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Extendability of the complementary prism of bipartite graphs

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Abstract

For a nonnegative integer k, a connected graph G of order at least $2k+2$ is k-extendable if G has a perfect matching and every set of k independent edges extends to a perfect matching in G. The largest integer k such that G is k-extendable is called the extendability of G. The complementary prism $G\overline{G}$ of G is the graph constructed from G and its complement $\overline{G}$ defined on a set of vertices disjoint from $V(G)$ (i.e. $V(G) \cap V(\overline{G}) = \emptyset$) by joining each pair of corresponding vertices by an edge. Janseana and

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Ananchuen [*Thai J. Math.* 13 (2015), 703–721] gave a lower bound to the extendability of $G\overline{G}$ in terms of the extendabilities of G and $\overline{G}$ in the case that neither G nor $\overline{G}$ is a bipartite graph. In this paper, we consider the remaining case and give a sharp lower bound to the extendability of $G\overline{G}$ when G is a bipartite graph.

1 Introduction

In this paper we only consider finite simple undirected graphs. For a nonnegative integer k, a connected graph G of order at least $2k+2$ is said to be k-extendable if G has a perfect matching and every matching of size k extends to a pefect matching in G. This notion was first introduced by Plummer [7]. He proved a number of basic properties of k-extendable graphs. In particular, he proved that if $k > 0$, every k-extendable graph is $(k-1)$-extendable. By this result, we can naturally define the extendability of a graph G, which is the maximum integer k such that G is k-extendable. Extendability is one of the main topics in the theory of matchings in graphs.

The complementary prism is a specific complementary product introduced by Haynes et al. [3]. For a simple graph G and its complement $\overline{G}$, the complementary prism $G\overline{G}$ of G is the graph formed from the disjoint union of G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. More precisely, take a new vertex $\bar{x}$ for each $x \in V(G)$. Then the complementary prism $G\overline{G}$ of G is defined by

$$\begin{aligned} &V(G\overline{G}) = V(G) \cup V(\overline{G}) \quad \text{and} \\ &E(G\overline{G}) = E(G) \cup E(\overline{G}) \cup \{u\bar{u} \colon u \in V(G)\}, \end{aligned}$$

where $E(\overline{G}) = \{\bar{u}\bar{v} \colon u \neq v, uv \notin E(G)\}$. For $z \in V(\overline{G})$, let $\bar{z}$ be the vertex in $V(G)$ with $\bar{\bar{z}} = z$. Moreover, for $S \subset V(G\overline{G})$, we define $\overline{S}$ by $\overline{S} = \{\bar{u} \colon u \in S\}$. By the definition, we have $\bar{\bar{u}} = u$ for $u \in V(G\overline{G})$ and $\overline{\overline{S}} = S$ for $S \subset V(G\overline{G})$.

Haynes et al. [3] investigated several parameters of the complementary prism of a graph including the diameter, the independence number, the domination number and the total domination number. Janseana, Rueangthampisan and Ananchuen [6] and Janseana and Ananchuen [4] investigated the extendability of the complementary prism of a regular graph. They proved that for an integer r with $r \geq 2$, the complementary prism of an r-regular graph is 2-extendable with some exceptions for $2 \leq r \leq 3$. Janseana and Ananchuen [5] proved that for integers l_1 and l_2 with $l_1 \geq 2$ and $l_2 \geq 2$, if G is non-bipartite l_1-extendable and $\overline{G}$ is non-bipartite l_2-extendable, then $G\overline{G}$ is $(l+1)$-extendable, where $l = \min\{l_1, l_2\}$. This result shows a relationship between the extendability of the complementary prism $G\overline{G}$ of a graph G and those of G and $\overline{G}$. But it only applies to nonbipartite graphs. From this fact, we are motivated to study the extendability of the complementary prism of a bipartite graph. The result proved by Janseana and Ananchuen [5] involves the extendability of $\overline{G}$. On the other hand, if G is bipartite, the behavior of the matchings in $\overline{G}$ is simple. More precisely, $\overline{G}$ consists of a pair of disjoint cliques possibly with some edges between them. And this structure is retained even after we delete vertices. Because of this simple structure, we do not think it appropriate to put an assumption on the extendability of $\overline{G}$. We only concern ourselves with the basic parameters of G.

Based on the above background, we investigate the extendability of the complementary prism of a bipartite graph. Our main result is the following. We denote by P_n the path of order n.

