AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **63(3)** (2015), Pages 346–367

R-sequencings and strong half-cycles from narcissistic terraces

Gage N. Martin   M. A. Ollis*

Marlboro College
P.O. Box A, Marlboro
Vermont 05344
U.S.A.

Abstract

We construct narcissistic terraces for cyclic groups that have various properties which enable the construction of R-sequencings and strong half-cycles for many non-cyclic abelian groups. Among other results, we show: that an abelian group which is isomorphic to a direct product of cyclic factors such that the number of factors of order congruent to 3 (mod 4) of order at most 79 is at least as large as the number of such factors of order greater than 79 is R-sequenceable (including all groups that are the direct product of cyclic groups of orders congruent to 1 (mod 4)); that for any abelian 3-group there are infinitely many R-sequenceable groups whose Sylow 3-subgroups are of that form; and that abelian groups whose Sylow 3-subgroups are of the form $\mathbb{Z}_3^\rho \times \mathbb{Z}_9^\rho \times \mathbb{Z}_{27}^\sigma \times \mathbb{Z}_{81}^\tau$ or $\mathbb{Z}_3^\rho \times \mathbb{Z}_9^\rho \times \mathbb{Z}_{27}^\sigma \times \mathbb{Z}_{81}^\tau \times \mathbb{Z}_{3^k}$ where $k \equiv \rho + \sigma \pmod 2$ are R-sequenceable. For strong half-cycles we give the first constructions for non-cyclic and non-elementary-abelian groups, including for groups that can be written as the product of cyclic factors, all either of order congruent to 1 (mod 12) or order at most 81 with order congruent to 1 (mod 4). We also show that for composite n with $21 \leq n \leq 69$ there is a robust half-cycle for $\mathbb{Z}_n$.

1 Introduction

Complete mappings have long been studied, initially for their connection to sets of mutually orthogonal Latin squares. In this paper we investigate two types of complete mappings of abelian groups that have a particular cycle structure: R-sequencings, which also have a rich history, and strong half-cycles, which have been studied less. Our basic method involves constructing and combining narcissistic terraces and we extend and add some results on these objects that are of interest in their own right.

* Corresponding author, `matt@marlboro.edu`.

We are only concerned with abelian groups here, so we restrict our definitions to this case and use additive notation throughout. However, many of the concepts have natural extensions to nonabelian groups. We denote the cyclic group by $\mathbb{Z}_n$ and take it to be addition modulo n on the symbols $\{0, 1, \dots, n-1\}$.

Let G be an abelian group with identity element e and let $\theta : G \to G$ be a bijection. If the map $\phi : g \mapsto g+\theta(g)$ is also a bijection then θ is a *complete mapping* of G and ϕ is an *orthomorphism* of G. If $\theta(e) = e$ then also $\phi(e) = e$ and both the complete mapping and the orthomorphism are *normalized*. From any complete mapping θ' we can construct a normalized complete mapping $\theta : g \mapsto \theta'(g) - \theta'(e)$.

An abelian group has a complete mapping if and only if it does not have exactly one involution [15]. For more on complete mappings of groups (including the nonabelian case) see the survey [6].

Viewing a normalized orthomorphism as a member of the symmetric group on the elements of G we see that it has a cycle of length 1 as the identity element is a fixed element. We are interested in two possibilities for the cycle structure of the remainder of the permutation: that the elements form a single cycle of length $|G|-1$ ("R-sequenceability") and that they form two cycles of equal length $(|G|-1)/2$ ("strong half-cycles"). In the latter case $|G|$ must be odd and, in fact, we limit ourselves to groups of odd order in both cases.

We will often consider a list of elements $(a_1, a_2, \dots, a_l)$ of a group to "wrap-around" so that a_1 is thought of as immediately following a_l to give a circular sequence. To emphasise this, we include a hooked arrow in such lists: $(a_1, a_2, \dots, a_l, \hookleftarrow)$. When working with such sequences, subscripts of the elements are taken modulo the length of the sequence. In particular, when working with a circular sequence $(a_1, a_2, \dots a_l, \hookleftarrow)$ the element a_{l+1} is a_1.

Let G be an abelian group of order n. Let $\mathbf{a} = (a_1, a_2, \dots, a_{n-1}, \hookleftarrow)$ be a circular arrangement of the non-identity elements of G and define $\mathbf{b} = (b_1, b_2, \dots, b_{n-1}, \hookleftarrow)$ by $b_i = a_{i+1} - a_i$ for each i (note that $b_{n-1} = a_1 - a_{n-1}$ as $a_n = a_1$ for the purposes of this calculation). If $\mathbf{b}$ also consists of all of the non-identity elements of G then $\mathbf{a}$ is a *directed rotational terrace* or *directed R-terrace* and $\mathbf{b}$ is a *rotational sequencing* or *R-sequencing*. Several alternative but equivalent definitions appear elsewhere [1, 7, 10, 12, 16].

To see the connection to complete mappings, given a directed rotational terrace with this notation, define a map $\phi : G \to G$ that fixes e and sends a_i to a_{i+1} for each i. Then ϕ is a normalized orthomorphism.

Example 1.1 *In* $\mathbb{Z}_{25}$

$$(2, 21, 6, 17, 10, 13, 11, 16, 7, 20, 3, 24, 23, 4, 19, 8, 15, 12, 14, 9, 18, 5, 22, 1, \hookleftarrow)$$

is a directed R-terrace with the R-sequencing

$$(19, 10, 11, 18, 3, 23, 5, 16, 13, 8, 21, 24, 6, 15, 14, 7, 22, 2, 20, 9, 12, 17, 4, 1, \hookleftarrow).$$

In this paper we are only concerned with abelian groups of odd order. It is known that abelian groups of orders congruent to 1 or 5 $\pmod 6$ are R-sequenceable and several infinite families of R-sequenceable groups of order congruent to 3 $\pmod 6$ are known. Friedlander, Gordon and Miller conjecture (as part of a wider conjecture including the even case) that all abelian groups of odd order are R-sequenceable. We discuss these past results further in the next section and make significant progress with the unknown cases in Section 5, including finding R-sequencings for the following families of abelian groups of odd order:

- those which are isomorphic to a direct product of cyclic factors such that the number of factors of order congruent to 3 $\pmod 4$ of order at most 79 is at least as large as the number of such factors of order greater than 79,
- those whose Sylow 3-subgroups are of the form $\mathbb{Z}_3^\rho \times \mathbb{Z}_9^\rho \times \mathbb{Z}_{27}^\sigma \times \mathbb{Z}_{81}^\tau$,
- those whose Sylow 3-subgroups are of the form $\mathbb{Z}_3^\rho \times \mathbb{Z}_9^\rho \times \mathbb{Z}_{27}^\sigma \times \mathbb{Z}_{81}^\tau \times \mathbb{Z}_{3^k}$ provided that $k > 1$ and $k \equiv \rho + \sigma \pmod 2$.

We also show that for any abelian 3-group S there are infinitely many R-sequenceable abelian groups with Sylow subgroups isomorphic to S.

A *half-cycle* for an abelian group G of odd order $n = 2m+1$ is a circular sequence $\mathbf{a} = (a_1, a_2, \dots, a_m, \hookleftarrow)$ of distinct elements of G such that the sequence of differences $\mathbf{b} = (b_1, b_2, \dots b_m, \hookleftarrow)$ also contains no repeated elements. If we strengthen our definition to require that $\mathbf{a}$ and $\mathbf{b}$ each consist of exactly one occurrence from each pair $\{x, -x : x \neq e\}$, then $\mathbf{a}$ is a *strong half-cycle*.

The connection to complete mappings is similar to that for R-sequencings. Given a strong half-cycle with this notation, define a map $\phi : G \to G$ that fixes e, sends a_i to a_{i+1}, and $-a_i$ to $-a_{i+1}$ for each i. Then ϕ is a normalized orthomorphism.

