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Small snarks and 6-chromatic triangulations on the Klein bottle

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Abstract

It is known that for every nonorientable surface there are infinitely many (large) snarks that can be polyhedrally embedded on that surface. We take a dual approach to the embedding of snarks on the Klein bottle, and investigate edge-colorings of 6-chromatic triangulations of the Klein bottle. In the process, we discover the smallest snarks that can be polyhedrally embedded on the Klein bottle. Additionally, we show that every triangulation containing certain 6-critical graphs on the Klein bottle must have a Grünbaum coloring and thus cannot admit a dual embedded snark.

1 Introduction and Summary

We begin by recalling the definition of a snark:

Definition 1.1. A *snark* is a bridgeless, 3-regular graph G that has edge-chromatic number $\chi'(G) = 4$, has girth at least 5, and is cyclically 4-edge connected.

Snarks originally arose as potential counterexamples to the Four Color Conjecture [7] and continue to be of interest in their own right for structural study as well as because of their role in important conjectures. For example, the Berge-Fulkerson conjecture [6] states that every 2-connected cubic graph has a double-edge covering of six perfect matchings. This may be framed in terms of colorings, so that every 2-connected cubic graph has a proper 6-edge coloring in which every edge receives two colors. More closely related to the current work is Grünbaum's Conjecture [8], which generalized the Four Color Conjecture to say that no cubic graph with edge-chromatic number 4 could be embedded in a particular way (*polyhedrally*, defined below) on any orientable surface. For more aspects of the significance of snarks, see [4]. Our interest here is in the embedding properties of snarks.

An *embedding* of a graph G into a surface S (as in [14]) is formally an isomorphism $\pi : G \to G' \subset S$, where the vertices of G' are distinct points of S, and the edges of G' are simple arcs in S between vertices of G' such that the relative interiors of any two edges are disjoint and the relative interior of any edge is disjoint from all vertices. Here S is called the *embedding surface*. The faces of the embedding are the components of $S \setminus G'$.

We think of the embedded graph as including its faces, so that combinatorially we have a vertex set, an edge set (of vertex pairs), and a face set (of facial cycle lists). We will only consider *cellular* embeddings, in which every face is equivalent to a topological disk. In practice, we visualize a cellular embedding by drawing the graph G on a polygon representation of the embedding surface S. We abuse notation by referring to the embedded graph together with its faces as $\pi(G)$. A *triangulation* T of a surface is an embedded graph $\pi(G)$ such that every face is a triangle.

A graph embedding on a surface other than the sphere is *polyhedral* if G is 3-connected and any simple noncontractible curve on the embedding surface intersects at least three points of G. For a 3-regular graph, an embedding is polyhedral if no face uses an edge twice and no two faces share more than one edge.

The dual of a 3-regular graph G relative to its embedding surface is a triangulation T of that surface. This gives a correspondence between 3-edge colorings of G and *Grünbaum colorings* of T, which are 3-colorings of the edges such that every facial triangle is incident with three different colors. We will adopt both perspectives here.

The motivation for the present work was a theorem in [15] that shows the existence of a polyhedrally embedded snark for every nonorientable surface *except* the Klein bottle; [15, Problem 5.3] asked whether there exists a snark with a polyhedral embedding in the Klein bottle. The answer is 'yes'; our presentation proceeds from a different perspective than [13], where the authors exploit Kochol's superposition technique [10] to exhibit an infinite family of snarks polyhedrally embedded on the Klein Bottle.

In [1], it was shown that any toroidal triangulation T with $\chi(T) \neq 5$ has a Grünbaum coloring. There, the cases of $\chi(T) \leq 4, \chi(T) = 7$, and $\chi(T) = 6$ were treated separately; the last was the most challenging. We will take a similar approach of analyzing triangulations of the Klein bottle by chromatic number. This is possible because the chromatic number for the Klein bottle is six [16], and because the 6-critical graphs for the Klein bottle have been characterized [5, 9].

The bulk of the paper is contained in Section 3, where we analyze triangulations containing the various 6-critical graphs for the Klein bottle. Some 6-chromatic triangulations have Grünbaum colorings and others do not, depending on which 6-critical graph(s) the triangulation contains. We find six small snarks that can be polyhedrally embedded on the Klein bottle, some of which have duals that contain more than one of the 6-critical graphs. Among these are the smallest snarks that can be polyhedrally embedded on the Klein bottle.

2 Triangulations T of the Klein bottle with $\chi(T) < 6$

Every graph G on the Klein bottle must have chromatic number $\chi(G) \leq 6$ by [16, Theorem 4.12]. The focus of this paper is on the case of triangulations T with

$\chi(T) = 6$, but we briefly address lower chromatic-number cases.

Consider first triangulations T of the Klein bottle with $\chi(T) \leq 4$.

Lemma 2.1 ([1], Lemma 2.1). *If T is a triangulation of any surface and $\chi(T) \leq 4$, then T has a Grünbaum coloring.*

The proof uses a homomorphism from T to K_4 to produce a Grünbaum coloring of T.

Corollary 2.2. *Any triangulation T of the Klein bottle with $\chi(T) \leq 4$ has a Grünbaum coloring.*

The 5-chromatic graphs of the Klein bottle have not been classified (and by [12, p. 502] no such classification could be finite), so we cannot fully address the case of $\chi(T) = 5$. However, the case of *even* triangulations (those with all vertices of even degree) of the Klein bottle with $\chi(T) = 5$ has been resolved:

Theorem 2.3 ([11]). *Every even triangulation of the Klein bottle has a Grünbaum coloring.*

3 Triangulations T of the Klein bottle with $\chi(T) = 6$

Every triangulation T of the Klein bottle with $\chi(T) = 6$ contains a 6-critical graph.

Theorem 3.1 ([5, 9]). *There are nine 6-critical graphs on the Klein bottle, namely six special graphs L_1–L_6, $K_2 + H_7, C_3 + C_5$, and K_6.*

The nine 6-critical graphs of Theorem 3.1 are shown in Figure 1.

There are a range of embedding properties that a 6-critical graph G embedded on the Klein bottle may have, and these divide embeddings into categories. In order from *no-snark* to *snark!*, these are

Category (1): $\pi(G)$ is a triangulation with a Grünbaum coloring. By Lemma 3.2, no triangulation containing it can be dual to a 4-edge chromatic graph (or, therefore, a snark).

Category (2): $\pi(G)$ is a *near*-triangulation and every triangulation containing this graph has a Grünbaum coloring.

Category (3): Some triangulations containing the near-triangulation $\pi(G)$ have no Grünbaum coloring, but none has a dual that is a snark (e.g. the dual has low girth or low cyclic edge-connectivity).

Category (4): Some triangulations containing the near-triangulation $\pi(G)$ have no Grünbaum coloring, but none has a dual that is a polyhedrally embedded snark.

