the Raney numbers to ordered partitions.

Theorem 2.6. *Let k, p, r be positive integers, and consider the Raney number $R_{p,r}(k)$. Then:*

$$R_{p,r}(k) = \sum_{\lambda} \binom{r}{\lambda_1}\binom{p\lambda_1}{\lambda_2}\binom{p\lambda_2}{\lambda_3}\ldots\binom{p\lambda_{j-1}}{\lambda_j}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_j)$ varies over all ordered partitions of k (of any length $j \geq 1$).

Notice that this is a distinct reduction of $R_{p,r}(k)$ into a summation over ordered partitions of k than the one presented in Lemma 2.4, even though both summations involve terms of the form $\binom{p\lambda_j}{m}$. The decomposition of Lemma 2.4 ranges over weak partitions and follows from "horizontally" dividing our coral diagram into a tuple of attached trees, whereas the summation of Theorem 2.6 ranges over (strong) partitions and follows from "vertically" dividing our coral diagram into tiers.

3   Combinatorial Interpretations of $R_{p,r}(k)$

The remainder of this paper is devoted to combinatorial interpretations of the Raney numbers $R_{p,r}$ for specific choices of p, r. Our primary tools are Theorem 2.5 and the coral diagram framework that it suggests. To begin with the least derived (and least informative) result we have:

Proposition 3.1. *Let k, p, r be positive integers, then $R_{p,r}(k)$ equals the number of distinct planar embeddings of trees with $k + 1$ internal vertices such that all internal vertices are $(p + 1)$-valent except for the vertex incident upon the leftmost terminal edge, which is $(r + 1)$-valent.*

Proof. Follows directly from Proposition 2.5 and the definition of coral diagram. □

Obviously, the situation becomes far more interesting if $p = r$, with $R_{p,p}(k)$ enumerating distinct planar embeddings of wholly $(p+1)$-valent trees. This specialization also yields a new proof of the following Raney number identity:

Proposition 3.2. *Let p be a positive integer. Then $R_{p,p}(k) = R_{p,1}(k + 1)$, with both quantities equaling the number of distinct planar embeddings of $(p + 1)$-valent trees with $k + 1$ internal vertices.*

Proof. Consider a (p, p)-coral diagram with k total p-stars. We divide the leftmost edge emanating from the base vertex by adding an additional 2-valent vertex, and then isotope so that this new vertex lies at the base. This is now a $(p, 1)$-coral diagram with $k + 1$ total p-stars. Since there is only one attachment site for the first p-star, we are able to get every coral diagram of type $(p, 1)$ with $k + 1$ stars in this manner. Also, it is clear that distinct (p, p)-coral diagrams with k p-stars are transformed into distinct $(p, 1)$-coral diagrams with $k + 1$ p-stars. □

Also of interest is the situation where $p = 1$, where coral diagrams provide a new "geometric" proof of the result that $R_{1,r}(k)$ is related to ordered weak partitions of k:

Proposition 3.3. *Let r be a positive integer. Then $R_{1,r}(k)$ equals the number of ordered weak partitions of the positive integer k into r pieces.*

Proof. $(1, r)$-coral diagrams take the form shown in Figure 3. Adding k total 1-stars then amounts to choosing a partition of k into r pieces with $\lambda_j \geq 0$ 1-stars each. This partition is ordered because of the unused edge at the left. □
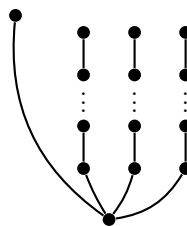

Figure 3: A (1,3)-coral diagram, giving an ordered partition in three pieces

Further interpretations of the Raney numbers can be made when one orients the edges of the planar embeddings. Here we only consider orientations that are coherent in the sense that every vertex is either a source or a sink. In such a configuration, a rooted tree satisfies the *source-sink property* if every vertex is either a source or a sink, the neighborhood of every source consists only of sinks, and the neighborhood of every sink consists only of sources.