Example 1.2 [18] *In* $\mathbb{Z}_{19}$

$$(1, 17, 3, 15, 14, 6, 12, 8, 10, \hookleftarrow)$$

is a strong half cycle with the differences

$$(16, 5, 12, 18, 11, 6, 15, 2, 10, \hookleftarrow).$$

Half-cycles for cyclic groups were investigated by Preece in [18], who credits the concept to Azaïs [3] and also cites Buratti and Del Fra's use of them as generators of cyclic m-cycle systems [5]. However the idea for arbitrary abelian groups can be traced back at least as far as a paper by Friedlander, Gordon and Tannenbaum [8] where they appear as m-regular complete mappings, where $m = (|G|-1)/2$ as above.

Preece also introduces two notions that are stronger still: "robust" and "champion" half-cycles. A strong half-cycle $(a_1, \dots, a_n, \hookleftarrow)$ with differences $(b_1, \dots, b_n, \hookleftarrow)$ is *robust* if the set $\{a_1, \dots, a_n\}$ is equal to either $\{b_1, \dots, b_n\}$ or $\{-b_1, \dots, -b_n\}$. A robust half cycle is *champion* if when $a_{i+c} = \pm b_i$ then $a_{i+c+j} = \pm b_{i+j}$ for all j.

Strong half-cycles are known to exist for cyclic groups and elementary abelian p-groups of odd order [8]; we shall see a construction from [5] that proves this for cyclic groups in Section 3. Friedlander, Gordon and Tannenbaum conjecture that all abelian groups of odd order have a strong half-cycle (as part of a more general conjecture about regular complete mappings). In the next three sections we make the first inroads of which we are aware into other cases, including constructing strong half-cycles for odd order abelian groups that can be written as the product of cyclic factors, all either of order congruent to $1 \pmod{12}$ or order at most 81 with order congruent to $1 \pmod 4$.

Preece [18] shows that cyclic groups of prime order have robust (in fact, champion) half-cycles. In Section 6 we report on computer searches for robust half-cycles, finding that they exist in cyclic groups all composite orders n with $21 \leq n \leq 69$. Given this, it seems likely that robust half-cycles exist for cyclic groups of all odd orders $n > 15$.

The unifying tool in our constructions is the "narcissistic terrace."

Let G be an abelian group of odd order $n = 2m+1$. A *terrace* for G is a linear arrangement $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ of the elements of G such that the associated sequence $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$, defined by $b_i = a_{i+1} - a_i$, has exactly two occurrences from each set $\{x, -x : x \neq e\}$. The sequence $\mathbf{b}$ is called a *2-sequencing* of G. If the 2-sequencing is equal to its reverse then it is called *reflective* and the terrace is *narcissistic.*

Terraces were introduced by Bailey [4] in order to determine when the rows and columns of the Cayley table of a group may be permuted to give a quasi-complete Latin square. Similar ideas had been used earlier by Williams [20] for cyclic groups and by Gordon [9] with the stronger condition that each non-identity element appears exactly once in the 2-sequencing (in which case it is a *sequencing*, the terrace is *directed*, and one may permute the rows and columns of the Cayley table to produce a complete Latin square).

Translating all of the elements of a terrace by adding a fixed element of the group to each produces another terrace with the same 2-sequencing. If the group has odd order (the only case with which we are concerned in this paper) we can therefore assume that the element in the center of the terrace is the identity and when this is the case we call the terrace *centered.* It is convenient to index a centered narcissistic terrace as $(a_{-m}, \ldots, a_{-1}, a_0, a_1, \ldots, a_m)$. When we do this we have that $a_0 = e$ and $a_{-i} = -a_i$ for each i. Given this, it is sufficient when describing a centered narcissistic terrace to list $(\ldots, a_0, a_1, \ldots, a_m)$, a practice we often employ. Similarly, the associated reflective 2-sequencing is fully captured by $(\ldots, b_1, b_2, \ldots, b_{m-1})$.

Example 1.3 *In* $\mathbb{Z}_{19}$*,*

$$(\ldots, 0, 10, 8, 12, 6, 14, 15, 3, 17, 1)$$

is a centered narcissistic terrace with the reflective 2*-sequencing*

$$(\ldots, 10, 17, 4, 13, 8, 1, 7, 14, 3).$$

"Zigzags" were explicitly defined and used in the construction of a wide array of terraces by Preece [17]. He notes that specific zigzag terraces have previously been constructed including the Lucas-Walecki-Williams terraces (described below), possibly the first instances of terraces (although with different terminology) in 1892 [11].

A *zigzag* in $\mathbb{Z}_n$ is defined by four numbers, which we list in angle brackets, $\langle x, y, s, l\rangle$ and is a sequence of length l where the ith entry is given by $y - s(i-2)/2$ if i is even and $x + s(i-1)/2$ if i is odd. Perhaps more intuitively, the zigzag $\langle x, y, s, l\rangle$ starts as

$$x, y, x + s, y - s, x + 2s, y - 2s, \dots$$

and ends with $y - s(l-2)/2$ if l is even and $x + s(l-1)/2$ if l is odd.

Example 1.4 *In $\mathbb{Z}_{25}$ the zigzag $\langle 1, 22, 4, 6\rangle$ is*

$$(1, 22, 5, 18, 9, 14)$$

and the one given by $\langle 12, 15, 21, 6\rangle$ is

$$(12, 15, 8, 19, 4, 23).$$

The ith internal difference produced by a zigzag is given by $y - x - s(i-1)$ if i is odd and $x - y + s(i-1)$ if i is even. That is, the sequence of differences is

$$y - x,\ -(y - x - s),\ y - x - 2s,\ -(y - x - 3s), \dots$$

The *Lucas-Walecki-Williams terrace* or *LWW terrace* for $\mathbb{Z}_n$ consists of the element 0 followed by a single zigzag:

$$(0, \langle 1, n-1, 1, n-1\rangle) = (0, 1, n-1, 2, n-2, \dots, \lceil n/2\rceil).$$

See [18], for example, for more details.

In the next section we review the results that allow us to construct R-sequencings from narcissistic terraces and introduce similar ones that allow us to construct strong half-cycles. In Sections 3 and 4 we give new direct constructions of narcissistic terraces for cyclic groups, using zigzags and number theoretic methods respectively. These terraces are of interest in their own right, extending results in [2, 17], but our attention here is on their usefulness in building R-sequencings and half-cycles.

In Section 5 we see what the implications are for the existence of R-sequencings and strong half-cycles, including both the results stated in the abstract and somewhat more general cases that are not easy to express succinctly.

2 Preliminaries

The method we will use requires finding narcissistic terraces for cyclic (or other) groups that have various properties that allow them to be turned into R-sequencings or strong half-cycles. This can be used in conjunction with the product theorem for narcissistic terraces from [13] to create narcissistic terraces with the required properties for larger, more complicated groups. With the exception of the strong half-cycle construction in Theorem 2.1, the material in this section is taken from [14] and the reader is referred there for proofs and constructions.

Let G be an abelian group of order $2m+1$ and let $\mathbf{a} = (\ldots, e, a_1, a_2, \ldots, a_m)$ be a centered narcissistic terrace for G. If $a_m = -2a_1$ then we say that $\mathbf{a}$ has *Property A* and if $a_m = 2a_1$ then we say that $\mathbf{a}$ has *Property B*.

Theorem 2.1 *Let G be an abelian group of odd order. If G has a centered narcissistic terrace with Property A then G has a directed R-terrace and if G has a centered narcissistic terrace with Property B then G has a strong half-cycle.*

Proof. Suppose $|G| = 2m+1$ and let $\mathbf{a} = (\ldots, e, a_1, a_2, \ldots, a_m)$ be a centered narcissistic terrace for G with 2-sequencing $\mathbf{b} = (\ldots, b_1, b_2, \ldots, b_{m-1})$.

