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Nonagon quadruple systems: existence, balance, embeddings

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Abstract

A cycle of length 9 of vertices $(x_1, x_2, \dots, x_9)$, in the cyclical order, with the three edges $\{x_1, x_4\}, \{x_4, x_7\}, \{x_1, x_7\}$ is called an NQ-graph or also a nonagon quadruple graph. A nonagon quadruple system, briefly NQS, of order v and index λ is an NQ-decomposition of the complete multigraph λK_v. An NQS is said to be perfect if the inside K_3, generated by the vertices x_1, x_4, x_7, forms a Steiner triple system; it is said to be balanced if all the vertices have the same degree. In this paper, the spectrum of NQSs, the spectrum of perfect NQSs and the spectrum of balanced NQSs are completely determined.

1 Introduction

Let λK_v be the complete multigraph defined in a vertex-set X, with $|X| = v$, so that there are exactly λ edges joining each pair of vertices. Let G be a subgraph of λK_v. A *G-decomposition* of λK_v, of order v and index λ, is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge-set of λK_v into subsets all of which yield subgraphs isomorphic to G. A *G-decomposition* of λK_v is also called a G-design, of order v and index λ. The classes of the partition $\mathcal{B}$ are said the *blocks* of Σ.

A cycle of length 9 of vertices $(x_1, x_2, \dots, x_9)$, in the cyclical order, with the three *edges* $\{x_1, x_4\}, \{x_4, x_7\}, \{x_1, x_7\}$ is called a *nonagon quadruple graph*, briefly a *NQ-graph*. Such a graph will be indicated by $[(x_1), x_2, x_3, (x_4), x_5, x_6, (x_7), x_8, x_9]$.

A *nonagon quadruple system*, briefly a NQS, of order v and index λ is a NQ-decomposition of the complete multigraph λK_v. A NQS is said *perfect*, briefly a $PNQS$, if the *inside* graphs K_3, generated by the vertices x_1, x_4, x_7, form a 3-cycle system of order v, which we say is embedded (or also nested) in the NQS. If a perfect NQS has index λ and the inside triples form a 3-cycle system of order v and index μ, we will say that the $PNQS$ has indices (λ, μ).

A *nonagon quadruple system* NQS is said to be *balanced*, briefly denoted $BNQS$, if all the vertices have the same *degree*: in other words, there exists a constant d, such that every vertex is contained in d blocks.

In this paper, the spectrum of NQSs, the spectrum of *perfect* NQSs and the spectrum of *balanced* NQSs are completely determined.

The problem of studying the embedding of a Steiner triple system in another design started with the case of the embedding of a Steiner triple system in an hexagon triple system [9] or in a dexagon triple system case [10]. Later similar ideas, dealing also with the problem of embedding a 4-cycle system, have been developed in [3, 6, 1, 2, 4, 5].

Observe that, in what follows, we will use *difference methods* to construct G-designs. This means that, fixed as vertex-set $X = \mathbb{Z}_v$, the ring of integers modulo v, the translates of a *base-block* of vertices $\{x, y, \ldots, z\}$ will be all the blocks having for vertices $\{x+i, y+i, \ldots, z+i\}$, for every $i \in \mathbb{Z}_v$. For a given v, we will consider the difference set $D(v) = \{|x-y| \colon x, y \in \mathbb{Z}_v,\ x \neq y\}$. Further, when the index of a system is not indicated, it means that the index is equal to one.

2 Necessary existence conditions

In this section we determine necessary existence conditions for NQSs, $PNQS$s and $BNQS$s.

2.1 ***NQS*s**:

Theorem 2.1 *If $\Sigma = (X, \mathcal{B})$ is a NQS of order v and index λ, then:*
1) $|\mathcal{B}| = \frac{\lambda v(v-1)}{24}$;
2) $\lambda = 1 \Longrightarrow v \equiv 1, 9 \pmod{24}, v \geq 9.$

Proof: 1) Immediate. 2) If $\lambda = 1$, since all the vertices have even degree in the blocks of Σ, it follows that v is odd. Therefore $v = 8k+1$, for $k \in \mathbb{N}$, $v \geq 9$. Further, necessarily $v(v-1) \equiv 0 \pmod 3$. From this, by simple calculations, it follows $v \equiv 1, 9 \pmod{24}$. □

2.2 ***BNQS*s**:

If $\Sigma = (X, \mathcal{B})$ is a balanced NQS of order v, we indicate by d the constant number of blocks containing any vertex $x \in X$, by C_x the number of blocks in which a vertex x occupies a central position as x_1, x_4, x_7, by E_x the number of blocks in which a vertex x occupies a non-central position.

Theorem 2.2 *If $\Sigma = (X, \mathcal{B})$ is a balanced NQS of order v, then:*

1) $d = \frac{3(v-1)}{8}$;

2) *for every vertex* $x \in X$, $C_x = \frac{v-1}{8}$, $E_x = \frac{v-1}{4}$;

3) *necessarily* $v \equiv 1, 9 \pmod{24}$, $v \geq 9$.

Proof: Let $\Sigma = (X, \mathcal{B})$ be a balanced NQS of order v.