Proposition 3.4. *Let p be a positive integer; then $R_{p^2,p}(k)$ equals the number of distinct planar embeddings of $(p + 1)$-valent trees, coherently oriented according to the source-sink property, with $k(p^2 - 1) + (p + 1)$ 1-valent boundary vertices that are all sinks.*

Proof. We look to establish a bijection between planar embeddings that satisfy the hypothesis of the proposition and (p^2, p)-coral diagrams whose "stars" take the modified form shown in Figure 4. We induct on the number of internal vertices in the tree:

For the base step, notice that a $(p + 1)$-valent tree with one internal vertex and $(p + 1)$ external sinks is merely the base of a (p^2, p)-coral diagram where the base vertex is a source. For the inductive step, notice that adding p edges to an external sink produces p new external vertices that are all sources. If the resulting tree is to satisfy the hypotheses of the proposition, this necessitates the addition of p additional edges to each of these new vertices. This yields p^2 new external sinks, and shows that any qualifying tree must be built up via the attachment of entire p^2-stars of the type shown in Figure 4. It follows that every planar embedding created in this way is homeomorphic to a unique (p^2, p)-coral diagram. $\square$

Figure 4 shows our modified p^2-stars for source-sink directed trees in the cases of $p = 2$ and $p = 3$. Notice that the top of those stars exhibit a constant upward orientation, and that the base of those stars can only be attached to an upward oriented edge.
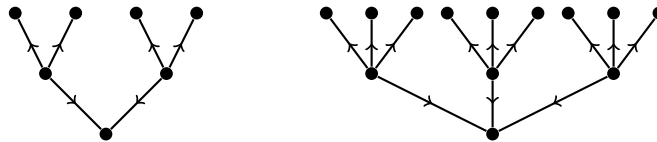

Figure 4: Oriented "stars" for $(2^2, 2)$- and $(3^2, 3)$-coral diagrams

A quick inductive argument shows that the only $(p + 1)$-valent trees with the properties required by Proposition 3.4 have $k(p^2 - 1) + (p + 1)$ boundary points. Thus, ranging over $R_{p^2,p}(k)$ for all $k \geq 0$ accounts for all source-sink oriented $(p+1)$-valent trees with constant boundary vertex orientation.

An equivalent interpretation to the one in Proposition 3.4 is that $R_{p^2,p}(k)$ counts the number of distinct planar embeddings of $(p+1)$-valent trees with $k(p^2-1)+(p+1)$ 1-valent boundary points such that, for any fixed vertex, any path from that vertex to the boundary passes through an equivalent number of edges modulo two.

The primary reason we present the specific interpretation of Proposition 3.4 is that the $(p, r) = (4, 2)$ case gives planar embeddings that qualify as (non-elliptic) A_2 webs (referred to by some as simply sl_3 webs). A_2 webs constitute the morphisms in the braided monoidal category introduced by Kuperberg [8] to diagrammatically present the representation theory of the quantum enveloping algebra $U_q(sl_3)$. Objects in this category are finite words in the alphabet $\{+, -\}$, corresponding to the two (dual) irreducible three-dimensional sl_3-modules V^+ and V^-. These words are encoded via the orientation of the boundary vertices, so that all webs can be represented

as elements of $Hom(\vec{s}, \emptyset)$ for boundary word $\vec{s}$. Non-elliptic webs are those webs that lack internal squares and bigons. Non-elliptic webs form a linear basis for all A_2 webs over $\mathbb{Z}[q, q^{-1}]$. For a constant boundary string of $3k$ total pluses, the total number of non-elliptic webs is known to be in bijection with standard Young tableaux of size $3 \times k$ ([11],[14]). Our $(4, 2)$-coral diagrams with k total p-stars then form a subset of non-elliptic webs with a constant boundary string of $k(4-1) + (2+1) = 3(k+1)$ total pluses, and are in bijection with an interesting subset of standard Young tableaux of size $3 \times (k+1)$.

In terms of A_2 web terminology, Proposition 3.4 can be restated in the specific case of $p = 2$ as follows:

Corollary 3.5. *For all positive integers k, $R_{4,2}(k)$ equals the number of connected non-elliptic A_2 webs that lack an internal face and have a constant boundary string with $3(k+1)$ pluses.*