Category (5): A triangulation containing the near-triangulation $\pi(G)$ has a dual that is a polyhedrally embedded snark.

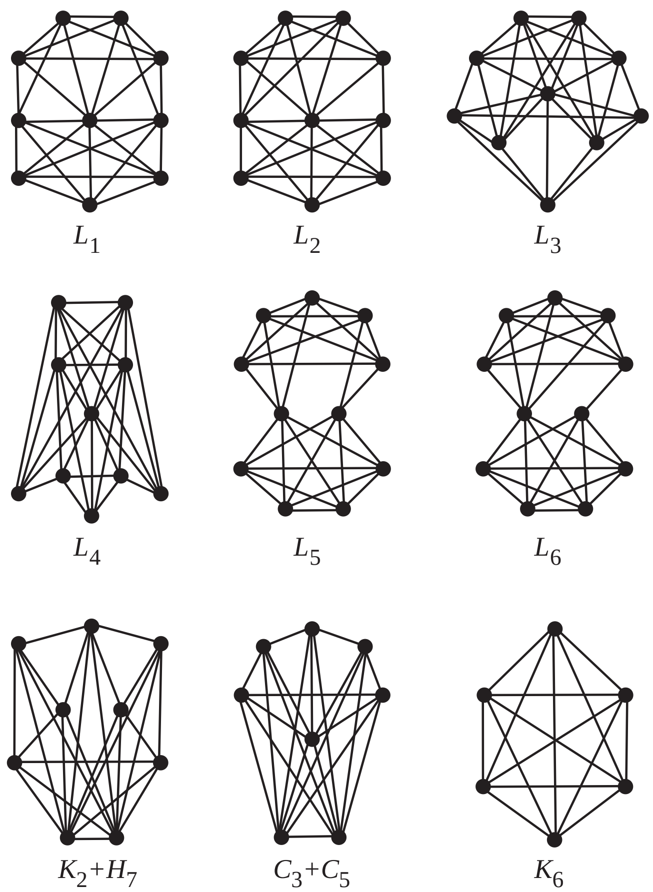

Figure 1: The nine 6-critical graphs on the Klein bottle.

In Section 3.1 we show that L_3, L_4 are in Category (1) and C_3+C_5 is in Category (2); in Section 3.2 we show that K_2+H_7 is in Category (4); and, in Section 3.3 we show that L_1, L_2, L_5, and L_6 are in Category (5). Note that we do not have an example of an embedding in Category (3). Our limited results on K_6 are given in Section 3.4.

3.1 Embeddings with, or extending to, triangulations with Grünbaum colorings

We will use the following result frequently in this section.

Lemma 3.2 ([1], Lemma 2.3). *Suppose H is a triangulation of a surface S that has a Grünbaum coloring. If G is a triangulation of S that contains H as a subgraph, i.e. G is a refinement of H, then G has a Grünbaum coloring.*

Now we can proceed to this section's main result.

Theorem 3.3. *Every triangulation containing L_3, L_4, or C_3+C_5 has a Grünbaum coloring.*

We treat the cases L_3 and L_4 separately from C_3+C_5 because of their embeddings' structural differences.

Proof of Theorem 3.3, the cases of L_3 and L_4. Each of L_3 and L_4 has a unique embedding on the Klein bottle [9], and each embedding is a triangulation. By Lemma 3.2, it suffices to exhibit a Grünbaum coloring for each of these two embeddings; they are given in Figure 2. $\square$

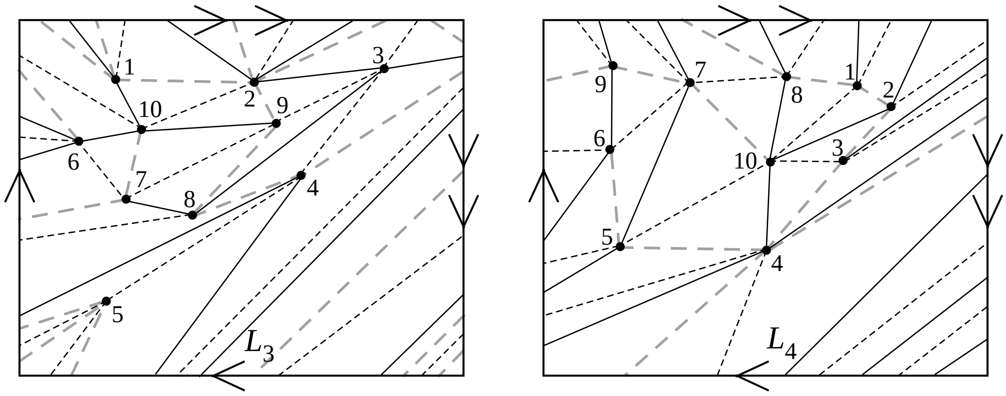

Figure 2: Grünbaum colorings of L_3 (left) and L_4 (right) embedded on the Klein bottle.

We summarize here the technique introduced in [1] for examining near-triangulations. Suppose we have a near-triangulation N of the Klein bottle with a single non-triangular face P. Triangulating P to produce $T(P)$ completes N to a triangulation. We may construct a triangulation of the plane by drawing P in the plane and triangulating both the interior (as $T(P)$) and the exterior. This resulting triangulation has a Grünbaum coloring by the Four Color Theorem, and this induces an edge-coloring on P.

If we enumerate all possible such induced edge-colorings of P, and check that each can be extended to a Grünbaum coloring of the remainder of N, then we know that any completion of N to a triangulation has a Grünbaum coloring as well. Together with Lemma 3.2, this shows that any triangulation containing N has a Grünbaum coloring.

Lemma 3.4 ([1])**.** *Suppose a near-triangulation N of a surface has a single non-triangular face P with four sides. Denote edge colors as s (solid), d (dash), and g (grey). If the edge-colorings ssss, ssdd, and sdds of P each extend to a Grünbaum coloring of the remainder of the embedding, then every triangulation containing N has a Grünbaum coloring.*

Proof. Draw P in the plane and triangulate the exterior as a 4-wheel (that is, add a single vertex incident to edges leading to each vertex of P). It is straightforward to see that the only possible edge-colorings of P are *ssss*, *ssdd*, and *sdds* (up to global color permutation). Thus, applying the Four Color Theorem and Tait's Theorem ([18]) and dualizing, any triangulation $T(P)$ of P must have a Grünbaum coloring compatible with one of the colorings *ssss*, *ssdd*, or *sdds* on P. Therefore, if each of the colorings *ssss*, *ssdd*, or *sdds* on P extends to a Grünbaum coloring on the remainder of N, any augmentation of N by some $T(P)$ to a triangulation has a Grünbaum coloring. Finally, it follows by Lemma 3.2 that any triangulation containing N has a Grünbaum coloring. □

We now apply Lemma 3.4.