1) The technique to calculate the degree d of the vertices is well-known. Easily, since every block contains 9 vertices, it follows that:

$d \cdot v = 9 \cdot |\mathcal{B}|$,

from which: $d \cdot v = 9v(v-1)/24$, and then $d = \frac{3(v-1)}{8}$.

2) For every $x \in X$, we have:

$4C_x + 2E_x = v - 1$,

$C_x + E_x = d = \frac{3(v-1)}{8}$,

and hence: $C_x = \frac{v-1}{8}$, $E_x = \frac{v-1}{4}$.

3) From Theorem 2.1: $v \equiv 1$ or $9 \pmod{24}$. From 1) and 2), we must have $v \equiv 1 \pmod 8$. Necessarily $v \equiv 1$ or $9 \pmod{24}$, $v \geq 9$. □

From Theorem 2.2 it follows that the possible spectrum of NQSs is the same of the possible spectrum of $BNQS$s. Further, always from Theorem 2.2, we have that: *in a balanced NQS there are two constants $C, E \in N$, such that $C_x = C$, $E_x = E$, for every vertex x.* This means that *every balanced NQS is strongly balanced*; which is equivalent to saying that in balanced NQSs every vertex has constant degree *inside* the two automorphism classes of the graph NQ ([7]).

2.3 ***PNQS*s:**

Theorem 2.3 *If $\Sigma = (X, \mathcal{B})$ is a perfect NQS of order v and indices (λ, μ), then:*

1) $\lambda = 4\mu$;

2) *for $\mu = 1$, necessarily* $v \equiv 1, 3 \mod 6, v \geq 9$.

Proof: 1) Let $\Sigma = (X, \mathcal{B})$ be a perfect NQS of order v and indices (λ, μ). If $\Sigma' = (X, B')$ is the STS nested in Σ, necessarily $|\mathcal{B}| = |\mathcal{B}'|$. Therefore, since

$$|\mathcal{B}| = \lambda \frac{v(v-1)}{24}, \ |\mathcal{B}'| = \mu \frac{v(v-1)}{6},$$

we easily have $\lambda = 4\mu$. To prove 2), let $\mu = 1$. It is sufficient to consider that Σ' is a Steiner triple system of index 1 and it is well-known ([8]) that the spectrum consists of all $v \equiv 1, 3 \mod 6$, $v \geq 3$. In this case, necessarily $v \geq 9$. □

3 The spectrum of NQSs and $BNQS$s

In this section we completely determine the spectrum of NQSs and the spectrum of balanced NQSs. We will see that the two spectra coincide.

Theorem 3.1 *There exist balanced NQSs of order $v = 24k + 9$, $v \geq 9$.*

Proof: Let $X = \mathbb{Z}_v$.
1) Let $v = 9$. Observe that the difference set is $D(9) = \{1, 2, 3, 4\}$. Therefore, consider the following base-blocks:

$B_1 = [(0), 7, 8, (3), 1, 2, (6), 4, 5]$,
$B_2 = [(1), 8, 0, (4), 2, 3, (7), 5, 6]$,
$B_3 = [(2), 0, 1, (5), 3, 4, (8), 6, 7]$.

If $\mathcal{B} = \{B_1, B_2, B_3\}$, we can verify that $\Sigma = (\mathbb{Z}_9, \mathcal{B})$ is a balanced NQS of order 9 and index $\lambda = 1$, consisting of $|\mathcal{B}| = \frac{1 \cdot 9 \cdot 8}{24} = 3$ blocks and every vertex appears in $d = \frac{3(9-1)}{8} = 3$ blocks.

2) Let $v = 33$. Observe that the difference set is $D(33) = \{1, 2, \ldots, 16\}$. At first, consider the family $\mathcal{F} = \{F_1, F_2, \ldots, F_{11},\}$ of NQs defined as follows:

$F_1 = [(0), 1, 15, (14), 12, 32, (13), 11, 31]$,
$F_2 = [(3), 4, 18, (17), 15, 2, (16), 14, 1]$,
$F_3 = [(6), 7, 21, (20), 18, 5, (19), 17, 4]$,
$F_4 = [(9), 10, 24, (23), 21, 8, (22), 20, 7]$,
$F_5 = [(12), 13, 27, (26), 24, 11, (25), 23, 10]$,
$F_6 = [(15), 16, 30, (29), 27, 14, (28), 26, 13]$,
$F_7 = [(18), 19, 0, (32), 30, 17, (31), 29, 16]$,
$F_8 = [(21), 22, 3, (2), 0, 20, (1), 32, 19]$,
$F_9 = [(24), 25, 6, (5), 3, 23, (4), 2, 22]$,
$F_{10} = [(27), 28, 9, (8), 6, 26, (7), 5, 25]$,
$F_{11} = [(30), 31, 12, (11), 9, 29, (10), 8, 28]$.

Further, consider the following base-block:

$B = [(0), 15, 10, (16), 13, 3, (12), 4, 11]$.

If $\mathcal{B}$ is the collection of all the translates of B and $\Gamma = \mathcal{F} \cup \mathcal{B}$, we can see that $\Sigma = (X, \Gamma)$ is a NQS of order $v = 33$.

