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A generalization of the Erdős-Ko-Rado theorem

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Abstract

Our main result is a new upper bound for the size of k-uniform, L-intersecting families of sets, where L contains only positive integers. We characterize extremal families in this setting. Our proof is based on the Ray-Chaudhuri–Wilson Theorem [*Osaka J. Math.* 12 (1975),737–744]. As an application, we give a new proof for the Erdős-Ko-Rado Theorem, improve Fisher's inequality in the uniform case and give a uniform version of the Frankl-Füredi conjecture.

1 Introduction

First we introduce some notation.

Let $[n]$ stand for the set $\{1, 2, \ldots, n\}$. We denote the family of all subsets of $[n]$ by $2^{[n]}$. For k an integer with $0 \le k \le n$ we denote by $\binom{[n]}{k}$ the family of all k element subsets of $[n]$. We say that a family $\mathcal{F}$ of subsets of $[n]$ is *k-uniform* if $|F| = k$ for each $F \in \mathcal{F}$.

Bose proved the following result in [1].

Theorem 1.1 *Let $\lambda > 0$ be a positive integer. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a k-uniform family of subsets of $[n]$ such that $|F_i \cap F_j| = \lambda$ for each $1 \le i, j \le m,\ i \ne j$. Then $m \le n$.*

Majumdar generalized this result in [8] and proved the following nonuniform version of Theorem 1.1.

Theorem 1.2 *Let $\lambda > 0$ be a positive integer. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a family of subsets of $[n]$ such that $|F_i \cap F_j| = \lambda$ for each $1 \le i, j \le m,\ i \ne j$. Then $m \le n$.*

Frankl and Füredi conjectured in [6], and Ramanan proved in [9], the following statement.

Theorem 1.3 *Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of subsets of $[n]$ such that $1 \le |F_i \cap F_j| \le s$ for each $1 \le i, j \le m,\ i \ne j$. Then*

$$m \le \sum_{i=0}^{s} \binom{n-1}{i}.$$

Later Snevily conjectured the following statement in his doctoral dissertation (see [12]). Finally he proved this result in [11].

Theorem 1.4 *Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of subsets of $[n]$. Let $L = \{\ell_1, \dots, \ell_s\}$ be a collection of s positive integers. If $|F_i \cap F_j| \in L$ for each $1 \le i, j \le m,\ i \ne j$, then*

$$m \le \sum_{i=0}^{s} \binom{n-1}{i}.$$

A family $\mathcal{F}$ is said to be *t-intersecting* if $|F \cap F'| \ge t$ whenever $F, F' \in \mathcal{F}$. In particular, $\mathcal{F}$ is an *intersecting* family if $F \cap F' \ne \emptyset$ whenever $F, F' \in \mathcal{F}$.

Erdős, Ko and Rado proved the following well-known result in [5]:

Theorem 1.5 *Let n, k, t be integers with $0 < t < k < n$. Suppose $\mathcal{F}$ is a t-intersecting, k-uniform family of subsets of $[n]$. Then for $n > n_0(k, t)$,*

$$|\mathcal{F}| \le \binom{n-t}{k-t}.$$

Further, $|\mathcal{F}| = \binom{n-t}{k-t}$ if and only if for some $T \in \binom{[n]}{t}$ we have

$$\mathcal{F} = \{F \in \binom{[n]}{k} :\ T \subseteq F\}.$$

Let L be a set of nonnegative integers. A family $\mathcal{F}$ is *L-intersecting* if $|E \cap F| \in L$ for every pair E, F of distinct members of $\mathcal{F}$. The following theorem gives a remarkable upper bound for the size of a k-uniform L-intersecting family.

Theorem 1.6 (Ray-Chaudhuri–Wilson [10]) *Let s, k, n be positive integers such that $0 < s \le k \le n$. Let L be a set of s nonnegative integers and $\mathcal{F} = \{F_1, \dots, F_m\}$ an L-intersecting, k-uniform family of subsets of $[n]$. Then*

$$m \le \binom{n}{s}.$$

Erdős, Deza and Frankl improved Theorem 1.6 in [4]. They used the theory of Δ-systems in their proof.

Theorem 1.7 *Let s, k, n be positive integers satisfying $0 < s \le k \le n$. Let L be a set of s nonnegative integers and let $\mathcal{F} = \{F_1, \dots, F_m\}$ be an L-intersecting, k-uniform family of subsets of $[n]$. Then, for $n > n_0(k, L)$,*

$$m \le \prod_{i=1}^{s} \frac{n - \ell_i}{k - \ell_i}.$$

Barg and Musin gave an improved version of Theorem 1.6 in [2].