Suppose that $\mathbf{a}$ has Property B and consider the sequence $(a_1, a_2, \ldots, a_m, \hookleftarrow)$. We claim that this is a strong half-cycle for G. The differences are $(b_2, b_3 \ldots, b_{m-1}, a_1 - a_m, \hookleftarrow)$. As

$$a_1 - a_m = -a_1 = -b_1$$

and $\mathbf{b}$ is a reflective 2-sequencing, these differences satisfy the strong half-cycle constraints.

If $\mathbf{a}$ has Property A then a similar argument shows that

$$(a_{-m}, a_{-(m-1)}, \ldots, a_{-1}, a_m, a_{m-1}, \ldots, a_1, \hookleftarrow)$$

is a directed R-terrace for G [14]. □

If we take the centered narcissistic terrace from Example 1.3 and apply this process to it we get the strong half-cycle from Example 1.2.

Theorem 2.2 [14] *Let $G_1, G_2, \ldots, G_\alpha$ and $H_1, H_2, \ldots, H_\beta$ be abelian groups with centered narcissistic terraces that have Property A, where $|G_i| \equiv 1 \pmod 4$ and $|H_j| \equiv 3 \pmod 4$ for each i and j. Let $J_1, J_2, \ldots, J_\gamma$ and $K_1, K_2, \ldots, K_\delta$ be abelian groups with centered narcissistic terraces that have Property B, where $|J_i| \equiv 1 \pmod 4$ and $|K_j| \equiv 3 \pmod 4$ for each i and j. Let*

$$L = \left(\prod_{i=1}^{\alpha} G_i\right) \times \left(\prod_{i=1}^{\beta} H_i\right) \times \left(\prod_{i=1}^{\gamma} J_i\right) \times \left(\prod_{i=1}^{\delta} K_i\right).$$

Then L has a terrace with Property A provided that $\alpha + \beta \geq 1$ and $\delta \in \{\beta - 1, \beta\}$, subject to the extra condition that if $\beta = 0$ then $\gamma = \delta = 0$. Further, L has a terrace with Property B provided that $\gamma + \delta \geq 1$ and $\beta \in \{\delta - 1, \delta\}$, subject to the extra condition that if $\delta = 0$ then $\alpha = \beta = 0$.

The last sentence of Theorem 2.2 regarding L having Property B is not explicitly stated in [14] but the argument is essentially the same as that for the Property A clause.

Therefore, finding centered narcissistic terraces with Property A or B leads to more constructions of R-sequencings and strong half-cycles. We might hope that the following corollary would reduce the problems to cyclic groups:

Corollary 2.3 [14] *Let G be an abelian group of odd order and suppose*

$$G \cong A_1 \times A_2 \times \cdots \times A_s.$$

If each A_i has both a centered narcissistic terrace with Property A and one with Property B then G also has both a centered narcissistic terrace with Property A and one with Property B.

However, $\mathbb{Z}_3$ and $\mathbb{Z}_5$ each have a centered narcissistic terrace with Property A but none with Property B and $\mathbb{Z}_7$ has a centered narcissistic terrace with Property B but not one with Property A. Further, elementary abelian 3-groups of order at least 9 do not have centered narcissistic terraces with either property.

If n is prime and x has multiplicative order $n-1$ in $\mathbb{Z}_n$ (considered as a ring) then x is a *primitive root* of n.

Other orders of cyclic groups for which there is a centered narcissistic terrace with Property A include: n prime with 2 as a primitive root; $n \equiv 5 \pmod 8$ and $n-2$ is prime with 2 as a primitive root; and n with $9 \leq n \leq 29$. For Property B the list is: $n \equiv 7 \pmod 8$ and $n-2$ is prime with 2 as a primitive root; $n \equiv 1 \pmod 8$ and $n-2$ is prime for which 2 has multiplicative order $(n-3)/2$; and n with $9 \leq n \leq 29$. Also $\mathbb{Z}_5^2$ and $\mathbb{Z}_9 \times \mathbb{Z}_3$ both have centered narcissistic terraces with each property [14].

Two additional constraints that a centered narcissistic terrace might meet let us join the approach for finding R-sequencings given here into other existing theory.

We call a sequence $(a_1, a_2, \ldots, a_n)$ or $(a_1, a_2, \ldots, a_n, \hookleftarrow)$ *starry* if $a_i = a_{i-1} + a_{i+1}$ for some a_i in the sequence. For example, the terrace $(\ldots, 0, 1, 12, 5, 8, 6, 11, 2)$ for $\mathbb{Z}_{15}$ is starry because $5 = 12 + 8$. If a centered narcissistic terrace has Property A (or B) and is also starry then we say it has *Property A** (or *Property B**). The terrace for $\mathbb{Z}_{15}$ here has Property B*.

A centered narcissistic terrace where $a_i = 2a_{i+1}$ for some a_i has the property of being *doublesome*. A centered narcissistic terrace satisfies the doublesome property in one half if and only if $a_j = 2a_{j-1}$ for some a_j in the other half. For example, the terrace $(\ldots, 0, 1, 10, 5, 7, 4, 11)$ for $\mathbb{Z}_{13}$ has this property because $2 \cdot 5 = 10$. If a centered narcissistic terrace has Property A (or B) and is also doublesome then we say it has *Property A*$_\dagger$ (or *Property B*$_\dagger$). The terrace for $\mathbb{Z}_{13}$ here has Property A$_\dagger$.

The next two results show how we can combine terraces with the various properties into ones that have Property A* or B*.

Theorem 2.4 [14] *Let $G_1, G_2, \ldots, G_\alpha$ and $H_1, H_2, \ldots, H_\beta$ be abelian groups with centered narcissistic terraces that have Property $A_\dagger$, where $|G_i| \equiv 1 \pmod 4$ and $|H_j| \equiv 3 \pmod 4$ for each i and j. Let $J_1, J_2, \ldots, J_\gamma$ and $K_1, K_2, \ldots, K_\delta$ be abelian groups with centered narcissistic terraces that have Property $B_\dagger$, where $|J_i| \equiv 1 \pmod 4$ and $|K_j| \equiv 3 \pmod 4$ for each i and j. Let*

$$L = \left(\prod_{i=1}^{\alpha} G_i\right) \times \left(\prod_{i=1}^{\beta} H_i\right) \times \left(\prod_{i=1}^{\gamma} J_i\right) \times \left(\prod_{i=1}^{\delta} K_i\right).$$

If $\alpha + \beta \geq 1$ and $\delta \in \{\beta - 1, \beta\}$, subject to the extra condition that if $\beta = 0$ then $\gamma = \delta = 0$, then L has a centered narcissistic terrace with Property $A_\dagger$. If $\gamma + \delta \geq 1$ and $\beta \in \{\delta - 1, \delta\}$, subject to the extra condition that if $\delta = 0$ then $\alpha = \beta = 0$, then L has a centered narcissistic terrace with Property $B_\dagger$.

Let M and N be abelian groups with centered narcissistic terraces that have Property A^* and B^* respectively. Then

- *If L has order congruent to $1 \pmod 4$ and its centered narcissistic terrace has Property $A_\dagger$, then $L \times M$ has a centered narcissistic terrace with Property A^*.*
- *If L has order congruent to $3 \pmod 4$ and its centered narcissistic terrace has Property $A_\dagger$, then $L \times N$ has a centered narcissistic terrace with Property A^*.*
- *If L has order congruent to $1 \pmod 4$ and its centered narcissistic terrace has Property $B_\dagger$, then $L \times N$ has a centered narcissistic terrace with Property B^*.*
- *If L has order congruent to $3 \pmod 4$ and its centered narcissistic terrace has Property $B_\dagger$, then $L \times M$ has a centered narcissistic terrace with Property B^*.*

A less general but more succinct version of Theorem 2.4 is:

Corollary 2.5 [14] *Let G be an abelian group of odd order and suppose*

$$G \cong A_1 \times A_2 \times \cdots \times A_s.$$

If each A_i, for $1 \leq i \leq s$, has both a centered narcissistic terrace with Property $A_\dagger$ and one with Property $B_\dagger$ and A_{s+1} has a centered narcissistic terrace with Property A^* and one with Property B^*, then G has a centered narcissistic terrace with Property A^* and one with Property B^*.