To verify this, observe that in the blocks of $\mathcal{F}$ there are all the edges $\{x, y\}$, for $x, y \in X$, such that $|x - y| = 1, 2, 13, 14$. In the blocks of $\mathcal{B}$ there are all the edges in which the differences between the extremes belongs to $D(33) - \{1, 2, 13, 14\}$.

We can also see that Σ is balanced. Indeed, every vertex has degree 3 in $\mathcal{F}$ and degree 9 in $\mathcal{B}$. At last, every vertex has degree $d = 12$.

3) Let $v = 24k+9$, $k \geq 2$. Observe that the difference set is $D(v) = \{1, 2, \ldots, 12k+4\}$. At first, consider the family $\mathcal{A}$ of NQs defined as follows:

$\mathcal{A} = \{A_i : i = 0, 1, 2, \ldots, 8k+2\}$,

where

$A_i = [(3i), 3i+1, 3i+12k+3, (3i+12k+2), 3i+12k, 3i-1, (3i+12k+1), 3i+12k-1, 3i-2]$.

Then, define the following base-blocks:

$B = [(0), 12k, 2, (8k), 20k-1, 10k, (2k+1), 14k+5, 12k+3]$,

and

$C_j = [(0), 6k-2j-1, 18k-4j-2, (8k-2j-1), 2j-1, 12k-2j-1, (2j+1), 22k+9, 2j+2]$

for every $j = 1, 2, \ldots, k-1$.

If $\mathcal{B}$ is the family of all the translates of the blocks B, $\mathcal{C}$ is the family of all the translates of the blocks C_j, for every $j = 1, 2, \ldots, k-1$, and $\Gamma = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, we can see that $\Sigma = (X, \Gamma)$ is a NQS of order $v = 24k+9$.

To verify, observe that:

- in the blocks of $\mathcal{A}$ there are all the edges $\{x, y\}$, for $x, y \in X$, such that $|x-y| = 1, 2, 12k+1, 12k+2$;
- in the blocks of $\mathcal{B}$ there are all the edges $\{x, y\}$, for $x, y \in X$, such that $|x-y| = 2k+1, 2k+2, 6k-1, 8k-2, 8k-1, 8k, 10k-1, 12k-2, 12k-1, 12k, 12k+3, 12k+4$;
- in the blocks of $\mathcal{C}$ there are all the edges in which the differences between the extremes are all the others in D.

We can also see that Σ is balanced. Indeed, every vertex has degree 3 in $\mathcal{A}$, degree 9 in $\mathcal{B}$ and degree $9(k-1)$ in $\mathcal{C}$. Finally, every vertex has degree $d = 9k+3$. □

Theorem 3.2 *There exist balanced $NQSs$ of order $v = 24k+1$, $v \geq 25$.*

Proof: Let $X = \mathbb{Z}_v$. Consider the following base-blocks defined in $\mathbb{Z}_v$:

$B_i = [(0), 6k-2i-1, 18k-4i-2, (8k-2i-1), 2i-1, 12k-2i-1, (2i+1), 22k+1, 2i+2]$,

for every $i = 0, 1, 2, \ldots, k-1$. If $\mathcal{B}$ is the collection of all the translates of $B_0, B_1, \ldots, B_{k-1}$, we can verify that $\Sigma = (X, \mathcal{B})$ is a NQS of order $v = 24k+1$. Further, since all the blocks of Σ are obtained by the fixed base-blocks so that every vertex is contained in nine translates of any base-block, every vertex has degree $d = 9k$ and Σ is balanced. □

Collecting together the results of this section, it follows that:

Theorem 3.3 *There exists a NQS of order v if and only if $v \equiv 1$ or $9 \pmod{24}$.*

Theorem 3.4 *There exists a balanced NQS of order v if and only if $v \equiv 1$ or $9 \pmod{24}$.*

4 The spectrum of perfect NQSs

In this section we determine the spectrum of $PNQS$s.

We recall that an *automorphism* of a STS(v) $\Sigma = (X, \mathcal{B})$ is a bijection $\varphi : X \to X$ such that $B = \{x, y, z\} \in \mathcal{B}$ if and only if $\varphi(B) = \{\varphi(x), \varphi(y), \varphi(z)\} \in \mathcal{B}$. An STS($v$) is said to be *cyclic* if it admits an automorphism that is a permutation consisting of a single cycle of length v. The following is well-known.

Theorem 4.1 ([12]) *For every $v \geq 3$ and $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$, there exists a cyclic STS(v).*