Theorem 1.1. *Let k and t be nonnegative integers, and let G be a k-extendable bipartite graph of order at least $4(k+t)$ and minimum degree at least $k+2t-1$. Then $G\overline{G}$ is $(k+t)$-extendable except for the case $(k,t) = (0,1)$ and $G \simeq P_4$.*

According to this theorem, when we consider the class of k-extendable bipartite graphs, as the minimum degree and the order of G increases, the lower bound of the extendabilty of $G\overline{G}$ also increases. We cannot observe a similar behavior for nonbipartite graphs in general. For given integers k, d and n, take an even integer p which is greater than $\max\{2k+2, d+1, n\}$ and let G be the complete graph of order p. Then G is a k-extendable graph of minimum degree greater than d and order greater than n. But $G\overline{G}$ contains a pendant edge and hence it is not even 1-extendable. This example shows that the type of phenomenon described in Theorem 1.1 is observed only for bipartite graphs.

Theorem 1.1 describes one exception in the case $(k,t) = (0,1)$. It is easy to see that P_4 is a 0-extendable graph of order four and minimum degree one, but its complementary prism is not 1-extendable.

The main theorem is proved in Section 3 and in Section 4, we discuss the sharpness of Theorem 1.1.

For basic graph theoretical terminology and definitions not explained in this paper, we refer the reader to [2]. Let G be a graph. For $e = uv \in E(G)$ and $F \subset E(G)$, we define $V(e)$ and $V(F)$ by $V(e) = \{u, v\}$ and $V(F) = \bigcup_{f\in F} V(f)$. Note that $|V(F)| \le 2|F|$ and that the equality holds if and only if F is a matching in G. For X, $Y \subset V(G)$ with $X \cap Y = \emptyset$, we write $E_G(X,Y)$ for the set of edges joining X and Y: $E_G(X,Y) = \{e \in E(G)\colon V(e) \cap X \ne \emptyset, V(e) \cap Y \ne \emptyset\}$. For $x \in V(G)$, we denote by $\deg_G x$ and $N_G(x)$ the degree and the neighborhood of G, respectively. The minimum degree of G is denoted by $\delta(G)$.

2 Preliminaries

Before we prove Theorem 1.1, we introduce several lemmas. We use the following result due to Plummer [8].

Theorem 2.1 ([8])**.** *Let G be a balanced bipartite graph with partite sets X and Y and suppose k is a positive integer such that $k \le (|V(G)|-2)/2$. Then the following statements (1) and (2) are equivalent:*

(1) *G is k-extendable.*

(2) *For all $x_1, \ldots, x_k \in X$ and $y_1, \ldots, y_k \in Y$, $G' = G - \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ has a perfect matching.*

The above theorem immediately yields the following corollary.

Corollary 2.2. *Let k and m be nonnegative integers with $m \leq k$, and let G be a k-extendable bipartite graph with partite sets X and Y. Then for every $X' \subset X$ and $Y' \subset Y$ with $|X'| = |Y'| = m$, $G - (X' \cup Y')$ is $(k - m)$-extendable.*

As we have observed in the introduction, the complement of a bipartite graph consists of two disjoint cliques possibly with some edges between them. From this fact, we make the following easy observation.

Lemma 2.3. *Let G be a bipartite graph of even order with partite sets X and Y. Then $\overline{G}$ does not contain a perfect matching if and only if $|X| \equiv |Y| \equiv 1 \pmod 2$ and $E_{\overline{G}}(\overline{X}, \overline{Y}) = \emptyset$.*

Note that in the above lemma, we do not assume that G is balanced. The conclusion holds even if $|X| \neq |Y|$.

The following theorem was proved by Ananchuen and Caccetta [1]. Note that although they assume $m \geq 3$, it is easy to see that the result also holds in the cases $m = 1$ and $m = 2$.

Theorem 2.4 ([1])**.** *For a positive integer m, every balanced bipartite graph of order exactly $2(m+1)$ and minimum degree at least m is $(m-1)$-extendable.*

We prove one more easy lemma.