In an upcoming paper [1], the authors apply Corollary 3.5 and other combinatorial results to enumerate webs with distinct geometric structures. In Figure 5 we show how Corollary 3.5 applies to A_2 webs with boundary word $(++++++)$. Here we have 5 total non-elliptic webs, $R_{4,2}(1) = 2$ of which are connected trees.
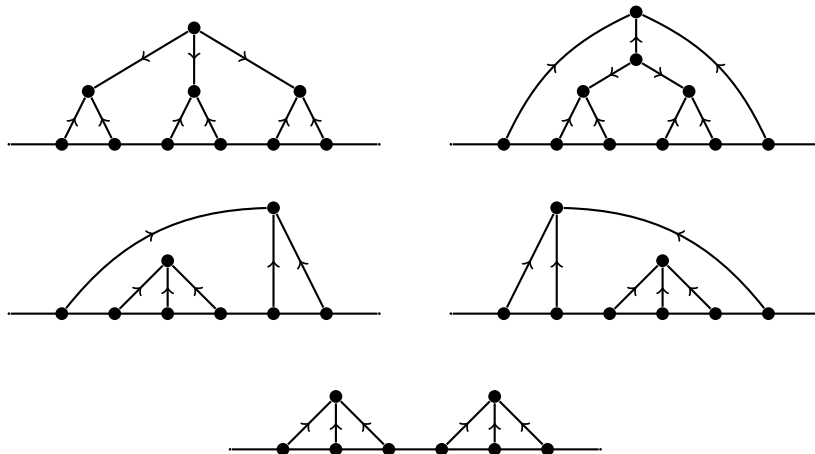

Figure 5: The 5 non-elliptic sl_3 webs with boundary $(++++++)$

Raney numbers can also be applied to enumerate non-elliptic A_2 webs with boundary word of the form $(-+++\ldots+)$. The construction here is similar to that in Proposition 3.4, with the same form of "modified" 4-stars. The only difference is that we now give the leftmost unused edge the opposite orientation and directly wrap it around to form a single attachment site for our 4-stars (as opposed to adding a single trivalent vertex in the base that flips the orientation of attachment sites). This directly proves the following:

Proposition 3.6. *$R_{4,1}(k)$ equals the number of connected non-elliptic A_2 webs that lack an internal face and have a constant boundary string with one minus followed by $3k + 1$ pluses.*

An application of Proposition 3.6 in the case of $k = 1$ is shown in Figure 6. Here we have 3 total non-elliptic webs with boundary word $(- + + + +)$, $R_{4,1}(1) = 1$ of which is a connected tree. Notice that, since $R_{4,1}(k)$ is simply the kth entry in the 4-Catalan sequence, this result also gives a new interpretation of the 4-Catalan numbers.
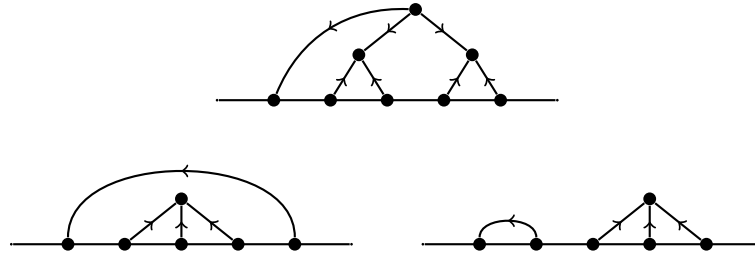

Figure 6: The three non-elliptic sl_3 webs with boundary $(- + + + +)$

We close this paper by commenting upon possible interpretations of $R_{p,r}$ for other choices of p and r. We focus on how our results about A_2 webs in Corollary 3.5 and Proposition 3.6 may be generalized to enumerate connected trees in the wider class of sl_n webs.

Much as A_2 webs are used to diagrammatically present the representation theory of $U_q(sl_3)$, sl_n webs do the same for $U_q(sl_n)$. In addition to carrying an orientation, edges in these sl_n webs are now labelled by one of the $(n-1)$ fundamental representations, while reversal of orientation corresponds to taking the dual of the given representation (in sl_3 webs there are two fundamental representations that are duals of one another, so the additional edge labellings are dropped because the orientations carry all necessary information). The vertices of these webs must obey a more complicated set of conditions that depend on both orientation and edge-labelling, giving a far more complicated theory. See [10] for a detailed introduction to this topic, [3] and [15] for constructive algorithms producing sl_n web bases, and the more recent [2] for a generating set of relations for sl_n webs.

The following conjecture is a direct generalization of Corollary 3.5 to the enumeration of sl_n "tree webs" with a constant boundary string.