Peltesohn, in her construction (see [7, 11, 12]) considers the cases $v = 18k + 1$, $v = 18k + 7$, $v = 18k + 13$, if $v \equiv 1 \pmod 6$, and the cases $v = 18k + 3$, $v = 18k + 9$ and $v = 18k + 15$, if $v \equiv 3 \pmod 6$. Given, on the vertex set Z_v, the difference set $D(v) = \{1, 2, \dots, \frac{v-1}{2}\}$, if $v \equiv 1 \pmod 6$, and $D(v) = \{1, 2, \dots, \frac{v-1}{2}\} - \{\frac{v}{3}\}$, if $v \equiv 3 \pmod 6$, Peltesohn determines a partition of $D(v)$ into triples of differences. These triples are of type $\{3h+1, a-3h-1, a\}$, $\{3h+2, b-3h-2, b\}$ and $\{3h+3, c-3h-3, c\}$, for some a , b, c and h. In all the cases we will consider the following blocks:

$$\begin{cases} A_h = [(0), b, a+b, (a), a+b-3h-2, b-1, (3h+1), 6h+3, 3h+2], \\ B_h = [(0), c, b+c, (b), b+c-3h-3, c-1, (3h+2), 6h+5, 3h+3], \\ C_h = [(0), a, a+c, (c), a+c-3h-1, a+2, (3h+3), 6h+4, 3h+1], \end{cases} \qquad (1)$$

in such a way that all the differences $3h + 1$, $3h + 2$, $3h + 3$, $a - 3h - 1$, $b - 3h - 2$, $c - 3h - 3$, a, b and c appear globally four times in A_h, B_h and C_h and once in the inside triples. Note also that, given two of the above differences x and y, we must have $x - y \not\equiv 0 \pmod v$ and $x + y \not\equiv 0 \pmod v$. So we can immediately say, without any computation, that some of the vertices in the blocks above are pairwise distinct.

Using this notation we will prove the following theorem.

Theorem 4.2 *If $v \equiv 1 \pmod 6$, $v \geq 13$, there exists a PNQS of order v and with indices ($\lambda = 4$, $\mu = 1$).*

Proof: It is known that for every $v \equiv 1$ or $3 \pmod 6$, $v \geq 3$, there exist Steiner triple systems STS of order v. Let $\Sigma' = (X, \mathcal{B}')$ be an STS(v) with index $\mu = 1$, whose blocks are the *inside* triples contained in the blocks of Σ. From Theorem 2.3, Σ has index $\lambda = 4$.

1) Let $v = 18k + 1$ for some $k \geq 1$.

Let $v = 19$. Consider the base-blocks defined on $\mathbb{Z}_{19}$:

$$D_1 = [(0), 10, 16, (6), 14, 9, (1), 3, 2],$$
$$D_2 = [(0), 7, 17, (10), 14, 6, (2), 5, 3],$$
$$D_3 = [(0), 6, 13, (7), 12, 8, (3), 4, 1].$$

If $\mathcal{B}$ is the collection of all the translates of the blocks D_1, D_2 and D_3, then one can verify that the system $\Sigma = (\mathbb{Z}_{19}, \mathcal{B})$ is a $PNQS$ of order 19 and indices $(4, 1)$, whose inside triples determine an STS of order 19.

Let $k \geq 2$. In this case the partition of $D(v)$ given by Petelsohn is the following:

- $\{3h + 1, 4k - h + 1, 4k + 2h + 2\}$ for $h \in \{0, \ldots, k - 1\}$
- $\{3h + 2, 8k - h, 8k + 2h + 2\}$ for $h \in \{0, \ldots, k - 1\}$
- $\{3h + 3, 6k - 2h - 1, 6k + h + 2\}$ for $h \in \{0, \ldots, k - 2\}$
- $\{3k, 3k + 1, 6k + 1\}$.

So a cyclic STS on $\mathbb{Z}_v$ can be constructed with triples having these as differences and, keeping the notation given in (1), $a = 4k + 2h + 2$, $b = 8k + 2h + 2$, $c = 6k + h + 2$.

Consider now the following base-blocks on $X = \mathbb{Z}_v$:

$$A_h \text{ for } 0 \leq h \leq k - 1 \text{ and } B_h \text{ and } C_h \text{ for } 0 \leq h \leq k - 2,$$
$$D_1 = [(0), 6k + 1, 16k + 1, (10k), 13k + 1, 6k, (3k - 1), 6k - 1, 3k],$$
$$D_2 = [(0), 6k, 12k + 1, (6k + 1), 9k + 3, 6k + 2, (3k), 6k - 2, 3k - 2].$$

Note that in this case we have $4k+2 \leq a \leq 6k$, $8k+2 \leq b \leq 10k$ and $6k+2 \leq c \leq 7k$. So we easily see that all the vertices in the blocks A_h, B_h and C_h are distinct.

If $\mathcal{B}$ is the collection of all the translates of the blocks in the set:

$$\{A_h : 0 \leq h \leq k - 1\} \cup \{B_h : 0 \leq h \leq k - 2\} \cup \{C_h : 0 \leq h \leq k - 2\} \cup \{D_1, D_2\},$$

then one can verify that the system $\Sigma = (\mathbb{Z}_v, \mathcal{B})$ is a $PNQS$ of order v with indices $(4, 1)$, whose inside triples determine an STS of order v.

2) Let $v = 18k + 7$ for some $k \geq 1$. In this case the partition of $D(v)$ given by Petelsohn is the following:

- $\{3h + 1, 8k - h + 3, 8k + 2h + 4\}$ for $h \in \{0, \ldots, k - 1\}$
- $\{3h + 2, 6k - 2h + 1, 6k + h + 3\}$ for $h \in \{0, \ldots, k - 1\}$
- $\{3h + 3, 4k - h + 1, 4k + 2h + 4\}$ for $h \in \{0, \ldots, k - 1\}$
- $\{3k + 1, 4k + 2, 7k + 3\}$.