If a directed R-terrace is starry then it is called a *directed R^*-terrace* and its associated R-sequencing is an *R^*-sequencing.* The reason for the importance of Property A* is that from such terraces we can create directed R*-terraces and a result of Friedlander, Gordon and Miller lets us then show that many more abelian groups are R-sequenceable.

Theorem 2.6 [14] *Let G be an abelian group of odd order. If G has a centered narcissistic terrace with Property A^* then G has a directed R^*-terrace.*

Proof. Use the method of Theorem 2.1. □

Theorem 2.7 [7] *If G is an R^*-sequenceable abelian group of odd order and H is an abelian group of order coprime to* 6*, then $G \times H$ is R^*-sequenceable.*

We are therefore especially interested in starriness and doublesomeness when the order of the group is a multiple of 3 in order to be able to construct R*-sequencings for further classes of groups.

The groups $\mathbb{Z}_5$ and $\mathbb{Z}_7$ have centered narcissistic terraces with Properties $A_\dagger$ and $B_\dagger$ respectively and $\mathbb{Z}_9$ has both a centered narcissistic terrace with Property A^* and one with Property B^*. For odd n with $11 \leq n \leq 29$ each abelian group of order n other than $\mathbb{Z}_3^3$ has a centered narcissistic terrace with Property $A_\dagger^*$ and one with Property $B_\dagger^*$ [14].

Our goal now is to construct as many narcissistic terraces with the desired properties as we can.

3 Zigzag constructions

There are many constructions in the literature for terraces using zigzags, some described before zigzags were codified (such as the LWW terrace) and many appear in [17] where zigzags are explicitly introduced. Following the convention from [17] we talk about a terrace with four zigzags as *tetrazetal*, one with five zigzags as *pentazetal*, and so on.

In this section we present two new constructions for families of centered narcissistic terraces using zigzags, one tetrazetal and one hexazetal. First, the tetrazetal one:

Theorem 3.1 *If $n \equiv 1 \pmod 4$ then $\mathbb{Z}_n$ has a tetrazetal centered narcissistic terrace with Property A. If $n \equiv 3 \pmod 4$ then $\mathbb{Z}_n$ has a tetrazetal centered narcissistic terrace with Property B.*

Proof. Define a sequence as follows. This sequence will be our tetrazetal terrace. There are slightly different formulations as n varies modulo 8. We use the zigzag notation introduced in Section 1.

When $n = 8k + 1$:

$$(\ldots 0, \langle 1, 8k-2, 4, 2k \rangle, \langle 4k, 4k+3, -4, 2k \rangle)\,.$$

When $n = 8k + 3$:

$$(\ldots 0, \langle 1, 8k, 4, 2k+1 \rangle, \langle 4k+3, 4k-2, 4, 2k \rangle)\,.$$

When $n = 8k + 5$:

$$(\ldots 0, \langle 1, 8k+2, 4, 2k+1 \rangle, \langle 4k+3, 4k, 4, 2k+1 \rangle)\,.$$

When $n = 8k + 7$:

$$(\ldots 0, \langle 1, 8k+5, 4, 2k+2\rangle, \langle 4k+2, 4k+7, -4, 2k+1\rangle).$$

These constructions produce narcissistic terraces. Considering the zigzags in all but the $n = 8k+7$ case, in the first zigzag we have all of the numbers (or their negatives) following the pattern $1, -3, \ldots$ up to $\pm(n-1)/2$ or $\pm(n-3)/2$ and in the second zigzag we have all of the numbers (or their negatives) from the other of $\pm(n-1)/2$ and $\pm(n-3)/2$ and ending $\ldots, \pm 4, \pm 2$. For $n = 8k+7$ the situation is very similar but the first zigzag also includes $(n+1)/2$ (the negative of $(n-1)/2$) and the second zigzag starts at $\pm(n-5)/2$ instead. Hence the lists contain each element of $\mathbb{Z}_n$ exactly once.

Now consider the differences. In each case we have: 1 as the first difference; ± 2 as the difference that comes from the join between the first and second zigzags; $\pm 4, \pm 8 \ldots$ up to either $(n-3)$ or $(n-5)$ as the internal differences of the first zigzag; whichever of $(n-3)$ and $(n-5)$ that did not appear in the first zigzag differences down to $\ldots \pm 10, \pm 6$ as the internal differences of the second zigzag.

Therefore we do indeed have a centered narcissistic terrace for $\mathbb{Z}_n$.

Each of the terraces has $a_1 = 1$. When $n \equiv 1 \pmod 4$ the terraces have $a_{(n-1)/2} = -2$ and so these have Property A. When $n \equiv 3 \pmod 4$ they have $a_{(n-1)/2} = 2$ and so have Property B. □

We refer to the terrace constructed in the proof of Theorem 3.1 as the *tetrazetal terrace* for $\mathbb{Z}_n$.

For $n \equiv 3 \pmod 4$ the tetrazetal terraces for $\mathbb{Z}_n$ give strong half-cycles via Theorem 2.1 that are equivalent to the direct construction of strong half-cycles in [18]. The terraces are also equivalent to the "double hiccup" terraces of [17].

When $n \equiv 3 \pmod 6$ the tetrazetal terrace is starry: for $n > 9$ we have that the entry in position $(n+3)/6$ is the sum of its neighbours. For $n = 9$ the terrace is

$$(\ldots 0, 1, 6, 4, 7)$$

which is starry because $6 + 7 = 4$.

When $n \equiv 5 \pmod 6$ the tetrazetal terrace is doublesome: the relevant entries are in positions $(n+1)/6$ and $(n+7)/6$.

The second construction gives hexazetal centered narcissistic terraces:

Theorem 3.2 *If* $n \equiv 1 \pmod 6$ *then* $\mathbb{Z}_n$ *has a hexazetal centered narcissistic terrace with Property B.*

Proof. Define a sequence as follows. This sequence will be our hexazetal terrace. As in the proof of Theorem 3.1, there are slightly different formulations as n varies modulo 8.

When $n = 24k + 1$:

$$(\ldots 0, \langle 8k, 16k+2, -2, 2k+1\rangle, \langle 6k-1, 18k+3, -2, 6k-1\rangle, \langle 12k, 12k+2, -2, 4k\rangle)\,.$$

When $n = 24k + 7$:

$$(\ldots 0, \langle 8k+2, 18k+6, -2, 2k+1\rangle, \langle 6k+1, 18k+7, -2, 6k+1\rangle, \\ \langle 12k+4, 12k+2, 2, 4k+1\rangle)\,.$$

When $n = 24k + 13$:

$$(\ldots 0, \langle 8k+4, 16k+10, -2, 2k+2\rangle, \langle 18k+11, 6k+1, 2, 6k+2\rangle, \\ \langle 12k+6, 12k+8, -2, 4k+2\rangle)\,.$$

When $n = 24k + 19$:

$$(\ldots 0, \langle 8k+6, 16k+14, -2, 2k+2\rangle, \langle 18k+15, 6k+3, 2, 6k+4\rangle, \\ \langle 12k+10, 12k+8, 2, 4k+3\rangle)\,.$$

The first two zigzags contain the elements

$$(n-1)/3, -(n-4)/3, \ldots, 1$$

and the final zigzag gives us

$$\pm(n-1)/2, \pm(n-3)/2, \ldots, -(n+2)/3.$$

For the differences, in the first zigzag we get

$$(n-1)/3, (n+5)/3, \ldots \pm (n-3)/2$$

when $n \equiv 1, 13 \pmod{24}$ and

$$(n-1)/3, (n+5)/3, \ldots \pm (n-1)/2$$

when $n \equiv 7, 19 \pmod{24}$.
In the second zigzag we get

$$\pm(n+5)/2, \pm(n+9)/2, \ldots, -(n-3)$$

when $n \equiv 1, 13 \pmod{24}$ and

$$\pm(n+7)/2, \pm(n+11)/2, \ldots, -(n-3)$$

when $n \equiv 7, 19 \pmod{24}$.