Proof of Theorem 3.3, the case of $C_3 + C_5$*.* There are three embeddings of $C_3 + C_5$ on the Klein bottle, and each has all faces triangles except for one quadrilateral [9]. By Lemma 3.4, it suffices to show, for each embedding, that each of the edge-colorings *ssss*, *ssdd*, and *sdds* of the quadrilateral face extends to a Grünbaum coloring of the remainder of the embedding. Figures 3, 4, and 5 give the required near-Grünbaum colorings for the three embeddings of $C_3 + C_5$ on the Klein bottle.

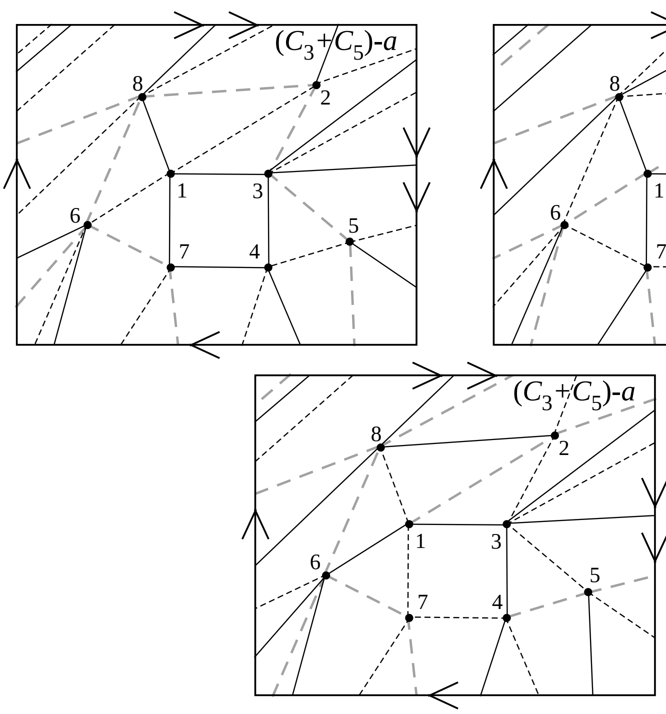

Figure 3: Partial Grünbaum colorings of the embedding $(C_3 + C_5)$-a.

By Lemma 3.2, this shows that every triangulation containing $C_3 + C_5$ has a Grünbaum coloring. □

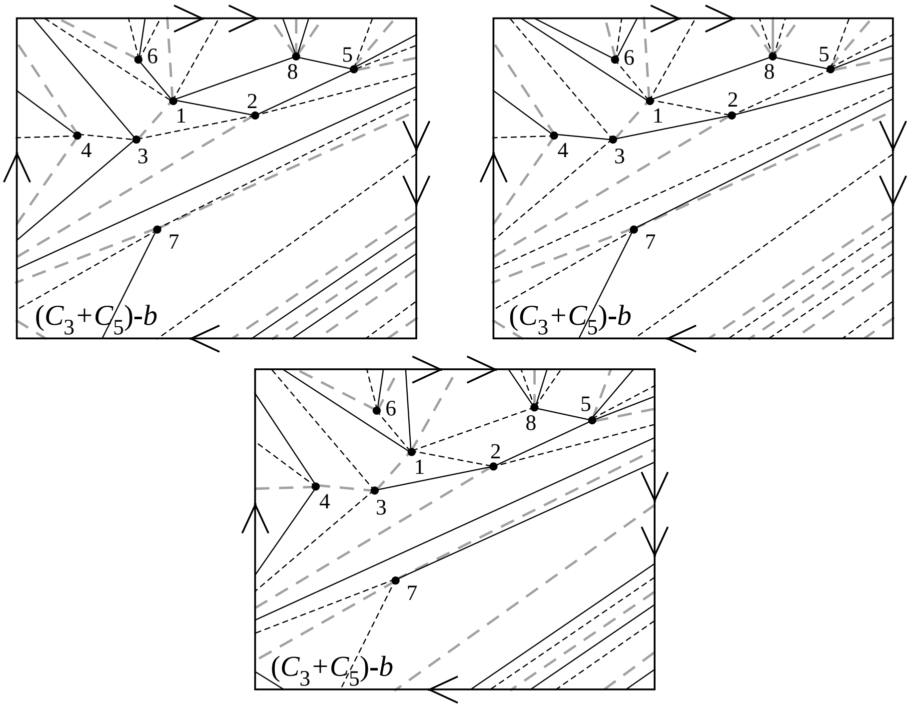

Figure 4: Partial Grünbaum colorings of the embedding $(C_3 + C_5)$-b.

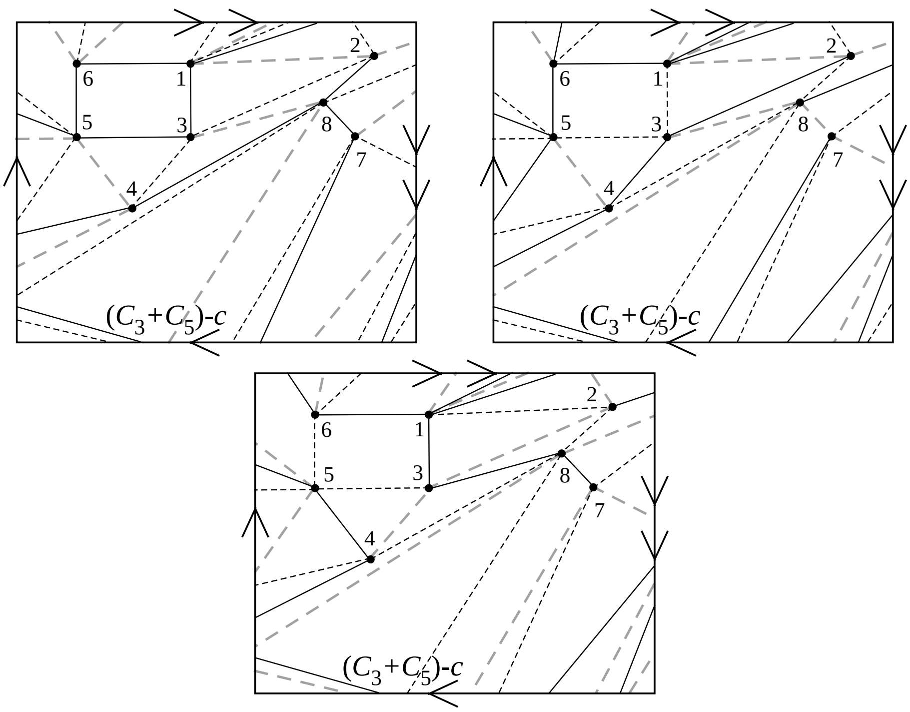

Figure 5: Partial Grünbaum colorings of the embedding $(C_3 + C_5)$-c.