So a cyclic STS on $\mathbb{Z}_v$ can be constructed with triples having these as differences and, keeping the notation given in (1), $a = 8k + 2h + 4$, $b = 6k + h + 3$, $c = 4k + 2h + 4$.

Consider now the following base-blocks on $X = \mathbb{Z}_v$:

A_h and B_h for $0 \leq h \leq k-1$ and C_h for $1 \leq h \leq k-1$, in the case $k \geq 2$,
$D_1 = [(0), 7k+3, 11k+7, (4k+4), 8k+6, 4k+5, (3), 3k+4, 3k+1]$,
$D_2 = [(0), 8k+4, 15k+7, (7k+3), 15k+6, 11k+4, (3k+1), 3k+2, 1]$.

Note that in this case we have $8k + 4 \leq a \leq 10k + 2$, $6k + 3 \leq b \leq 7k + 2$ and $4k + 4 \leq c \leq 6k + 2$. So we easily see that all the vertices in the blocks A_h, B_h and C_h are distinct.

If $k = 1$, let $\mathcal{B}$ the collection of all the translates of the blocks in the set $\{A_0, B_0, D_1, D_2\}$, while for $k \geq 2$ let $\mathcal{B}$ the collection of all the translates of the blocks in the set:

$$\{A_h : 0 \leq h \leq k-1\} \cup \{B_h : 0 \leq h \leq k-1\} \cup \{C_h : 1 \leq h \leq k-1\} \cup \{D_1, D_2\}.$$

Then one can verify that the system $\Sigma = (\mathbb{Z}_v, \mathcal{B})$ is a $PNQS$ of order v and indices $(4, 1)$, whose inside triples determine a STS of order v.

3) Let $v = 18k + 13$ for some $k \geq 0$.

Let $v = 13$. Consider the base-blocks defined on $\mathbb{Z}_{13}$:

$$D_1 = [(0), 7, 11, (4), 9, 6, (1), 3, 2],$$
$$D_2 = [(0), 4, 11, (7), 10, 5, (2), 3, 1].$$

If $\mathcal{B}$ is the collection of all the translates of the blocks D_1 and D_2, then one can verify that the system $\Sigma = (\mathbb{Z}_{13}, \mathcal{B})$ is a $PNQS$ of order 13 and indices $(4, 1)$, whose inside triples determine a STS of order 13.

Let $k \geq 1$. In this case the partition of $D(v)$ given by Petelsohn is the following:

- $\{3h+1, 4k-h+3, 4k+2h+4\}$ for $h \in \{0, \ldots, k\}$
- $\{3h+2, 6k-2h+3, 6k+h+5\}$ for $h \in \{0, \ldots, k-1\}$
- $\{3h+3, 8k-h+5, 8k+2h+8\}$ for $h \in \{0, \ldots, k-1\}$
- $\{3k+2, 7k+5, 8k+6\}$.

So a cyclic STS on Z_v can be constructed with triples having these as differences and, keeping the notation given in (1), $a = 4k + 2h + 4$, $b = 6k + h + 5$, $c = 8k + 2h + 8$.

Consider now the following base-blocks on $X = \mathbb{Z}_v$:

A_h, B_h and C_h for $0 \leq h \leq k-1$,
$D_1 = [(0), 7k+5, 13k+9, (6k+4), 14k+10, 11k+7, (3k+1), 6k+3, 3k+2]$,
$D_2 = [(0), 6k+4, 16k+11, (10k+7), 13k+10, 6k+5, (3k+2), 6k+3, 3k+1]$.

Note that in this case we have $4k+4 \leq a \leq 6k+4$, $6k+5 \leq b \leq 7k+4$ and $8k+8 \leq c \leq 10k+6$. So we easily see that all the vertices in the blocks A_h, B_h and C_h are distinct.

If $\mathcal{B}$ is the collection of all the translates of the blocks in the set

$$\{A_h \colon 0 \leq h \leq k-1\} \cup \{B_h \colon 0 \leq h \leq k-1\} \cup \{C_h \colon 0 \leq h \leq k-1\} \cup \{D_1, D_2\},$$

then one can verify that the system $\Sigma = (\mathbb{Z}_v, \mathcal{B})$ is a $PNQS$ of order v and with indices $(4,1)$, whose inside triples determine an STS of order v. □

Using the same technique we will give a construction in the case $v \equiv 3 \pmod 6$.