In the final zigzag we get

$$\pm 2, \pm 4, \dots, (n-7)/3$$

and at the two joins we get 1 and then $-(n+1)/2$ if $n \equiv 1, 13 \pmod{24}$ and $-(n+3)/2$ when $n \equiv 7, 19 \pmod{24}$. Hence we do have a centered narcissistic terrace.

As $-(n+2)/3 = (2n-2)/3$ these centered narcissistic terraces have Property B. □

We refer to the terrace constructed in the proof of Theorem 3.2 as the *hexazetal terrace* for $\mathbb{Z}_n$.

Example 3.3 *For* $\mathbb{Z}_{25}$ *the tetrazetal terrace is*

$$(\dots, 0, 1, 22, 5, 18, 9, 14, 12, 15, 8, 19, 4, 23)$$

and the hexazetal terrace is

$$(\dots, 0, 8, 18, 6, 5, 21, 3, 23, 1, 12, 14, 10, 16).$$

The former has Property $A_\dagger$ *and the latter has Property B.*

4 Number theoretic constructions

There are many constructions in the literature for strong half-cycles and narcissistic terraces that rely on number theoretic properties. In this section we show how one of the longest known strong half-cycle constructions gives centered narcissistic terraces with Property $B_\dagger$ for many primes and see how a method for constructing narcissistic terraces of order three times a prime can be extended to give results of use here.

An element $x \in \mathbb{Z}_p$ (considered as a ring) is *negating* if $p-1 \in \langle x \rangle$ and is *non-negating* otherwise.

A strong half-cycle with a doublesome point is equivalent to a centered narcissistic terrace with Property B via the proof of Theorem 2.1 and if it has more than one doublesome point then we have Property $B_\dagger$.

The theorems of [18, Section 3] give "robust chaplets" for various orders; when the order is prime the notions of a chaplet and a half-cycle coincide, so the results give robust (and hence strong) half-cycles in that case. Two are of use to us, although the second does not cover any values not already covered by the first.

Theorem 4.1 *Let* $p > 3$ *be an odd prime congruent to* $3 \pmod 4$ *such that* 2 *is a non-negating element of* $\mathbb{Z}_p$ *of order* $(p-1)/2$. *Then* $\mathbb{Z}_p$ *has a centered narcissistic terrace with Property* $B_\dagger$.

Proof. Theorem 3.1 of [18], where credit is given to Rees [19], with $x = 2$ shows that the sequence $(2^0, 2^1, \dots, 2^{(p-3)/2}, \hookleftarrow)$ is a strong half-cycle for $\mathbb{Z}_p$. It is clearly doublesome at every point. □

The values covered by Theorem 4.1 up to 600 are:

$$7, 23, 47, 71, 79, 103, 167, 191, 199, 239, 263, 271, 311, 359, 367, 383, 463, 479, 487, 503, 599.$$

Theorem 3.2 of [18] gives alternative constructions of centered narcissistic terraces with Property $B_\dagger$ for $\mathbb{Z}_p$ when $p < 600$ for the values:

$$71, 79, 103, 191, 199, 239, 271, 311, 367, 463, 599.$$

Next we turn to the methods of [2, Section 2], where narcissistic terraces for $\mathbb{Z}_{3p}$, where p is prime, are constructed. The constructions there are all of the form $(\ldots, 0, \pm p, \ldots, 3k)$ for some k and so cannot have Property A or B. However, we extract the construction of the central piece of each of their terraces, which deals primarily with the elements that are not multiples of 3, and then build from that in a different way.

Lemma 4.2 *Let $p > 3$ be an odd prime and set $n = 3p$. If there is a non-negating λ of order $p-1 \pmod{n}$ with $\lambda \equiv 2 \pmod 3$ and $\lambda \neq 2$ then there is a sequence of elements of $\mathbb{Z}_n$ such that*

- *the first and last elements, x and y, are multiples of 3 and $x \neq y$,*
- *the elements other than the first and last have exactly one occurrence from each set $\{g, -g : 3 \nmid g\}$,*
- *the internal differences are $x + y$ and exactly one occurrence from each set $\{g, -g : 3 \nmid g\}$.*

Proof. Suppose first that $p \equiv 2 \pmod 3$. Our sequence is

$$x, p, \lambda, \lambda^2, \lambda^3, \ldots, \lambda^{p-1}, y$$

where we set $x = \lambda + p - 1$ and $y = p + 1$. As p and λ are congruent to $2 \pmod 3$, both x and y are multiples of 3. As $\lambda \neq 2$ we have $x \neq y$.

The properties of λ ensure that there is exactly one occurrence from each set of the form $\{g, -g : 3 \nmid g, p \nmid g\}$ in the sequence and that the other internal element is p validates the second item.

Similarly, the properties of λ give exactly one occurrence as differences from each set $\{g, -g : 3 \nmid g, p \nmid g\}$, with the exception of $\pm(\lambda - \lambda^{p-1}) = \pm(\lambda - 1)$. The three unaccounted for differences are

$$p - x = p - (\lambda + p - 1) = -(\lambda - 1),$$

$$\lambda - p = (\lambda + p - 1) + (p + 1) = x + y$$

and

$$y - \lambda^{p-1} = y - 1 = p$$

fulfilling the requirements of the third item.

In the case $p \equiv 1 \pmod 3$ a similar argument shows that the sequence

$$x, 2p, \lambda, \lambda^2, \lambda^3, \ldots, \lambda^{p-1}, y$$

with $x = \lambda - p - 1$ and $y = 2p + 1$ suffices. □

We now use the sequence from Lemma 4.2 as a building block for a narcissistic terrace.

Theorem 4.3 *Let $n = 3p$ where $p > 3$ is prime and let λ, x and y be as in Lemma 4.2 with $\phi : \mathbb{Z}_p \to \mathbb{Z}_{3p}$ an injective homomorphism. If there is a centered narcissistic terrace* $\mathbf{a}$ *for $\mathbb{Z}_p$ such that $\phi^{-1}(x)$ and $\phi^{-1}(-y)$ are adjacent then there is a centered narcissistic terrace* $\mathbf{c}$ *for $\mathbb{Z}_n$.*

Further, if $\mathbf{a}$ *has Property A (respectively B) then* $\mathbf{c}$ *has Property B (respectively A) and if* $\mathbf{a}$ *is starry or doublesome then so is* $\mathbf{c}$*, with the exception of the case where both elements $\phi^{-1}(x)$ and $\phi^{-1}(-y)$ are required to meet the definition of starriness or doublesomeness.*

Proof. Let $\mathbf{a} = (\ldots, 0, a_1, a_2, \ldots, a_{(p-1)/2})$, with $a_i = \phi^{-1}(x)$ and $a_{i+1} = \phi^{-1}(-y)$ for some $i > 0$. Consider

$$\mathbf{c} = (\ldots, 0, \phi(a_1), \ldots, \phi(a_i), \delta p, \lambda, \lambda^2, \ldots, \lambda^{p-1}, \phi(-a_{i+1}), \phi(-a_{i+2}), \ldots, \phi(-a_{(p-1)/2}))$$

where $\delta = 1$ or 2 according as p is congruent to 2 or 1 $\pmod 3$ respectively.

It follows from the proof of Lemma 4.2 that the subsequence

$$\phi(a_i), \delta p, \lambda, \lambda^2, \ldots, \lambda^{p-1}, \phi(-a_{i+1})$$

has exactly one occurrence from each set $\{g, -g : 3 \nmid g\}$ among the elements other than $\phi(a_i)$ and $\phi(-a_{i+1})$ and the internal differences are $\phi(a_i)+\phi(-a_{i+1}) = -\phi(a_{i+1} - a_i)$ and exactly one occurrence from each set $\{g, -g : 3 \nmid g\}$.