3.2 The case of $K_2 + H_7$

As in Section 3.1, we will need an auxiliary result. Namely, if a triangulated quadrilateral has no diagonal edge, then it has a Grünbaum coloring with the exterior cycle colored *ssss* or *sdds*.

Lemma 3.5. *Let $\pi(G)$ be a near-triangulation of the sphere with outer 4-cycle abce such that a-c, b-e $\notin E(G)$. Then there exists a Grünbaum coloring of $\pi(G)$ such that the outer 4-cycle is colored ssss or sdds.*

Proof, from [2]. Consider a triangulated quadrilateral with exterior vertices labeled clockwise *abce* as at left in Figure 6. Identify the vertices b and e to obtain a planar

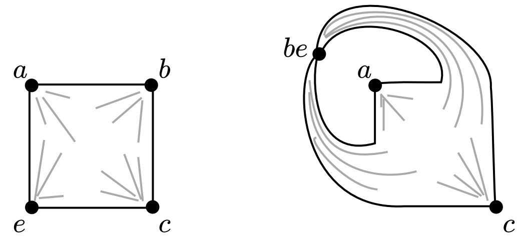

Figure 6: At left, the quadrilateral *abce*; at right, the degenerate quadrilateral resulting from identifying vertices b and e.

triangulated degenerate quadrilateral, as at right in Figure 6, which has no loop and can therefore be vertex-colored. Using the Four Color Theorem, assign vertex colors from $\mathbb{Z}_2 \times \mathbb{Z}_2$, and sum the colors incident to each edge to obtain a color for that edge. Only three colors are used because $(0,0)$ could only arise from a non-proper vertex coloring.

Use this same coloring on the original triangulated quadrilateral. It is a Grünbaum coloring because each triangle has three distinct vertex colors, and thus three distinct edge colors. However, edge a-b is the same color as edge a-e, and edge b-c is the same color as edge c-e. Therefore the exterior of the quadrilateral has coloring *ssss* or *sdds*. □

Theorem 3.6. *No snark polyhedrally embedded on the Klein bottle can have a dual triangulation containing $K_2 + H_7$ as a subgraph.*

Proof. There are two embeddings of K_2+H_7 on the Klein bottle [9]. Each embedding is a near-triangulation with one quadrilateral face.

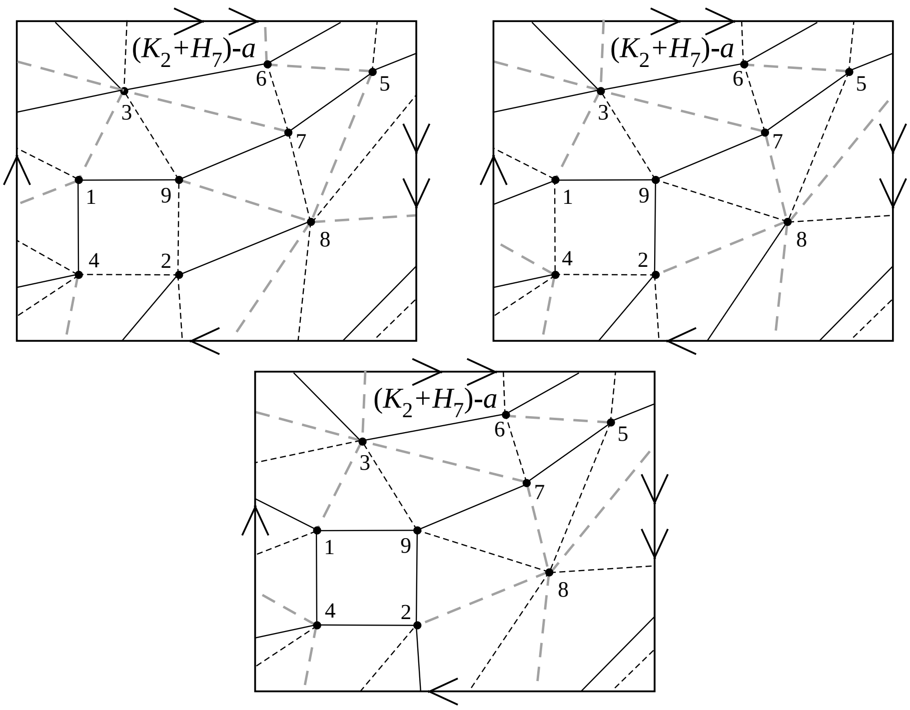

Figure 7: Partial Grünbaum colorings of the embedding $(K_2 + H_7)$-a.

For the embedding $(K_2 + H_7)$-a, each of the edge-colorings *ssss*, *ssdd*, and *sdds* of the quadrilateral face extends to a Grünbaum coloring of the remainder of the embedding (see Figure 7), and so by Lemma 3.2, every triangulation containing $(K_2 + H_7)$-a has a Grünbaum coloring.

Note (see Figure 8) that in the embedding $(K_2 + H_7)$-b, the quadrilateral face uses vertices 2, 6, 3, and 4, and the edges 2-3 and 4-6 are present. Therefore if either diagonal of the quadrilateral face is present in a triangulation containing $(K_2 + H_7)$-b, there are multiple edges and the dual graph cannot be polyhedrally embedded. If neither diagonal of the quadrilateral face is present in a triangulation containing $(K_2 + H_7)$-b, Lemma 3.5 applies.

As we see in Figure 8, the edge colorings *ssss* and *sdds* of the quadrilateral face extend to a Grünbaum coloring of the remainder of the embedding. Therefore no triangulation containing $(K_2 + H_7)$-b can be dual to a polyhedrally embedded snark, and this completes the proof of the theorem. □

However, there does exist a snark embedded on the Klein bottle (not polyhedrally) whose dual contains $K_2 + H_7$ as a subgraph.

Example 3.7. There is an infinite family of snarks of order $18 + 8k$ that can be embedded on the Klein bottle. Figure 9 shows $(K_2 + H_7)$-b with a second copy of the edge 4-6 added and the graph dual to this embedding drawn. The dual graph

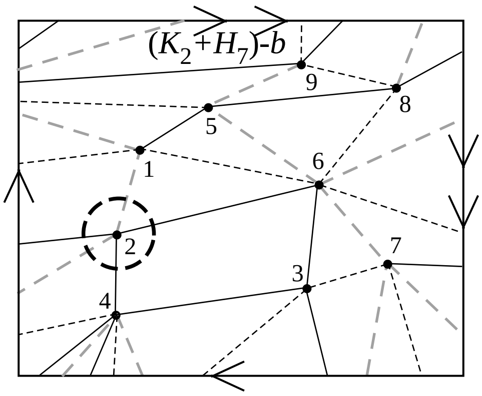

Figure 8: Partial *ssss* Grünbaum coloring of the embedding $(K_2 + H_7)$-b. A circle encloses a vertex around which a Kempe-change will convert the coloring to *sdds*.