Theorem 4.3 *If $v \equiv 3 \pmod 6$, $v \geq 13$, there exists a $PNQS$ of order v with indices $(\lambda = 4, \mu = 1)$.*

Proof: Let $v = 9$. Consider the following blocks defined on $\mathbb{Z}_9$:

$$\begin{aligned}
D_1 =& [(1), 9, 8, (3), 6, 7, (2), 5, 4] \\
D_2 =& [(4), 2, 8, (6), 7, 1, (5), 3, 9] \\
D_3 =& [(7), 4, 2, (9), 6, 1, (8), 3, 5] \\
D_4 =& [(1), 2, 5, (7), 3, 6, (4), 8, 9] \\
D_5 =& [(2), 3, 4, (8), 7, 1, (5), 9, 6] \\
D_6 =& [(3), 5, 8, (9), 2, 4, (6), 1, 7] \\
D_7 =& [(1), 3, 6, (9), 4, 7, (5), 8, 2] \\
D_8 =& [(2), 3, 9, (7), 4, 5, (6), 8, 1] \\
D_9 =& [(3), 7, 9, (4), 5, 2, (8), 6, 1] \\
D_{10} =& [(1), 4, 3, (8), 7, 2, (6), 5, 9] \\
D_{11} =& [(2), 6, 7, (9), 5, 8, (4), 1, 3] \\
D_{12} =& [(3), 9, 2, (7), 8, 1, (5), 6, 4].
\end{aligned}$$

If $\mathcal{B}$ is the collection of the blocks $D_1,\ldots,D_{12}$, then one can verify that the system $\Sigma = (\mathbb{Z}_9, \mathcal{B})$ is a $PNQS$ of order 9 and indices $(4,1)$.

Now we will prove the statement in the other cases, using the notation given previously in (1).

1) Let $v = 18k+3$, for some $k \geq 1$. In this case the partition of $D(v)$ given by Petelsohn is the following:

- $\{3h+1, 8k-h+1, 8k+2h+2\}$ for $h \in \{0, \ldots, k-1\}$,
- $\{3h+2, 4k-h, 4k+2h+2\}$ for $h \in \{0, \ldots, k-1\}$,
- $\{3h+3, 6k-2h-1, 6k+h+2\}$ for $h \in \{0, \ldots, k-1\}$.

So $a = 8k + 2h + 2$, $b = 4k + 2h + 2$ and $c = 6k + h + 2$. Moreover, a cyclic STS on $\mathbb{Z}_v$ can be constructed with triples having these are differences, plus the triples having differences $\{6k + 1, 6k + 1, 6k + 1\}$.

Consider now the following base-blocks defined on $X = \mathbb{Z}_v$:

$$\begin{aligned}
&A_h,\ B_h \text{ and } C_h, \text{ for } 1 \leq h \leq k-1, \text{ in the case } k \geq 2,\\
&D_1 = [(0), 8k+1, 16k+3, (8k+2), 16k+4, 8k+3, (1), 4, 2],\\
&D_2 = [(0), 6k+2, 10k+4, (4k+2), 10k+1, 6k+1, (2), 5, 3],\\
&D_3 = [(0), 6k+1, 12k+3, (6k+2), 2k+2, 6k+4, (3), 2, 4k+2].
\end{aligned}$$

Note that in this case for any h we have that $10h + 10 \leq a \leq 10k$, $6h + 6 \leq b \leq 6k$, $7h + 8 \leq c \leq 7k + 1$. So we easily see that all the vertices in the blocks A_h, B_h and C_h are distinct.

Consider also the following blocks:

$E_i = [(i), i+18k+2, i+18k+1, (i+12k+2), i+12k+1, i+12k, (i+6k+1), i+6k, i+6k-1]$.

for $i \in \{0, \ldots, 6k\}$. Let $\mathcal{E} = \{E_i \colon i = 0, \ldots, 6k\}$, let $\mathcal{F}$ be the collection of all the translates of the blocks in the set:

$$\{A_h \colon 1 \leq h \leq k-1\} \cup \{B_h \colon 1 \leq h \leq k-1\} \cup \{C_h \colon 1 \leq h \leq k-1\}$$

and let $\mathcal{G}$ be the collection of all the translates of the blocks in the set $\{D_1, D_2, D_3\}$. If $k = 1$, take $\mathcal{B} = \mathcal{E} \cup \mathcal{G}$, while for $k \geq 2$ take $\mathcal{B} = \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$. Then one can verify that the system $\Sigma = (Z_v, \mathcal{B})$ is a $PNQS$ of order v and indices $(4, 1)$, whose inside triples determine a cyclic STS of order v.

2) Let $v = 18k + 9$, for some $k \geq 1$.

Let $v = 27$. Consider the base-blocks defined on $\mathbb{Z}_{27}$:

$$\begin{aligned}
D_1 =& [(0), 7, 20, (13), 18, 6, (1), 3, 2],\\
D_2 =& [(0), 8, 15, (7), 18, 13, (2), 11, 9],\\
D_3 =& [(0), 10, 21, (11), 15, 7, (3), 12, 6],\\
D_4 =& [(0), 13, 23, (10), 22, 16, (4), 5, 1].
\end{aligned}$$

Furthermore consider the blocks:

$$E_i = [(i), i+24, i+21, (i+18), i+15, i+12, (i+9), i+6, i+3],$$

for $i \in \{0, \ldots, 8\}$. If $\mathcal{B}$ is the collection of the blocks E_i and of all the translates of D_1, D_2, D_3 and D_4, then one can verify that the system $\Sigma = (\mathbb{Z}_{27}, \mathcal{B})$ is a $PNQS$ of order 27 and indices $(4, 1)$, whose inside triples determine a cyclic STS of order 27.