As ϕ is a homomorphism and $\mathbf{a}$ is a centered narcissistic terrace, the required elements that are a multiple of 3 appear in both the sequence and the differences. Note that the missing $\phi(a_{i+1} - a_i)$ difference is replaced with $-\phi(a_{i+1} - a_i)$, as per the previous paragraph.

In the case $a_{i+1} = \phi^{-1}(x)$ and $a_i = \phi^{-1}(-y)$ we may instead use

$$\mathbf{c} = (\ldots, 0, \phi(-a_1), \ldots, \phi(-a_i), \lambda^{p-1}, \lambda^{p-2}, \ldots, \lambda, \delta p, \phi(a_{i+1}), \phi(a_{i+2}), \ldots, \phi(a_{(p-1)/2}))$$

and if $\phi^{-1}(x)$ and $\phi^{-1}(-y)$ are adjacent in the first half of the centered narcissistic terrace then reversing it moves them to the second half. □

Corollary 4.4 *Let $p > 3$ be prime. Suppose there exist λ, x and y as described in Lemma 4.2 such that $2x \equiv -y \pmod{3p}$ or $x \equiv -2y \pmod{3p}$. If 2 is a primitive root of p then $\mathbb{Z}_{3p}$ has a centered narcissistic terrace with Property $B_\dagger$. If p is congruent to $3 \pmod 4$ and 2 is a non-negating element of order $(p-1)/2$, then $\mathbb{Z}_{3p}$ has a centered narcissistic terrace with Property $A_\dagger$.*

Proof. For the first assertion, Theorem 2.7 of [14] gives a centered narcissistic terrace for $\mathbb{Z}_p$ such that any pair of elements with one double the other appear as neighbours (if either element is ± 1, an automorphism needs to be applied to achieve this). This terrace has Property $A_\dagger$. Applying Theorem 4.3 we find the required centered narcissistic terrace for $\mathbb{Z}_{3p}$ that has Property $B_\dagger$.

For the second assertion, the terrace of Theorem 4.1 above works similarly. □

Example 4.5 *Consider $\mathbb{Z}_{39}$. As 2 is a primitive root of 13 we have a centered narcissistic terrace with Property $A_\dagger$ for the cyclic subgroup of order 13:*

$$(\ldots, 0, 36, 18, 9, 24, 12, 6).$$

Further, we may take $\lambda = 20$, $x = 6$ and $y = 27 = -12$ to give the following centered narcissistic terrace with Property $B_\dagger$ for $\mathbb{Z}_{39}$:

$$(\ldots, 0, 3, 21, 30, 15, 27, 1, 2, 4, 8, 16, 32, 25, 11, 22, 5, 10, 20, 26, 6).$$

Here are the values of p up to 600 that may be used in Corollary 4.4 to give a centered narcissistic terrace with Property $B_\dagger$ for $\mathbb{Z}_{3p}$, along with the values of λ, x and y:

p	13	29	37	53	61	101	149	173	181	197	269
λ	20	44	56	80	92	152	224	260	272	296	404
x	6	72	18	132	30	252	372	432	90	492	672
y	27	30	75	54	123	102	150	174	363	198	270

p	293	317	349	373	389	421	461	509	541	557
λ	440	476	524	560	584	632	692	764	812	836
x	732	792	174	186	972	210	1152	1272	270	1392
y	294	318	699	747	390	843	462	510	1083	558

And here are the corresponding tables for Property $A_\dagger$:

p	7	23	47	71	79	103	167	191	199	239	263
λ	11	35	71	107	119	155	251	287	299	359	395
x	3	57	117	177	39	51	417	477	99	597	657
y	15	24	48	72	159	207	168	192	399	240	264

p	271	311	359	367	383	463	479	487	503	599
λ	407	467	539	551	575	695	719	731	755	899
x	135	777	897	183	957	231	1197	243	1257	1497
y	543	312	360	735	384	927	480	975	504	600

5 R-sequencings and strong half-cycles

In this section we investigate how the constructions of the last two sections connect with the results of Section 2. Before doing so, Theorem 5.1 summarizes the results of various computer searches.

Theorem 5.1 *Let n be in the range $11 \leq n \leq 81$. Then $\mathbb{Z}_n$ has a centered narcissistic terrace with Property A^*, one with $A_\dagger$, one with B^* and one with $B_\dagger$.*

Proof. For $n < 30$ the required terraces are given in [14]. Those for $n > 30$ were found by computer search and the file with the terraces may be accessed at:

`http://cs.marlboro.edu/courses/matt/narcissistic`

Many targets were covered by the theory of the last two sections and most of the remainder were constructed as hexazetal terraces. The last few were found by a straight exhaustive search. □

As the case $n = 81$ is of particular interest, we give the terraces at that order here:

Example 5.2 *Centered narcissistic terraces for $\mathbb{Z}_{81}$, with Property $A^*_\dagger$ and $B^*_\dagger$ respectively:*

$$(\ldots, 0, 1, 3, 6, 11, 5, 53, 17, 8, 74, 60, 35, 67, 63, 37, 47, 12, 59, 71, 31, \ldots$$
$$\ldots 39, 55, 33, 77, 49, 68, 45, 56, 29, 16, 66, 27, 57, 19, 43, 72, 51, 58, 41, 61, 79)$$

$$(\ldots, 0, 1, 3, 6, 11, 5, 46, 19, 23, 7, 49, 17, 24, 71, 45, 22, 31, 66, 77, 25, \ldots$$
$$\ldots 37, 68, 40, 65, 8, 52, 33, 55, 42, 27, 47, 14, 28, 18, 61, 69, 51, 72, 21, 38, 2)$$

The tetrazetal terrace given in Section 3 also has Property A^*.

We now turn to R-sequencings. This first result does not use starriness or doublesomeness of any sequences in its proof:

Theorem 5.3 *Let L be an abelian group written as a product of cyclic groups as follows:*

$$\left(\prod_{i=1}^{\alpha} G_i\right) \times \left(\prod_{i=1}^{\beta} H_i\right) \times \left(\prod_{i=1}^{\gamma} J_i\right) \times \left(\prod_{i=1}^{\delta} K_i\right)$$

where $\alpha + \beta \geq 1$ and $\delta \in \{\beta - 1, \beta\}$, subject to the extra condition that if $\beta = 0$ then $\gamma = \delta = 0$, and

- *for each G_i: $|G_i| \equiv 1 \pmod 4$,*
- *for each H_i: $|H_i| \leq 79$ and $|H_i| \equiv 3 \pmod 4$,*
- *for each J_i: $|J_i| = 3p$ for a prime $p \equiv 3 \pmod 4$ for which the conditions of Corollary 4.4 apply, or $|J_i| \equiv 1 \pmod{12}$, or $|J_i| \leq 81$ and $|J_i| \equiv 1 \pmod 4$,*
- *for each K_i: $|K_i| \equiv 3 \pmod 4$.*

Then L is R-sequenceable.

Proof. Apply Theorem 2.2 to the centered narcissistic terraces for cyclic groups constructed in [14], the previous two sections and Theorem 5.1. □

A slightly less general, but much cleaner, result that is an immediate consequence of Theorem 5.3 is:

Corollary 5.4 *Let L be an abelian group written as a direct product of cyclic groups. Among those cyclic groups of order congruent to $3 \pmod 4$, let r be the number that have order at most 79 and s be the number that have order greater than 79. If $r \geq s$ then L is R-sequenceable.*

In particular, if L can be written as the product of cyclic groups, each of which has order congruent to $1 \pmod 4$, then L is R-sequenceable.