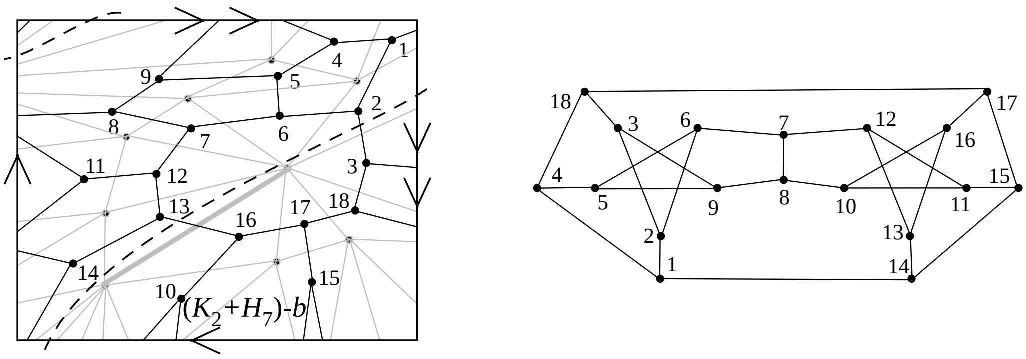

Figure 9: A Blanuša snark is shown embedded dual to $(K_2 + H_7)$-b (left) and drawn in the plane (right).

is a Blanuša snark; because the associated triangulation has a multiple edge, the embedding is not polyhedral. A dotted line indicates an orientable non-surface-separating loop that intersects two snark edges. Therefore, by [3, Theorem 2], we can use repeated dot product with the Petersen graph to construct an infinite family of snarks that can be embedded on the Klein bottle. The dual of each snark in this family contains a subdivision of $K_2 + H_7$.

3.3 Embeddings contained in triangulations dual to polyhedrally embedded small snarks

Theorem 3.8. *Each of the* 6*-critical graphs* L_1, L_2, L_5, L_6 *is a subgraph of at least one simple triangulation that is dual to a snark.*

We prove Theorem 3.8 by exhibiting the desired triangulations in Examples 3.9–3.13. Labelings of the graphs in question used throughout this section are given in Figure 10.

Note that because these triangulations have no multiple edges, each of the dual

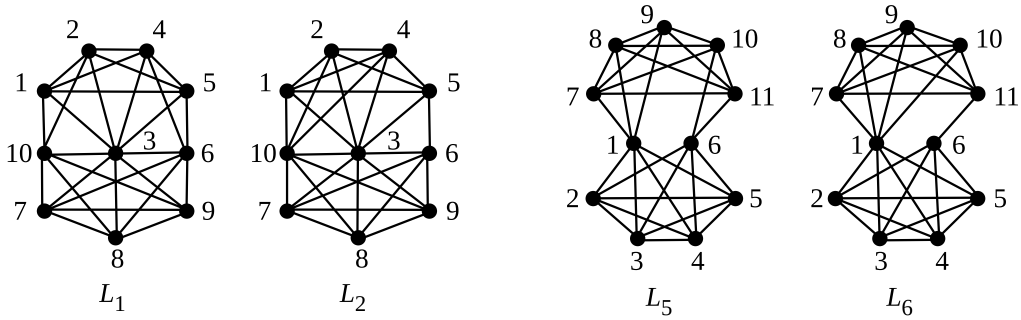

Figure 10: The labelings of L_1, L_2, L_5, L_6 used in this section.

snarks is polyhedrally embedded. Moreover, at 20 vertices, the snarks dual to triangulated embeddings of L_1, L_2 (see Examples 3.9 and 3.10) are the smallest snarks that can be polyhedrally embedded on the Klein bottle: neither of the Blanuša snarks (18 vertices) nor the Petersen graph (10 vertices) embeds polyhedrally on the Klein bottle. They are also new; Liu and Chen [13] exhibit a family of snarks that can be polyhedrally embedded on the Klein Bottle with one 22-vertex example that is included in Example 3.13 as well as members with $24+10k$ vertices and with $26+10k$ vertices (all larger than those given here).

Example 3.9. The graph L_1 has a single embedding in the Klein bottle [9], shown in Figure 11 with non-triangular face(s) shaded. We add edges 1-7 and 5-7 to form L_1+;

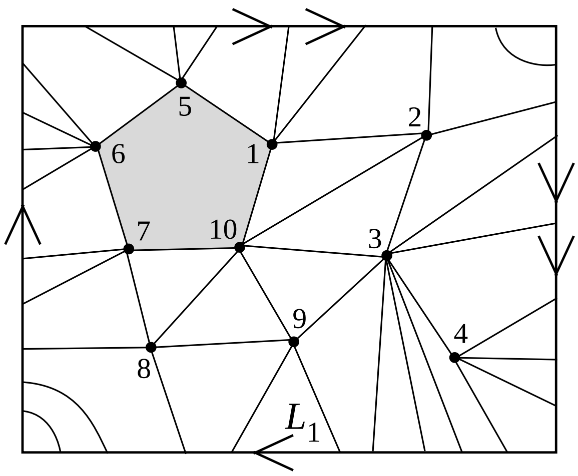

Figure 11: The unique embedding of L_1 on the Klein bottle with non-triangular face(s) shaded.

these do not create multiple edges. Figure 12 shows the dual graph embedded with L_1 and drawn in the plane. A check in *Mathematica* shows that $(L_1+)^*$ is isomorphic to the 4[th] snark with 20 vertices according to Gordon Royle's enumeration [17].

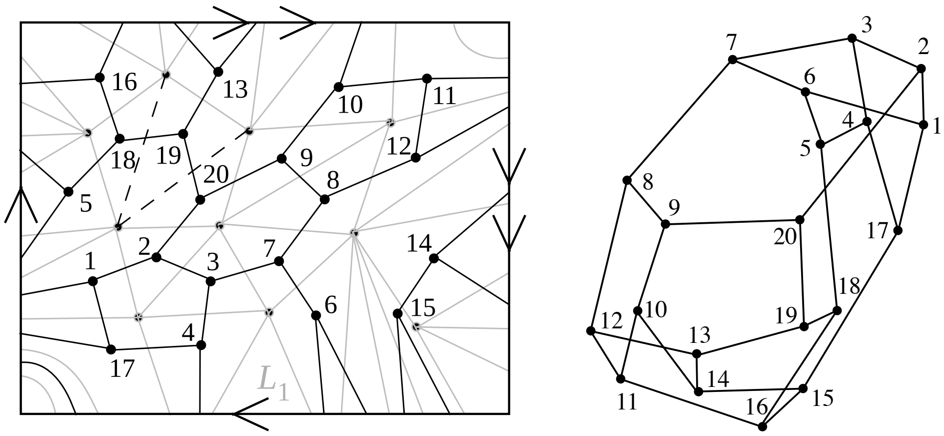

Figure 12: Dual graphs L_1+ and $(L_1+)^*$ embedded on the Klein bottle (left) and $(L_1+)^*$ drawn in the plane (right).