Let $v = 45$. Consider the base-blocks defined on $X = \mathbb{Z}_{45}$:

$$\begin{aligned} D_1 =& [(0), 19, 31, (12), 29, 18, (1), 3, 2],\\ D_2 =& [(0), 23, 42, (19), 39, 22, (2), 5, 3],\\ D_3 =& [(0), 12, 35, (23), 34, 14, (3), 4, 1],\\ D_4 =& [(0), 24, 38, (14), 32, 22, (4), 10, 6],\\ D_5 =& [(0), 8, 21, (13), 26, 18, (5), 20, 10],\\ D_6 =& [(0), 16, 40, (24), 33, 15, (6), 13, 7],\\ D_7 =& [(0), 14, 30, (16), 31, 22, (7), 11, 4]. \end{aligned}$$

Furthermore consider the blocks:

$$E_i = [(i), i + 40, i + 35, (i + 30), i + 25, i + 20, (i + 15), i + 10, i + 5],$$

for $i \in \{0, \dots, 14\}$. If $\mathcal{B}$ is the collection of the blocks E_i and of all the translates of $D_1, \dots, D_7$, let $\Sigma = (\mathbb{Z}_{45}, \mathcal{B})$. Note that the blocks D_1, D_2 and D_3 are constructed as A_h, B_h and C_h, so that with the previous notation $D_1 = A_0$, $D_2 = B_0$ and $D_3 = C_0$. So one can verify that the system Σ is a $PNQS$ of order 45 and indices $(4, 1)$, whose inside triples determine a cyclic STS of order 45.

Let $v = 63$. Consider the base-blocks defined on $X = \mathbb{Z}_{63}$:

$$\begin{aligned} D_1 =& [(0), 29, 45, (16), 43, 28, (1), 3, 2],\\ D_2 =& [(0), 28, 57, (29), 54, 27, (2), 5, 3],\\ D_3 =& [(0), 16, 44, (28), 43, 18, (3), 4, 1],\\ D_4 =& [(0), 31, 49, (18), 44, 30, (4), 9, 5],\\ D_5 =& [(0), 23, 54, (31), 48, 22, (5), 11, 6],\\ D_6 =& [(0), 18, 41, (23), 44, 27, (6), 10, 4],\\ D_7 =& [(0), 13, 33, (20), 40, 27, (7), 28, 14],\\ D_8 =& [(0), 33, 52, (19), 43, 32, (8), 17, 9],\\ D_9 =& [(0), 22, 55, (33), 45, 21, (9), 19, 10],\\ D_{10} =& [(0), 19, 41, (22), 33, 21, (10), 18, 8]. \end{aligned}$$

Furthermore consider the blocks:

$$E_i = [(i), i + 56, i + 49, (i + 42), i + 35, i + 28, (i + 21), i + 14, i + 7],$$

for $i \in \{0, \dots, 20\}$. If $\mathcal{B}$ is the collection of the blocks E_i and of all the translates of $D_1, \dots, D_{10}$, let $\Sigma = (\mathbb{Z}_{63}, \mathcal{B})$. Note that the blocks D_1, D_2 and D_3 and D_8, D_9 and D_{10} are constructed as A_h, B_h and C_h. So one can verify that the system Σ is a $PNQS$ of order 63 and indices $(4, 1)$, whose inside triples determine a cyclic STS of order 63.

Let $v = 18k + 9$, for some $k \geq 4$. In this case the partition of $D(v)$ given by Petelsohn is the following:

- $\{3h+1, 4k-h+3, 4k+2h+4\}$ for $0 \le h \le k$,
- $\{3h+2, 8k-h+2, 8k+2h+4\}$ for $2 \le h \le k-2$,
- $\{3h+3, 6k-2h+1, 6k+h+4\}$ for $1 \le h \le k-2$,
- $\{2, 8k+3, 8k+5\}$, $\{3, 8k+1, 8k+4\}$, $\{5, 8k+2, 8k+7\}$, $\{3k-1, 3k+2, 6k+1\}$, $\{3k, 7k+3, 8k+6\}$.

So $a = 4k+2h+4$, $b = 8k+2h+4$ and $c = 6k+h+4$. Moreover, a cyclic STS on $\mathbb{Z}_v$ can be constructed with triples having these are differences, plus the triples having differences $\{6k+3, 6k+3, 6k+3\}$.

Consider now the following base-blocks defined on $X = \mathbb{Z}_v$:

A_h and B_h for $2 \le h \le k-2$ and C_h, for $1 \le h \le k-2$,
$D_1 = [(0), 8k+5, 14k+7, (6k+2), 14k+5, 11k+1, (3k-2), 3k, 2]$,
$D_2 = [(0), 8k+4, 16k+9, (8k+5), 16k+6, 8k+3, (2), 5, 3]$,
$D_3 = [(0), 6k+2, 14k+6, (8k+4), 11k+8, 3k+7, (3), 3k+1, 3k-2]$,
$D_4 = [(0), 6k+1, 14k+8, (8k+7), 11k+9, 3k+7, (5), 3k+4, 3k-1]$,
$D_5 = [(0), 10k+3, 16k+4, (6k+1), 13k+4, 10k+2, (3k-1), 6k-1, 3k]$,
$D_6 = [(0), 8k+7, 1, (10k+3), 18k+5, 11k+2, (3k), 3k+5, 5]$,
$D_7 = [(0), 6k+3, 12k+8, (6k+4), 10k+7, 7k+4, (3k+1), 7k+5, 4k+4]$,
$D_8 = [(0), 3k+3, 7k+9, (4k+6), 10k+10, 6k+8, (4), 3k+5, 3k+1]$,
$D_9 = [(0), 6k-1, 10k+3, (4k+4), 10k+7, 6k+4, (1), 7, 6,]$.