Lemma 2.5. *Let k be a nonnegative integer and let G be a k-extendable bipartite graph with partite sets X and Y with $|X| = |Y| \geq k + 2$. Then for each $X' \subset X$ and $Y' \subset Y$ with $|X'| = |Y'| = k + 1$, $G - (X' \cup Y')$ contains a matching F with $|X - (X' \cup V(F))| = |Y - (Y' \cup V(F))| = 1$.*

Proof. Let $x \in X'$ and $y \in Y'$, and let $X'' = X' - \{x\}$ and $Y'' = Y' - \{y\}$. Let $G' = G - (X'' \cup Y'')$. Since G is k-extendable and $|X''| = |Y''| = k$, G' is 0-extendable by Corollary 2.2 and hence contains a perfect matching F'. If $xy \in F'$, then $F' - \{xy\}$ is a perfect matching in $G' - \{x, y\} = G - (X' \cup Y')$. Note $|F'| = |X| - k \geq 2$. Take $e \in F' - \{xy\}$. Then $F' - \{xy, e\}$ is a required matching. If $xy \notin F'$, let $\{xy', x'y\} \subset F'$ and $F = F' - \{xy', x'y\}$. Then $X - \big(X' \cup V(F)\big) = \{x'\}$ and $Y - \big(Y' \cup V(F)\big) = \{y'\}$, and hence F is a required matching. ■

3 Proof of the Main Theorem

In this section, we prove Theorem 1.1. Before we do this, we need to set up some notation used throughout this section, and give some lemmas.

For nonnegative integers k and t, let G be a k-extendable bipartite graph of order at least $4(k+t)$ and let M be a matching of size $k+t$ in $G\overline{G}$ which does not extend to a perfect matching in $G\overline{G}$. Further, let X and Y be the partite sets of G. Observe that $|X| = |Y| \geq 2(k+t)$ since G is 0-extendable. Now let

$$\begin{aligned}
M_G &= \{e \in M \colon V(e) \subset V(G)\},\\
M_{\overline{G}} &= \{e \in M \colon V(e) \subset V(\overline{G})\},\\
M_{G\overline{G}} &= \{e \in M \colon V(e) \cap V(G) \neq \emptyset, V(e) \cap V(\overline{G}) \neq \emptyset\},\\
M^X_{G\overline{G}} &= \{e \in M_{G\overline{G}} \colon V(e) \cap X \neq \emptyset\},\\
M^Y_{G\overline{G}} &= \{e \in M_{G\overline{G}} \colon V(e) \cap Y \neq \emptyset\},\\
M^{\overline{X}}_{\overline{G}} &= \{e \in M_{\overline{G}} \colon V(e) \subset \overline{X}\},\\
M^{\overline{Y}}_{\overline{G}} &= \{e \in M_{\overline{G}} \colon V(e) \subset \overline{Y}\} \text{ and}\\
M^{\overline{X}\,\overline{Y}}_{\overline{G}} &= \{e \in M_{\overline{G}} \colon V(e) \cap \overline{X} \neq \emptyset, V(e) \cap \overline{Y} \neq \emptyset\}.
\end{aligned}$$

Further, let $X_0 = X - (V(M_G) \cup V(M^X_{G\overline{G}}) \cup \overline{V(M_{\overline{G}})})$ and $Y_0 = Y - (V(M_G) \cup V(M^Y_{G\overline{G}}) \cup \overline{V(M_{\overline{G}})})$. Figure 3.1 illustrates our notation.

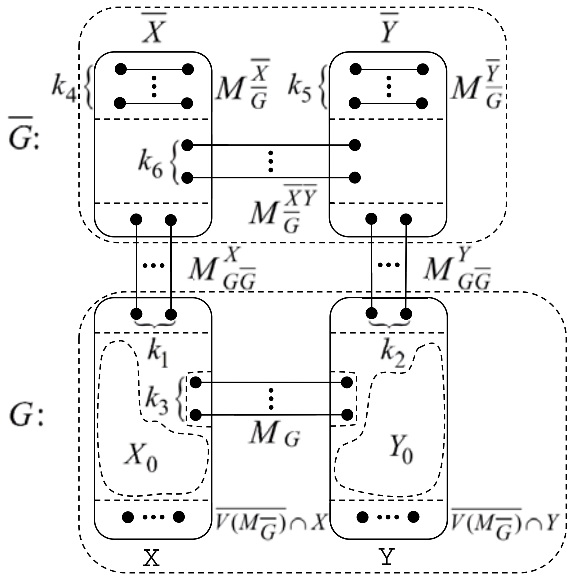

Figure 3.1: The illustration of our notation

Note that $M_{\overline{G}} = M^{\overline{X}}_{\overline{G}} \cup M^{\overline{Y}}_{\overline{G}} \cup M^{\overline{X}\,\overline{Y}}_{\overline{G}}$ and $M_{G\overline{G}} = M^X_{G\overline{G}} \cup M^Y_{G\overline{G}}$.