Example 3.10. The graph L_2 has a single embedding in the Klein bottle [9], shown in Figure 13 with non-triangular face(s) shaded. We add edges 1-6 and 1-7 to form

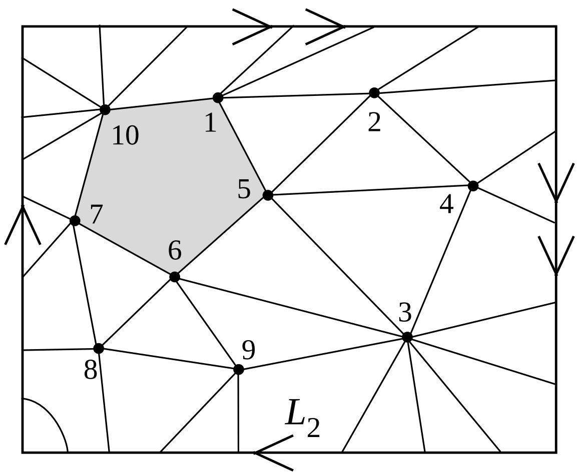

Figure 13: The unique embedding of L_2 on the Klein bottle with non-triangular face(s) shaded.

L_2+; these do not create multiple edges. Figure 14 shows the dual graph embedded with L_2 and drawn in the plane. A check in *Mathematica* shows that $(L_2+)^*$ is isomorphic to the 2nd snark with 20 vertices according to Gordon Royle's enumeration [17].

Fact 3.11. If we add to L_1 any one of the edges {1-6, 2-6, 4-10, 5-10}, and add to L_2 any one of the edges {1-6, 2-6, 4-6, 5-7, 5-8, 5-9}, we obtain isomorphic graphs. Denote such a graph $L_{1,2}$. Because each of L_1, L_2 have a single embedding on the Klein bottle, any embedding $\pi(L_{1,2})$ of $L_{1,2}$ on the Klein bottle is a refinement of both $\pi(L_1), \pi(L_2)$.

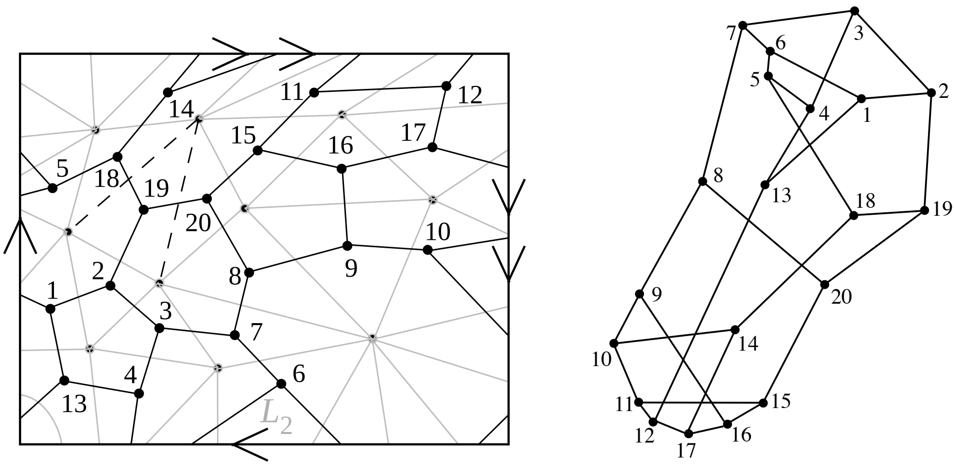

Figure 14: Dual graphs L_2+ and $(L_2+)^*$ embedded on the Klein bottle (left) and $(L_2+)^*$ drawn in the plane (right).

Adding edges 1-6 and 1-7, or edges 5-7 and 5-10, to L_1, we obtain a triangulation whose dual graph is isomorphic to the 2nd snark with 20 vertices according to Gordon Royle's enumeration [17].

Adding edges 1-7 and 5-7 (or edges 1-6 and 6-10 from Example 3.10) to L_2, we obtain a triangulation whose dual graph is also isomorphic to the 2nd snark with 20 vertices according to Gordon Royle's enumeration [17].

By Fact 3.11 all four dual embeddings of this snark contain $\pi(L_{1,2})$. Because $\pi(L_{1,2})$ has one non-triangular, quadrilateral face, there are exactly two 10-vertex triangulations that contain it. These triangulations have non-isomorphic dual graphs. Therefore all four embeddings of the 2nd snark with 20 vertices mentioned above are isomorphic.

Example 3.12. The graph L_5 has a single embedding in the Klein bottle [9], shown in Figure 15 with non-triangular faces shaded. We add edges 2-9, 2-10, 5-7, and 5-11 to form L_5+; these do not create multiple edges. Figure 16 shows the dual graph embedded with L_5 and drawn in the plane. A check in *Mathematica* shows that $(L_5+)^*$ is isomorphic to the 13th snark with 22 vertices according to the *Combinatorica* enumeration.

If we instead add edges 2-9, 2-10, 5-7, and 6-7 (embedding A); or, add edges 2-9, 6-9, 5-7, and 5-11 (embedding B), we obtain a triangulation L_5+' whose dual graph is isomorphic to the 6th snark with 22 vertices according to the *Combinatorica* enumeration. The first of these is shown in Figure 17. A manual check of rotation systems shows that embeddings A and B are equivalent under the vertex map

A vertex label	1	2	3	4	5	6	7	8	9	10	11
B vertex label	6	5	3	4	2	1	9	10	11	7	8

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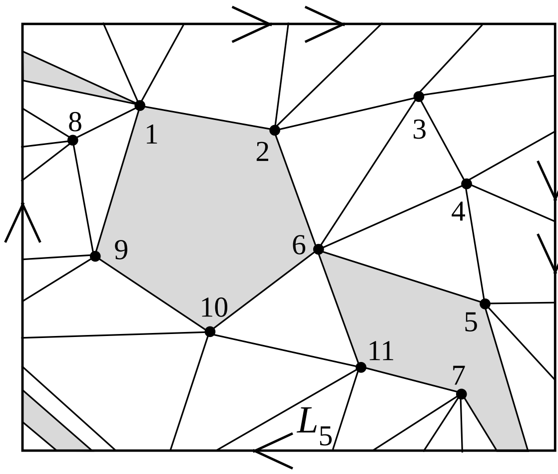

Figure 15: The unique embedding of L_5 on the Klein bottle with non-triangular face(s) shaded.