Note that in this case for any h we have that $6h+4 \le a \le 6k+4$, $10h+20 \le b \le 10k$, $7h+16 \le c \le 7k+2$. So we easily see that all the vertices in the blocks A_h, B_h and C_h are distinct.

Consider also the following blocks:

$$E_i = [(i), i+1, i+2, (i+12k+6), i+12k+7, i+12k+8, (i+6k+3), i+6k+4, i+6k+5]$$

for $i \in \{0, \dots, 6k+2\}$. Let $\mathcal{E} = \{E_i \colon i = 0, \dots, 6k+2\}$ and let $\mathcal{F}$ be the collection of all the translates of the blocks in the set:

$$\{A_h \colon 2 \le h \le k-2\} \cup \{B_h \colon 2 \le h \le k-2\} \cup \{C_h \colon 1 \le h \le k-2\} \cup \{D_1, \dots, D_9\}.$$

If $\mathcal{B} = \mathcal{E} \cup \mathcal{F}$, let $\Sigma = (\mathbb{Z}_v, \mathcal{B})$. Note that the blocks D_1, D_2 and D_3 and D_4, D_5 and D_6 are constructed as A_h, B_h and C_h. Then one can verify that the system Σ is a $PNQS$ of order v and indices $(4,1)$, whose inside triples determine a cyclic STS of order v.

3) Let $v = 18k+15$, for some $k \ge 0$.

Let $v = 15$. Consider the base-blocks defined on $X = \mathbb{Z}_{15}$:

$$D_1 = [(0), 9, 5, (4), 6, 2, (1), 8, 3],$$
$$D_2 = [(0), 4, 11, (8), 13, 7, (2), 9, 3].$$

Furthermore consider the blocks:

$$E_i = [(i), i+13, i+11, (i+10), i+8, i+6, (i+5), i+3, i+1],$$

for $i \in \{0, \ldots, 4\}$. If $\mathcal{B}$ is the collection of the blocks E_i and of all the translates of D_1 and D_2, then one can verify that the system $\Sigma = (\mathbb{Z}_{15}, \mathcal{B})$ is a $PNQS$ of order 15 and indices $(4, 1)$, whose inside triples determine a cyclic STS of order 15.

Let $v = 18k + 15$, for some $k \geq 1$. In this case the partition of $D(v)$ given by Petelsohn is the following:

- $\{3h+1, 4k-h+3, 4k+2h+4\}$ for $0 \leq h \leq k$,
- $\{3h+2, 8k-h+6, 8k+2h+8\}$ for $0 \leq h \leq k$,
- $\{3h+3, 6k-2h+3, 6k+h+6\}$ for $0 \leq h \leq k-1$.

So $a = 4k+2h+4$, $b = 8k+2h+8$ and $c = 6k+h+6$. Moreover, a cyclic STS on $\mathbb{Z}_v$ can be constructed with triples having these differences, plus the triples having differences $\{6k+5, 6k+5, 6k+5\}$.

Consider now the following base-blocks defined on $X = \mathbb{Z}_v$:

A_h for $0 \leq h \leq k$, B_h and C_h for $0 \leq h \leq k-1$,
$D = [(0), 6k+5, 16k+13, (10k+8), 13k+9, 6k+3, (3k+2), 9k+7, 3k+3]$.

Note that in this case for any h we have that $6h+4 \leq a \leq 6k+4$, $10h+8 \leq b \leq 10k+8$, $7h+12 \leq c \leq 7k+5$. So we easily see that all the vertices in the blocks A_h, B_h and C_h are distinct.

Consider also the following blocks:
$E_i = [(i), i+3k+2, i+18k+14, (i+12k+10), i+15k+12, i+12k+9, (i+6k+5), i+9k+7, i+6k+4]$
for $i \in \{0, \ldots, 6k+4\}$. Let $\mathcal{E} = \{E_i : i = 0, \ldots, 6k+4\}$ and let $\mathcal{F}$ be the collection of all the translates of the blocks in the set:

$$\{A_h : 0 \leq h \leq k\} \cup \{B_h : 0 \leq h \leq k-1\} \cup \{C_h : 0 \leq h \leq k-1\} \cup \{D\}.$$

If $\mathcal{B} = \mathcal{E} \cup \mathcal{F}$, then one can verify that the system $\Sigma = (\mathbb{Z}_v, \mathcal{B})$ is a $PNQS$ of order v and indices $(4, 1)$, whose inside triples determine a cyclic STS of order v. $\square$

Corollary 4.4 *There exists a* $PNQS$ *of order* v *and indices* $(\lambda = 4, \mu = 1)$ *if and only if* $v \equiv 1$ *or* $3 \pmod 6$, $v \geq 9$.

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