For simplicity, let $k_1 = |M^X_{G\overline{G}}|$, $k_2 = |M^Y_{G\overline{G}}|$, $k_3 = |M_G|$, $k_4 = |M^{\overline{X}}_{\overline{G}}|$, $k_5 = |M^{\overline{Y}}_{\overline{G}}|$ and $k_6 = |M^{\overline{X}\,\overline{Y}}_{\overline{G}}|$. Note that $k_1 + k_2 = |M_{G\overline{G}}|$, $k_4 + k_5 + k_6 = |M_{\overline{G}}|$ and $k_1 + k_2 + k_3 + k_4 + k_5 + k_6 = |M| = k + t$. Further, $|X_0| \geq |X| - (k_1 + k_3 + 2k_4 + k_6)$ and $|Y_0| \geq |Y| - (k_2 + k_3 + 2k_5 + k_6)$.

By symmetry, we may assume without loss of generality that $k_1 \geq k_2$. It then follows that $|Y_0| \geq k_1 - k_2$ otherwise $|Y| \leq |Y_0| + k_2 + k_3 + 2k_5 + k_6 \leq k_1 + k_3 + 2k_5 + k_6 - 1 \leq 2|M| - 1 = 2(k + t) - 1$.

We now introduce a specific subgraph of G. Take $Y_1 \subset Y_0$ with $|Y_1| = k_1 - k_2$, and let $H = G - \big(V(M_{G\overline{G}}) \cup Y_1\big)$. Observe that H is a balanced bipartite graph with partite sets

$$A = X - V(M^X_{G\overline{G}}) \text{ and } B = Y - \big(V(M^Y_{G\overline{G}}) \cup Y_1\big),$$

where $|A| = |B| = |X| - k_1$. Further, if $k_1 \leq k$, then H is $(k - k_1)$-extendable by Corollary 2.2.

Observation 3.1. *$M_G \cup M_{\overline{G}}$ does not extend to a perfect matching in $H\overline{H}$ otherwise a perfect matching F in $H\overline{H}$, containing $M_G \cup M_{\overline{G}}$, together with $M_{G\overline{G}}$ and $\{u\bar{u}\colon u \in Y_1\}$ is a perfect matching containing M in $G\overline{G}$, which contradicts the assumption that M does not extend to a perfect matching in $G\overline{G}$.*

We are now ready to establish some lemmas.

Lemma 3.1. *For nonnegative integers k and t, let G be a k-extendable bipartite graph of order at least $4(k + t)$ and minimum degree at least $k + 2t - 1$ and M a matching of size $k + t$ in $G\overline{G}$ which does not extend to a perfect matching in $G\overline{G}$. Further, let $H = G - \big(V(M_{G\overline{G}}) \cup Y_1\big)$, where $Y_1 \subset Y_0$ such that $|Y_1| = k_1 - k_2$, with partite sets A and B. If F is a matching in $H\overline{H}$ with $M_G \cup M_{\overline{G}} \subset F$ and $V(H) \subset V(F)$, then $|\overline{A} - V(F)| \equiv |\overline{B} - V(F)| \equiv 1 \pmod 2$ and $E_{\overline{H}}(\overline{A} - V(F), \overline{B} - V(F)) = \emptyset$. Moreover, for each $e \in F \cap E(G) - M_G$, $V(e) \cap \overline{V(M_{\overline{G}})} \neq \emptyset$.*

Proof. By Observation 3.1, $H\overline{H} - V(F)$ does not contain a perfect matching. Since $\overline{H\overline{H} - V(F)}$ is the complement of a bipartite graph with partite sets $\overline{\overline{A} - V(F)}$ and $\overline{\overline{B} - V(F)}$, the first part follows from Lemma 2.3.