Example 3.13. The graph L_6 has a single embedding in the Klein bottle [9], shown in Figure 18 with non-triangular faces shaded. We add edges 2-10, 2-11, 5-7, and 5-11 to form L_6+; these do not create multiple edges. Figure 19 shows the dual graph embedded with L_6 and drawn in the plane.

A check in *Mathematica* shows that $(L_6+)^*$ is isomorphic to the 2^{nd} snark with 22 vertices according to the *Combinatorica* enumeration.

If we instead add edges 2-10, 2-11, 5-7, and 6-7, we obtain a triangulation L_6+' whose dual graph is isomorphic to the 9^{th} snark with 22 vertices according to the *Combinatorica* enumeration; the embedding is shown in Figure 20. This is the same 22-vertex snark as exhibited in [13].

Note: There exist many so-called trivial snarks (that is, cyclically 3-edge connected 4-edge chromatic cubic graphs) that can be polyhedrally embedded on L_5 and on L_6.

Fact 3.14. If we add to L_5 any one of the edges {1-10, 1-11}, and add to L_6 any one of the edges {6-7, 6-8, 6-9, 6-10}, we obtain isomorphic graphs. Denote such a graph $L_{5,6}$. Because each of L_5, L_6 have a single embedding on the Klein bottle, any embedding $\pi(L_{5,6})$ of $L_{5,6}$ on the Klein bottle is a refinement of both $\pi(L_5), \pi(L_6)$.

Adding edges 2-9, 2-10, 1-11, and 5-11 to L_5; or, adding edges 1-10, 2-10, 5-7, and 5-11 to L_5, we obtain a triangulation whose dual graph is isomorphic to the 9^{th} snark with 22 vertices according to the *Combinatorica* enumeration.

Adding edges 2-10, 6-10, 5-7, and 5-11 to L_6 (or adding edges 2-10, 2-11, 5-7, and 6-7 to L_6 as in Example 3.13), we obtain a triangulation whose dual graph is isomorphic to the 9^{th} snark with 22 vertices according to the *Combinatorica* enumeration.

By Fact 3.14, all four dual embeddings of this snark contain $\pi(L_{5,6})$, which has one quadrilateral face and one pentagonal face that share two nonadjacent vertices v, w. Each dual embedding includes three additional edges; examination shows that

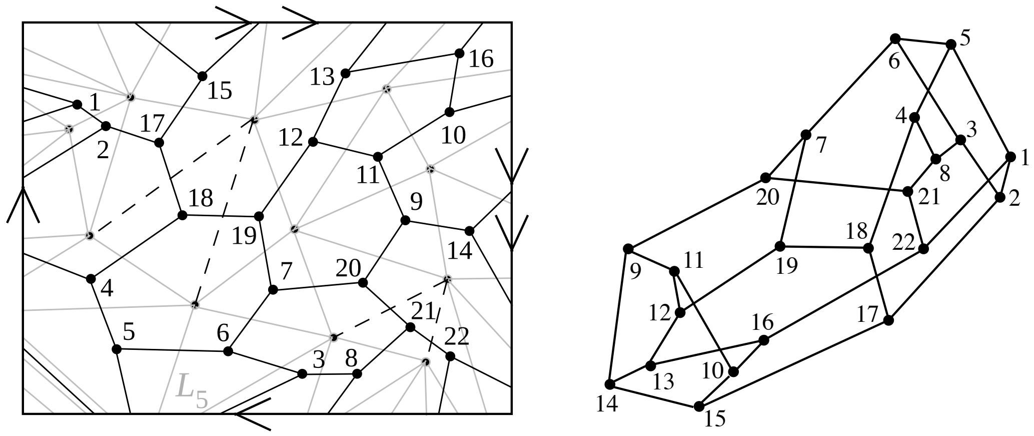

Figure 16: Dual graphs L_5+ and $(L_5+)^*$ embedded on the Klein bottle (left) and $(L_5+)^*$ drawn in the plane (right).

none of these edges is incident to v or w. Thus, the triangulations of the two faces are the same for each dual embedding. Therefore all four embeddings of the 9$^{\text{th}}$ snark with 22 vertices described above are isomorphic.

Adding edges 1-10, 2-10, 5-7, and 6-7 to L_5 (embedding C); or, adding edges 2-9, 6-9, 5-7, and 6-7 to L_5 (embedding D); or, adding edges 2-9, 6-9, 1-11, and 5-11 to L_5 (embedding E), we obtain a triangulation whose dual graph is isomorphic to the 2$^{\text{nd}}$ snark with 22 vertices according to the *Combinatorica* enumeration.

Adding edges 2-10, 6-10, 5-7, and 6-7 to L_6 (embedding F), or adding add edges 2-10, 2-11, 5-7, and 5-11 to L_6 as in Example 3.13 (embedding G), we obtain a triangulation whose dual graph is isomorphic to the 2$^{\text{nd}}$ snark with 22 vertices according to the *Combinatorica* enumeration.

By Fact 3.14, dual embeddings C, E, and F of this snark contain $\pi(L_{5,6})$, which has one quadrilateral face and one pentagonal face that share two nonadjacent vertices v, w. Each dual embedding includes three additional edges; examination shows that one edge (in the pentagonal face) is incident to v but not w, and the other two edges are incident to neither v or w. Thus, the triangulations of the two faces are the same for each dual embedding. Therefore dual embeddings C, E, and F of the 2$^{\text{nd}}$ snark with 22 vertices described above are isomorphic.

A manual check of rotation systems shows that embeddings C, D, and G are equivalent under the vertex maps

C vertex label	1	2	3	4	5	6	7	8	9	10	11
D vertex label	6	5	3	4	2	1	9	10	11	7	8
G vertex label	1	7	9	8	10	11	2	4	3	5	6

.

(Note that the A-B map is the same as the C-D map.)