Proof. If L is cyclic then it is known to be R-sequenceable by the results of [7]. Otherwise, apply Theorem 5.3 with the cyclic groups of order congruent to $1 \pmod 4$ as the G_i, the $r+s$ cyclic groups of order congruent to $3 \pmod 4$ as the H_i and K_i. □

Let $m, n \equiv 3 \pmod 4$, with m and n coprime. Noting that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ and $mn \equiv 1 \pmod 4$ gives this corollary much wider scope than it might at first appear.

Among groups of odd order it is those whose Sylow 3-subgroups are non-cyclic for which the question of R-sequenceability remains open. Many such groups are given by Theorem 5.3 (or Corollary 5.4) The next three results give some more progress on this front.

In [14] it was shown that for every abelian 3-group S with exponent at most 3^{12} there are infinitely many R-sequenceable abelian groups with Sylow 3-subgroups isomorphic to S. We are now able to both drop the condition on the exponent and give a much simpler proof.

Corollary 5.5 *Let S be an abelian 3-group. There are infinitely many R-sequenceable abelian groups with Sylow 3-subgroups isomorphic to S.*

Proof. Write S as the direct product of cyclic groups, each of order a power of 3. As in the statement of Corollary 5.4, among those cyclic groups in the direct product of order congruent to $3 \pmod 4$, let r be the number that have order at most 79 and s be the number that have order greater than 79.

If $r \geq s$ the corollary immediately gives that $S \times T$ has an R-sequencing for any cyclic groups T with $|T| \equiv 1 \pmod 4$. There are infinitely many such T that have order that is not a multiple of 3, proving the result in this case.

Now suppose that $r < s$. Consider $S \times \mathbb{Z}_{11}^{s-r} \times T$ where again T is any cyclic group with $|T| \equiv 1 \pmod 4$. Again, Corollary 5.4 tells us that such groups have R-sequencings, and again there are infinitely many such T that have order that is not a multiple of 3 as required. □

As well as the one given in the proof of Corollary 5.5, there are many other ways to use Corollary 5.4 to produce R-sequenceable groups with a given Sylow 3-subgroup. For example, one may add as many additional cyclic groups of order congruent to 1 (mod 4) (and order not a multiple of 3) as one wishes.

Next we consider R*-sequenceability.

Theorem 5.6 *Let L be an abelian group written as a product of cyclic groups as follows:*

$$L = \left(\prod_{i=1}^{\alpha} G_i \right) \times \left(\prod_{i=1}^{\beta} H_i \right) \times \left(\prod_{i=1}^{\gamma} J_i \right) \times \left(\prod_{i=1}^{\delta} K_i \right).$$

where $\alpha + \beta \geq 1$ and $\delta \in \{\beta - 1, \beta\}$, subject to the extra condition that if $\beta = 0$ then $\gamma = \delta = 0$, with:

- *for each G_i: $|G_i| \equiv 1, 5 \pmod{12}$, $|G_i| = 3p$ for a prime $p \equiv 3 \pmod 4$ for which the conditions of Corollary 4.4 apply, or $21 \leq |G_i| \leq 81$ and $|G_i| \equiv 9 \pmod{12}$,*
- *for each H_i: $|H_i| \leq 79$ and $|H_i| \equiv 3 \pmod 4$,*
- *for each J_i: $|J_i| = 3p$ for a prime $p \equiv 3 \pmod 4$ for which the conditions of Corollary 4.4 apply, or $|J_i| = 5$, or $13 \leq |J_i| \leq 81$ and $|J_i| \equiv 1 \pmod 4$,*
- *for each K_i: $|K_i| \equiv 7, 11 \pmod{12}$, $|K_i|$ is an odd prime with $|K_i| \equiv 3 \pmod{12}$ such that 2 is a non-negating element of order $(|K_i| - 1)/2$, or $|K_i| = 3p$ for a prime $p \equiv 1 \pmod 4$ for which the conditions of Corollary 4.4 apply, or $|J_i| \leq 71$ and $|J_i| \equiv 11 \pmod{12}$.*

Let M be a cyclic group with $|M| \equiv 9 \pmod{12}$ or $11 \leq |M| \leq 81$ and let N be a cyclic group with $|N| \equiv 3 \pmod{12}$ or $11 \leq |N| \leq 81$.

If L has order congruent to $1 \pmod 4$ then $L \times M$ is R-sequenceable. If L has order congruent to $3 \pmod 4$ then $L \times N$ is R*-sequenceable.*

Proof. We can construct a centered narcissistic terrace with Property A* using Theorem 2.4 and the constructions of [14] and the previous two sections. □

Extracting the results pertaining to 3-groups we find:

Corollary 5.7 *Let G be an abelian group of odd order whose Sylow 3-subgroups are of the form*

$$S \cong \mathbb{Z}_3^\rho \times \mathbb{Z}_9^\rho \times \mathbb{Z}_{27}^\sigma \times \mathbb{Z}_{81}^\tau$$

or

$$S \cong \mathbb{Z}_3^\rho \times \mathbb{Z}_9^\rho \times \mathbb{Z}_{27}^\sigma \times \mathbb{Z}_{81}^\tau \times \mathbb{Z}_{3^k}$$

where $k > 1$ and $k \equiv \rho + \sigma \pmod 2$. Then G is R-sequenceable.*

Proof. The Sylow subgroup S has a R*-sequencing by Theorem 5.6 and the constructions of [14]. Hence G has an R*-sequencing by Theorem 2.7. □

Turning to strong half-cycles, we do not have the benefit of existing theory that lets us use the starry property to extend beyond what we can construct directly from our centered narcissistic terraces with Properties A and B. The next result, parallel to Theorem 5.3 for R-sequencings, captures what we can construct, which we believe to be the first result regarding these structures in groups other than cyclic or elementary abelian ones:

Theorem 5.8 *Let L be an abelian group written as a product of cyclic groups as follows:*

$$\left(\prod_{i=1}^{\alpha} G_i\right) \times \left(\prod_{i=1}^{\beta} H_i\right) \times \left(\prod_{i=1}^{\gamma} J_i\right) \times \left(\prod_{i=1}^{\delta} K_i\right)$$

where $\gamma + \delta \geq 1$ and $\beta \in \{\delta - 1, \delta\}$, subject to the extra condition that if $\delta = 0$ then $\alpha = \beta = 0$, with:

- *for each G_i: $|G_i| \equiv 1 \pmod 4$,*
- *for each H_i: $|H_i| \leq 79$ and $|H_i| \equiv 3 \pmod 4$,*
- *for each J_i: $|J_i| = 3p$ for a prime $p \equiv 3 \pmod 4$ for which the conditions of Corollary 4.4 apply, or $|J_i| \equiv 1 \pmod{12}$, or $|J_i| \leq 81$ and $|J_i| \equiv 1 \pmod 4$,*
- *for each K_i: $|K_i| \equiv 3 \pmod 4$.*

Then L has a strong half-cycle.

Proof. Exactly as in Theorem 5.3, we apply Theorem 2.2 to the centered narcissistic terraces for cyclic groups constructed in [14] and the previous two sections. □

Mimicking Corollary 5.4 we get:

Corollary 5.9 *Let L be an abelian group written as a direct product of cyclic groups such that there is at least one cyclic group of order congruent to $3 \pmod 4$. Among those cyclic groups of order congruent to $3 \pmod 4$, let r be the number that have order at most 79 and s be the number that have order greater than 79. If $r \geq s$ then L is R-sequenceable.*

Alternatively, if L is the product of cyclic groups, each either of order congruent to $1 \pmod{12}$ or order at most 81 with order congruent to $1 \pmod 4$, then L has a strong half-cycle.