Theorem 1.8 *Let L be a set of s nonnegative integers and let $\mathcal{F} = \{F_1, \dots, F_m\}$ be an L-intersecting, k-uniform family of subsets of $[n]$. Suppose that*

$$\frac{s(k^2 - (s-1))(2k - n/2)}{n - 2(s-1)} \ \le\ \sum_{i=1}^{s} \ell_i.$$

Then

$$m \ \le\ \binom{n}{s} - \binom{n}{s-1} \frac{n - 2s + 3}{n - s + 2}.$$

First we prove a special case of our main result.

Proposition 1.9 *Let s, k, n be positive integers satisfying $0 < s \le k \le n$. Let $L = \{\ell_1, \dots, \ell_s\}$ be a set of s positive integers such that $0 < \ell_1 < \dots < \ell_s$. Suppose that $n \ge \binom{k^2}{\ell_1+1}s + \ell_1$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be an L-intersecting, k-uniform family of subsets of $[n]$. Suppose that $\bigcap_{F \in \mathcal{F}} \mathcal{F} = \emptyset$. Then*

$$m \le \binom{n - \ell_1}{s}.$$

We now state our main results.

Theorem 1.10 *Let s, k, n be positive integers satisfying $0 < s \le k \le n$. Let $L = \{\ell_1, \dots, \ell_s\}$ be a set of s positive integers such that $0 < \ell_1 < \dots < \ell_s$. Suppose that $n \ge \binom{k^2}{\ell_1+1}s + \ell_1$. Let $\mathcal{G} = \{G_1, \dots, G_m\}$ be an L-intersecting, k-uniform family of subsets of $[n]$. Then*

$$m \le \binom{n - \ell_1}{s}.$$

Further if $n > \binom{k^2}{\ell_1+1}s + \ell_1$ and

$$|\mathcal{G}| = \binom{n - \ell_1}{s},$$

then there exists a $T \in \binom{[n]}{\ell_1}$ subset such that $T \subseteq G$ for each $G \in \mathcal{G}$.

Clearly Theorem 1.10 implies the Ray-Chaudhuri–Wilson Theorem when n is sufficiently large.

In the proof of Theorem 1.10 we combine simple combinatorial arguments with the Ray-Chaudhuri–Wilson Theorem 1.6. Our proof was inspired by the proof of Proposition 8.8 in [7].

We give here some immediate consequences of Theorem 1.10. First we describe a uniform version of Theorem 1.3.

Corollary 1.11 *Let s, k, n be positive integers such that $0 < s < k \leq n$. Let $L = \{1, 2, \ldots, s\}$. Suppose that $n > \binom{k^2}{2}s$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be an L-intersecting, k-uniform family of subsets of $[n]$. Then*

$$m \leq \binom{n-1}{s}.$$

Further if $n > \binom{k^2}{2}s + 1$ and

$$|\mathcal{F}| = \binom{n-1}{s},$$

then $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$.

The following result is the uniform version of Theorem 1.1.

Corollary 1.12 *Let $\lambda > 0$ be a positive integer. Suppose that $n \geq \binom{k^2}{\lambda+1} + \lambda$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a k-uniform family of subsets of $[n]$ such that $|F_i \cap F_j| = \lambda$ for each $1 \leq i, j \leq m,\ i \neq j$. Then*

$$m \leq n - \lambda.$$

Further if $n > \binom{k^2}{\lambda+1} + \lambda$ and

$$|\mathcal{F}| = n - \lambda,$$

then there exists a $T \in \binom{[n]}{\lambda}$ subset such that $T \subseteq F$ for each $F \in \mathcal{F}$.

2 Proof of our results

The following Lemma is a well-known Helly-type result (see e.g. [3]).

Lemma 2.1 *If each family of at most $k+1$ members of a k-uniform set system intersect, then all members intersect.*

□

In our proof we use the following lemma.

Lemma 2.2 *Let ℓ_1 be a positive integer. Let $\mathcal{H}$ be a family of subsets of $[n]$. Suppose that $\bigcap_{H\in\mathcal{H}} H = \emptyset$. Let $F \subseteq [n]$, $F \notin \mathcal{H}$ be a subset such that $|F \cap H| \geq \ell_1$ for each $H \in \mathcal{H}$. Let $Q := \bigcup_{H\in\mathcal{H}} H$. Then*

$$|Q \cap F| \geq \ell_1 + 1.$$

Proof. Since $|F \cap H| \geq \ell_1$ for each $H \in \mathcal{H}$, thus $|Q \cap F| \geq \ell_1$. Indirectly, suppose that $|Q \cap F| = \ell_1$. Let $U := Q \cap F$. Then

$$U = Q \cap F = (\bigcup_{H\in\mathcal{H}} H) \cap F = \bigcup_{H\in\mathcal{H}} (H \cap F).$$