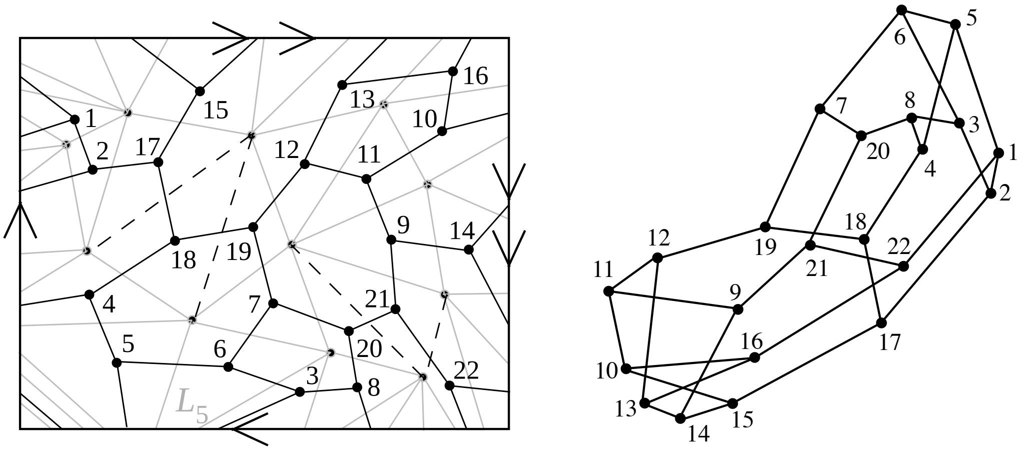

Figure 17: Dual graphs L_5+' and $(L_5+')^*$ embedded on the Klein bottle (left) and $(L_5+')^*$ drawn in the plane (right).

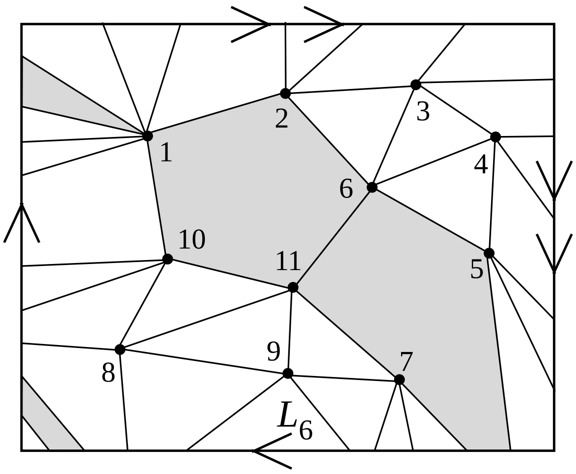

Figure 18: The unique embedding of L_6 on the Klein bottle with non-triangular face(s) shaded.

3.4 Commentary on K_6

There are seven 2-cell embeddings of K_6 on the Klein bottle [9], and one non-2-cell embedding in which K_6 sits in one of the two cross-caps of the Klein bottle. The non-2-cell embedding is unique (by symmetry of K_6).

There are several triangulations that contain the non-2-cell embedding of K_6 (and the embedding of L_1 or L_2 or L_5 or L_6) and are dual to polyhedral embeddings of 4-edge chromatic, girth 5, cyclically 3-edge connected cubic graphs.

Analysis of the 2-cell embeddings of K_6 is highly complex. Four have three four-sided faces (in different configurations); two have a single six-sided face (one where two edges are identified); one has one five-sided and one four-sided face. We have verified that for two of the 2-cell embeddings, there exists at least one precoloring of

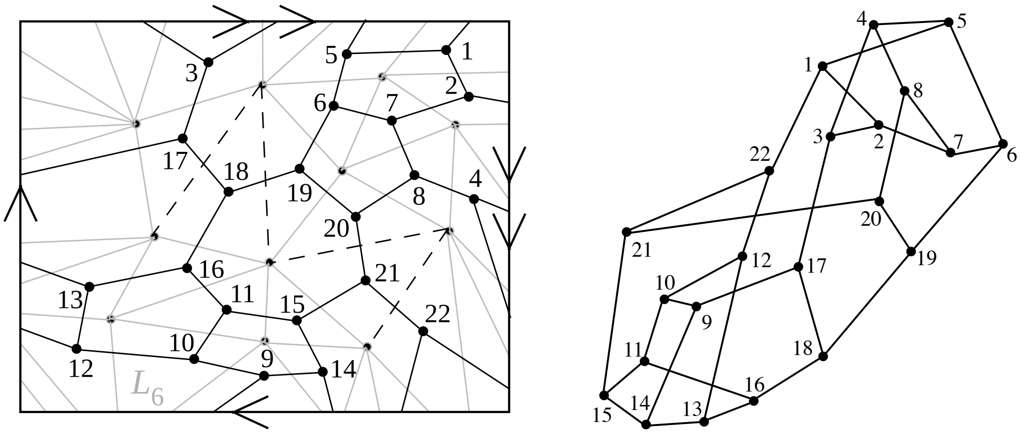

Figure 19: Dual graphs L_6+ and $(L_6+)^*$ embedded on the Klein bottle (left) and $(L_6+)^*$ drawn in the plane (right).

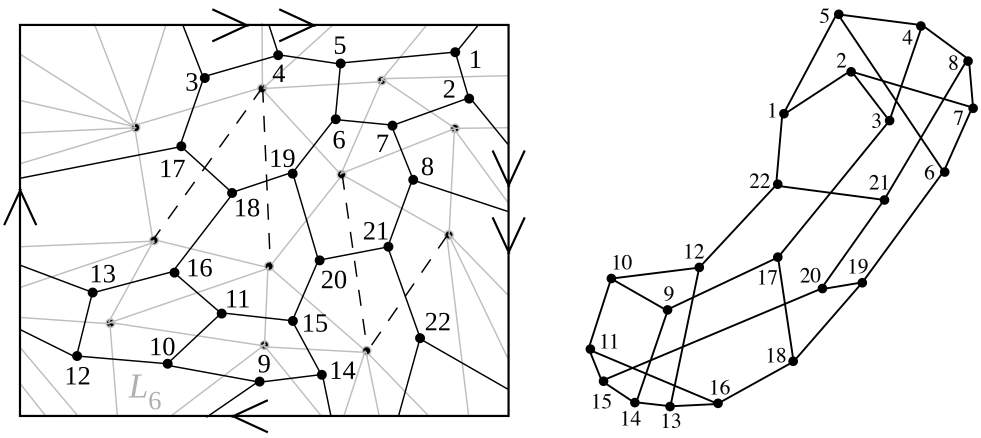

Figure 20: Dual graphs L_6+' and $(L_6+')^*$ embedded on the Klein bottle (left) and $(L_6+')^*$ drawn in the plane (right).

the non-triangular face edges that does not extend to a partial Grünbaum coloring of the remainder of the embedding. (These precolorings correspond to compatible colorings of common triangulations of the faces.) However, we have so far been unable to produce a triangulation containing a 2-cell embedding of K_6 that is dual to a 4-edge-chromatic cubic graph.

4 Conclusion

In this paper, we have completely analyzed triangulations containing eight of the nine 6-critical graphs for the Klein bottle. We have shown that every triangulation containing any of three of the 6-critical graphs has a Grünbaum coloring, that no triangulation containing a fourth of the 6-critical graphs can be dual to a polyhedrally embedded snark, and that each of the remaining 6-critical graphs is contained in at least one triangulation dual to a polyhedrally embedded snark. Among these are the smallest snarks that polyhedrally embed on the Klein bottle.

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