Finally for this section we give a more concise (but much less comprehensive) result that stems from Corollary 2.3:

Corollary 5.10 *Suppose L is expressible as a direct product of cyclic groups of odd order, each of which has order congruent to* $1 \pmod{12}$ *or order between* 9 *and* 81 *(inclusive). Then L has a strong half-cycle and is R-sequenceable.*

Proof. For the cyclic group of each these orders we have both a centered narcissistic terrace with Property A and one with Property B. Apply Corollary 2.3. □

6 Robust half-cycles

In this section we examine robust half-cycles. Preece proved the existence of robust half-cycles for cyclic groups of all prime orders less than 300 except possibly for 17, 193, and 257 [18].

Preece did not consider robust half-cycles for cyclic groups of composite order. We examined small cyclic groups of composite order and found that neither $\mathbb{Z}_9$ nor $\mathbb{Z}_{15}$ have a robust half-cycle but that there are robust half-cycles in cyclic groups of each composite order from 21 up to 69 where we stopped our search. The results for these orders can be found in Table 1.

Table 1: Some robust half-cycles for cyclic groups of composite order, up to order 69.

Order	Robust Half-Cycle
21	$(1, 3, 13, 9, 14, 11, 19, 4, 16, 15, \hookleftarrow)$
25	$(1, 3, 8, 4, 20, 10, 9, 6, 23, 11, 18, 12, \hookleftarrow)$
27	$(1, 3, 7, 12, 23, 17, 16, 9, 6, 25, 8, 22, 13, \hookleftarrow)$
33	$(1, 3, 7, 6, 11, 28, 19, 13, 31, 8, 29, 16, 9, 23, 15, 12, \hookleftarrow)$
35	$(1, 3, 7, 6, 11, 23, 17, 14, 33, 22, 30, 16, 31, 9, 27, 20, 10, \hookleftarrow)$
39	$(1, 3, 7, 6, 11, 8, 18, 35, 27, 9, 29, 22, 13, 25, 19, 34, 23, 37, 24, \hookleftarrow)$
45	$(1, 3, 7, 6, 11, 8, 17, 10, 22, 16, 40, 27, 13, 41, 30, 14, 33, 25, 43, 21, 36, 26, \hookleftarrow)$
49	$(1, 3, 7, 6, 11, 8, 17, 10, 22, 16, 30, 15, 40, 20, 47, 26, 45, 35, 24, \ldots$ $\ldots 37, 21, 44, 36, 18, \hookleftarrow)$
51	$(1, 3, 7, 6, 11, 8, 17, 10, 22, 14, 39, 26, 16, 46, 24, 47, 31, 49, 32, \ldots$ $\ldots 21, 15, 42, 28, 13, 33, \hookleftarrow)$
55	$(1, 3, 7, 6, 11, 8, 17, 10, 22, 14, 27, 16, 42, 15, 51, 31, 25, 46, 29, \ldots$ $\ldots 53, 43, 20, 50, 34, 19, 37, 23, \hookleftarrow)$
57	$(1, 3, 7, 6, 11, 8, 17, 10, 22, 14, 27, 16, 44, 29, 52, 21, 45, 20, 55, \ldots$ $\ldots 34, 18, 48, 31, 25, 15, 53, 33, 19, \hookleftarrow)$
63	$(1, 3, 7, 6, 11, 8, 17, 10, 22, 14, 27, 16, 31, 21, 40, 18, 54, 48, 24 \ldots$ $\ldots 61, 43, 26, 58, 44, 28, 51, 30, 50, 25, 59, 29, \hookleftarrow)$
65	$(1, 3, 7, 6, 11, 8, 17, 10, 22, 14, 27, 16, 31, 21, 50, 20, 52, 30, 56, \ldots$ $\ldots 33, 61, 41, 23, 63, 36, 60, 39, 25, 19, 53, 37, 18, \hookleftarrow)$
69	$(1, 3, 7, 6, 11, 8, 17, 10, 22, 14, 27, 16, 31, 21, 39, 19, 54, 33, 65, \ldots$ $\ldots 43, 37, 23, 67, 51, 20, 56, 29, 57, 34, 60, 41, 24, 64, 25, \hookleftarrow)$

Via an exhaustive computer search we know that there is no robust half-cycle for $\mathbb{Z}_{17}$. For $\mathbb{Z}_{193}$ and $\mathbb{Z}_{257}$ we looked for a robust half-cycle made up of sequences of the form

$$x\lambda^0, x\lambda^1, x\lambda^2, \ldots, x\lambda^{k-1}$$

for fixed λ and k, where k is a divisor of the multiplicative order of λ (these are called *power sequences*). All of the constructions in [18] for robust half-cycles have this form. However, we were unable to find any such half-cycles. If they exist for either of these groups it must be the case that $k \leq 8$.

Acknowledgements

We gratefully acknowledge the Marlboro College Faculty Professional Development Grant and the Marlboro College Town Meeting Scholarship Fund award that made this work possible.

Comments from the anonymous referees have led to significant improvements in the presentation of these results.

References

[1] A. Ahmed, M. I. Azimli, I. Anderson and D. A. Preece, Rotational terraces from rectangular arrays, *Bull. Inst. Combin. Applic.* **63** (2011), 4–12.

[2] I. Anderson and D. A. Preece, Narcissistic half-and-half power-sequence terraces for $\mathbb{Z}_n$ with $n = pq^t$, *Discrete Math.* **279** (2004), 33–60.

[3] J.-M. Azaïs, Design of experiments for studying intergenotypic competition, *J. Royal Statist. Soc. B* **49** (1987), 334–345.

[4] R. A. Bailey, Quasi-complete Latin squares: construction and randomization, *J. Royal Statist. Soc. Ser. B* **46** (1984), 323–334.

[5] M. Buratti and A. Del Fra, Existence of cyclic k-cycle systems of the complete graph, *Discrete Math.* **261** (2003), 113–125.

[6] A. B. Evans, Complete mappings and sequencings of finite groups, *The CRC Handbook of Combinatorial Designs, 2nd Ed.* (Eds. C. J. Colbourn and J. H. Dinitz), CRC Press (2007), 345–352.

[7] R. J. Friedlander, B. Gordon and M. D. Miller, On a group sequencing problem of Ringel, *Congr. Numer.* **21** (1978), 307–321.

[8] R. J. Friedlander, B. Gordon and P. Tannenbaum, Partitions of groups and complete mappings, *Pacific J. Math.* **92** (1981), 283–293.

[9] B. Gordon, Sequences in groups with distinct partial products, *Pacific J. Math.* **11** (1961), 1309–1313.

[10] A. D. Keedwell, On the R-sequenceability and R_h-sequenceability of groups, *Ann. Discrete Math.* **18** (1983), 535–548.

[11] É. Lucas, *Récréations Mathématiques*, Tôme II, Albert Blanchard, Paris, 1892 (reprinted 1975).

[12] M. A. Ollis, On terraces for abelian groups, *Discrete Math.* **305** (2005), 250–263.

[13] M. A. Ollis and P. Spiga, Every abelian group of odd order has a narcissistic terrace, *Ars Combin.* **76** (2005), 161–168.

[14] M. A. Ollis and D. T. Willmott, Constructions for terraces and R-sequencings, including a proof that Bailey's Conjecture holds for abelian groups, *J. Combin. Des.* **23** (2015), 1–17.

[15] L. J. Paige, A note on finite abelian groups, *Bull. Amer. Math. Soc.* **53** (1947), 590–593.

[16] L. J. Paige, Complete mappings of finite groups, *Pacific J. Math.* **1** (1951), 111–116.

[17] D. A. Preece, Zigzag and foxtrot terraces for $\mathbb{Z}_n$, *Australas. J. Combin.* **42** (2008), 261–278.

[18] D. A. Preece, Half-cycles and chaplets, *Australas. J. Combin.* **43** (2009), 253–280.

[19] D. H. Rees, Some designs of use in serology, *Biometrics* **23** (1967), 779–791.

[20] E. J. Williams, Experimental designs balanced for the estimation of residual effects of treatments, *Aust. J. Scient. Res. A*, **2** (1949), 149–168.

(Received 7 Jan 2015; revised 16 May 2015)