Hence $H \cap F \subseteq U$ for each $H \in \mathcal{H}$. Since $|U| = \ell_1$ and $|H \cap F| \geq \ell_1$ for each $H \in \mathcal{H}$, thus $U = H \cap F$ for each $H \in \mathcal{H}$. Hence $U \subseteq \bigcap_{H\in\mathcal{H}} H$, which is a contradiction with $\bigcap_{H\in\mathcal{H}} H = \emptyset$. □

Lemma 2.3 *Let $\mathcal{H}$ be a family of subsets of $[n]$. Suppose that $t := |\mathcal{H}| \geq 2$ and $\mathcal{H}$ is a k-uniform, intersecting family. Then*

$$|\bigcup_{H\in\mathcal{H}} H| \leq k + (t-1)(k-1). \tag{1}$$

Proof. We use induction on t. The inequality (1) is trivially true for $t = 2$.

Let $t \geq 3$. Suppose that the inequality (1) is true for $t-1$. Let $\mathcal{H}$ be an arbitrary k-uniform intersecting family such that $|\mathcal{H}| = t$. Let $\mathcal{G} \subseteq \mathcal{H}$ be a fixed subset of $\mathcal{H}$ such that $|\mathcal{G}| = t-1$. Clearly $\mathcal{G}$ is intersecting and k-uniform. It follows from the induction hypothesis that

$$|\bigcup_{G\in\mathcal{G}} G| \leq k + (t-2)(k-1).$$

Let $\{S\} = \mathcal{H} \setminus \mathcal{G}$. Then

$$\bigcup_{H\in\mathcal{H}} H = (\bigcup_{G\in\mathcal{G}} G) \cup S,$$

thus

$$|\bigcup_{H\in\mathcal{H}} H| = |\bigcup_{G\in\mathcal{G}} G| + |S| - |(\bigcup_{G\in\mathcal{G}} G) \cap S| \leq k + (t-2)(k-1) + k - 1 = k + (t-1)(k-1).$$

□

Proof of Proposition 1.9:

Consider the special case when $\bigcap_{F\in\mathcal{F}} F=\emptyset$. By Lemma 2.1 there exists a $\mathcal{G}\subseteq\mathcal{F}$ subset such that $\bigcap_{G\in\mathcal{G}} G=\emptyset$ and $|\mathcal{G}|=k+1$. Let

$$M:=\bigcup_{G\in\mathcal{G}} G.$$

It follows from Lemma 2.3 that $|M|\leq k+k(k-1)=k^2$. On the other hand it is easy to see that $|M\cap F|\geq \ell_1+1$ for each $F\in\mathcal{F}$ by Lemma 2.2.

Let T be a fixed subset of M such that $|T|=\ell_1+1$. Define

$$\mathcal{F}(T):=\{F\in\mathcal{F}:\ T\subseteq M\cap F\}.$$

Let $L':=\{\ell_2,\dots,\ell_s\}$. Clearly $|L'|=s-1$. Then $\mathcal{F}(T)$ is an L'-intersecting, k-uniform family, because $\mathcal{F}$ is an L-intersecting family and $|M\cap F|\geq \ell_1+1$ for each $F\in\mathcal{F}$.

Proposition 2.4

$$\mathcal{F}=\bigcup_{T\subseteq M,|T|=\ell_1+1}\mathcal{F}(T).$$

Proof. Let $\mathcal{M}:=\bigcup_{T\subseteq M,|T|=\ell_1+1}\mathcal{F}(T)$. Clearly $\mathcal{M}\subseteq\mathcal{F}$. We prove that $\mathcal{F}\subseteq\mathcal{M}$. Let $F\in\mathcal{F}$ be an arbitrary subset. Firstly, if $F\in\mathcal{G}$, then $F\cap M=F$, because $M=\bigcup_{G\in\mathcal{G}} G$. Let T be a fixed subset of F such that $|T|=\ell_1+1$. Then $F\in\mathcal{F}(T)$. Secondly, suppose that $F\notin\mathcal{G}$. Then $|F\cap M|\geq\ell_1+1$ by Lemma 2.2. Let T be a fixed subset of $F\cap M$ such that $|T|=\ell_1+1$. Then $F\in\mathcal{F}(T)$ again. □