Conjecture 3.7. *For any $n \geq 3$, $(n-2)^k R_{n+1,n-1}(k)$ equals the number of connected sl_n webs that lack an internal cycle and have a boundary string with $n(k+1)$ total 1's.*

The specific interpretation of $(n + 1, n - 1)$-coral diagrams that motivates Conjecture 3.7 is shown in Figure 7 for the relatively easy case of $n = 4$. On the left we show the base for our $(5, 3)$-coral diagrams, here interpreted as part of an sl_4 web, and on the right we show two non-equivalent pieces of sl_4 webs that both function as valid choices for each 5-star. In the sl_4 case, the reason that there aren't additional non-equivalent variations of these pieces follows from Kim's relations for sl_4 webs [6]. The relevant member of these relations is shown in Figure 8; notice that this relation also sees a direct analogue in the more general sl_n web relations of Cautis, Kamnitzer, and Morrison (Relation 2.6 of [2]). In the general sl_n case, we conjecture

the existence of one valid base and $n-2$ valid choices for each star. It is these $n-2$ non-equivalent choices for each of the k total $(n+1)$-stars in our coral diagrams that leads to the unusual "correction factor" of $(n-2)^k$ in Conjecture 3.7.
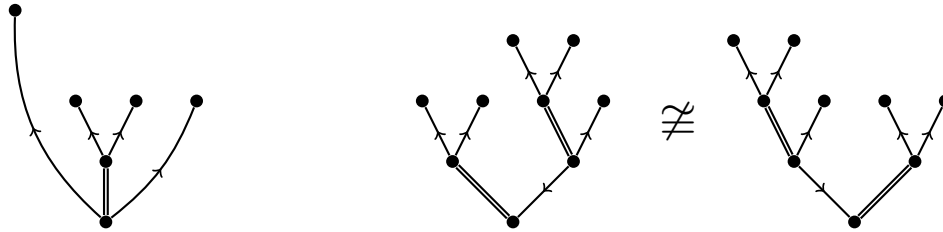

Figure 7: sl_4 web interpretation for $R_{5,3}$, with base (left) and two non-equivalent choices for each p-star (right)
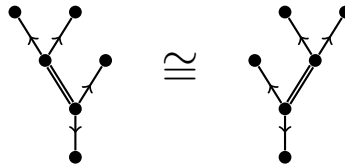

Figure 8: One of Kim's sl_4 web relations

The reason that we cannot prove Conjecture 3.7 at this point is that we lack a general proof of the fact that, in the $(n+1, n-1)$-coral diagram interpretation, there is precisely one valid choice for our base and precisely $(n-2)$ non-equivalent choices for each $(n+1)$-star. Furthermore, it would need to be shown that the different "pieces" of the coral diagram do not interact and allow for additional relations that cannot be localized to differences in the base or within a single coral.

In light of the generating sl_n web relations of [2], it appears that the only relation capable of directly relating a pair of distinct tree webs is the aforementioned $I = H$ relation (Relation 2.6, [2]). This relation should allow for a direct justification of the fact that there are $n-2$ distinct choices for each $(n+1)$-star in the above interpretation. However, it would still need to be shown that two tree webs cannot be connected via a string of other relations that pass through at least one non-tree sl_n web.

On a more basic level, notice that the local relations of [2] do not result in simple global conditions for determining whether an sl_n web is a member of a given basis: in the case of sl_3 webs the non-elliptic condition provides an easy way to recognize whether a given web is an element of the non-elliptic basis, but there is no similarly tractable condition for recognizing basis webs in the $n > 3$ case. Also notice that the constructive sl_n web bases developed in [15] and [3] aren't well-suited to proving our conjecture, as there isn't a reasonable way to determine which inputs to their growth algorithms yield webs that are connected trees. Even if these bases could be used to show that there are at least $n+2$ non-equivalent choices for each $(n+1)$-star, they cannot be easily applied to prove there are not additional tree webs that are equivalent to elements of the resulting set.

We also conjecture that Proposition 3.6 for sl_3 webs may be generalized to sl_n webs with a boundary string of the form $(j\ 1\ 1 \ldots 1)$. The desired result is given in Conjecture 3.8. If true, this result would give a combinatorial interpretation of $R_{p,r}(k)$ for all $k \geq 0$ whenever $1 \leq r \leq p - 2$.

Conjecture 3.8. *For any $n \geq 3$ and any $1 \leq j \leq n - 1$, $(n-2)^k R_{n-1,n-j}(k)$ equals the number of connected sl_n webs that lack an internal cycle and have a boundary string consisting of one j followed by $nk + n - j$ consecutive 1's.*

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