Let T be a fixed, but arbitrary subset of M such that $|T|=\ell_1+1$. Consider the set system

$$\mathcal{G}(T):=\{F\setminus T:\ F\in\mathcal{F}(T)\}.$$

Clearly $|\mathcal{G}(T)|=|\mathcal{F}(T)|$. Let $\overline{L}:=\{\ell_2-\ell_1-1,\dots,\ell_s-\ell_1-1\}$. Here $|\overline{L}|=s-1$. Since $\mathcal{F}(T)$ is an L'-intersecting, k-uniform family, thus $\mathcal{G}(T)$ is an $\overline{L}$-intersecting, $(k-\ell_1-1)$-uniform family and $G\subseteq[n]\setminus T$ for each $G\in\mathcal{G}(T)$. Hence it follows from Theorem 1.6 that

$$|\mathcal{F}(T)|=|\mathcal{G}(T)|\leq\binom{n-\ell_1-1}{s-1}.$$

Finally Proposition 2.4 implies that

$$|\mathcal{F}|\leq\sum_{T\subseteq M,|T|=\ell_1+1}|\mathcal{F}(T)|\leq\binom{k^2}{\ell_1+1}\binom{n-\ell_1-1}{s-1},$$

but

$$\binom{n-\ell_1-1}{s-1} = \frac{s}{n-\ell_1}\binom{n-\ell_1}{s},$$

hence

$$|\mathcal{F}| \le \binom{k^2}{\ell_1+1}\frac{s}{n-\ell_1}\binom{n-\ell_1}{s} \le \binom{n-\ell_1}{s}$$

because $n \ge \binom{k^2}{\ell_1+1}s + \ell_1$. □

Proof of Theorem 1.10:

First we handle the case when $|\bigcap_{G\in\mathcal{G}} G| \ge \ell_1$. Let T be a fixed subset of $\bigcap_{G\in\mathcal{G}} G$ such that $|T| = \ell_1$. Consider the set system

$$\mathcal{K} := \{G \setminus T :\ G \in \mathcal{G}\}.$$

Obviously $|\mathcal{G}| = |\mathcal{K}|$. Let $L' := \{0, \ell_2 - \ell_1, \dots, \ell_s - \ell_1\}$. Then clearly $\mathcal{K}$ is a $(k - \ell_1)$-uniform L'-intersecting set system of subsets of $[n] \setminus T$. It follows immediately from Ray-Chaudhuri–Wilson Theorem 1.6 that

$$|\mathcal{G}| = |\mathcal{K}| \le \binom{n-\ell_1}{s}.$$

Now suppose that $|\bigcap_{G\in\mathcal{G}} G| = t$, where $0 < t < \ell_1$. Let T be a fixed subset of $\bigcap_{G\in\mathcal{G}} G$ such that $|T| = t$. Then consider the set system

$$\mathcal{F} := \{G \setminus T :\ G \in \mathcal{G}\}.$$

Clearly $|\mathcal{F}| = |\mathcal{G}|$. Let $L' := \{\ell_1 - t, \ell_2 - t, \dots, \ell_s - t\}$. Then clearly $\mathcal{F}$ is a $(k - t)$-uniform L'-intersecting set system of subsets of $[n] \setminus T$. It follows from Proposition 1.9 that

$$|\mathcal{G}| = |\mathcal{F}| \le \binom{n-t-(\ell_1-t)}{s} = \binom{n-\ell_1}{s}.$$

Finally suppose that $\bigcap_{G\in\mathcal{G}} G = \emptyset$. Then Proposition 1.9 gives us immediately that

$$|\mathcal{G}| \le \binom{n-\ell_1}{s}.$$

□

Proof of Corollary 1.10:

It follows from the proof of Theorem 1.10 that if $|\mathcal{F}| = \binom{n-\ell_1}{s}$ and $n > \binom{k^2}{\ell_1+1}s + \ell_1$, then $|\bigcap_{F\in\mathcal{F}} F| \ge \ell_1$. Thus there exists a $T \in \binom{[n]}{\ell_1}$ such that $T \subseteq F$ for each $F \in \mathcal{F}$.

□

3 Remarks

Let $q \geq 2$ stand for a fixed prime power. Let $PG(2, q)$ denote the finite projective plane over the Galois field $GF(q)$. Denote by $\mathcal{L}$ the set of all lines of $PG(2, q)$. Let $k := q + 1$. Then $\mathcal{L}$ can be considered as a k-uniform family of subsets of the base set $[k^2 - k + 1]$. Clearly $|\mathcal{L}| = k$.

This example motivates our next conjecture.

Conjecture 1 Let $0 < s \leq k \leq n$ be positive integers. Let $L = \{\ell_1, \ldots, \ell_s\}$ be a set of s positive integers such that $0 < \ell_1 < \cdots < \ell_s$. Suppose that $n > k^2 - k + 1$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be an L-intersecting, k-uniform family of subsets of $[n]$. Then

$$m \leq \binom{n - \ell_1}{s}.$$

Further, if

$$|\mathcal{F}| = \binom{n - \ell_1}{s},$$

then there exists a $T \in \binom{[n]}{\ell_1}$ subset such that $T \subseteq F$ for each $F \in \mathcal{F}$.

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