For the second part, if there exists an edge $e = xy \in F \cap E(G) - M_G$ with $V(e) \cap \overline{V(M_{\overline{G}})} = \emptyset$, let $F' = \big(F - \{xy\}\big) \cup \{x\bar{x}, y\bar{y}\}$. Then F' is a matching in $H\overline{H}$ with $M_G \cup M_{\overline{G}} \subset F'$ and $V(F') = V(F) \cup \{\bar{x}, \bar{y}\}$. Furthermore, $|\overline{A} - V(F')| = |\overline{A} - V(F)| - 1 \equiv 0 \pmod 2$, which contradicts the first part of our lemma. ■

Lemma 3.2. *Let G, M and H be defined as in Lemma 3.1. Further, take a matching F_0 in H with $M_G \subset F_0$ so that*

(a) $|\overline{V(M_{\overline{G}})} \cap V(F_0)|$ is as large as possible, and

(b) $|F_0|$ is as large as possible, subject to (a).

Then

(1) $\overline{V(M_{\overline{G}})} \subset V(F_0)$.

(2) $|F_0| \leq |M_G| + 2|M_{\overline{G}}|$.

Proof. (1) Assume $\overline{V(M_{\overline{G}})} \not\subset V(F_0)$, and let $x \in \overline{V(M_{\overline{G}})} - V(F_0)$. By symmetry, we may assume $x \in A$. If $N_H(x) \not\subset V(F_0)$, then we can take a vertex $y \in N_H(x) - V(F_0)$, and let $F_1 = F_0 \cup \{xy\}$. Then F_1 is a matching in H with $M_G \subset F_0 \subset F_1$ and $\big(\overline{V(M_{\overline{G}})} \cap V(F_0)\big) \cup \{x\} \subset V(F_1)$. This contradicts the condition (a). Hence we have $N_H(x) \subset V(F_0)$.

Let $B_1 = N_H(x) - V(M_G)$ and $A_1 = \{u \in A\colon uv \in F_0, v \in B_1\}$. If $A_1 \not\subset \overline{V(M_{\overline{G}})}$, let $a \in A_1 - \overline{V(M_{\overline{G}})}$. Also let b be the vertex in B_1 with $ab \in F_0$, and let $F_2 = (F_0 - \{ab\}) \cup \{xb\}$. Then F_2 is a matching in H with $M_G \subset F_2$ and $\big(\overline{V(M_{\overline{G}})} \cap V(F_0)\big) \cup \{x\} \subset V(F_2)$. This again contradicts the condition (a). Hence we have $A_1 \subset \overline{V(M_{\overline{G}})}$.

Since $A_1 \cup \{x\} \subset \overline{V(M_{\overline{G}})}$, $|A_1| + 1 \leq |V(M_{\overline{G}})| = 2|M_{\overline{G}}|$. On the other hand, $|B_1| \geq \deg_H x - |M_G| \geq \delta(H) - |M_G| \geq \delta(G) - k_1 - |M_G|$. Since $|A_1| = |B_1|$, we have $2|M_{\overline{G}}| \geq \delta(G) - k_1 - |M_G| + 1 \geq k + 2t - 1 - k_1 - |M_G| + 1$. This implies $|M_G| + k_1 + 2|M_{\overline{G}}| \geq k + 2t$. Since $|M_G| + k_1 + |M_{\overline{G}}| \leq |M_G| + |M_{\overline{G}}| + k_1 + k_2 = |M| = k + t$, we have $|M_{\overline{G}}| \geq t$ and $|M_G| + k_1 + k_2 \leq k$. This implies $k_1 \leq k$ and hence H is $(k - k_1)$-extendable. Since $|M_G| \leq k - (k_1 + k_2) \leq k - k_1$, M_G extends to a perfect matching in H, and hence F_0 is a perfect matching in H, which yields $\overline{V(M_{\overline{G}})} \subset V(F_0)$. This contradicts the assumption, and (1) follows.

(2) Let $F_1 = F_0 \cup \{u\bar{u}\colon u \in V(H) - V(F_0)\} \cup M_{\overline{G}}$. By (1), F_1 is a matching in $H\overline{H}$ with $M_G \cup M_{\overline{G}} \subset F_1$ and $V(H) \subset V(F_1)$. Let $\overline{A_1} = \overline{A} - V(F_1)$ and $\overline{B_1} = \overline{B} - V(F_1)$. By Lemma 3.1, $|A_1| \equiv |B_1| \equiv 1 \pmod 2$ and $E_{\overline{H}}(\overline{A_1}, \overline{B_1}) = \emptyset$, and $V(e) \cap \overline{V(M_{\overline{G}})} \neq \emptyset$ for each $e \in F_0 - M_G$. Since $|V(M_{\overline{G}})| = 2|M_{\overline{G}}|$, this implies $|F_0| - |M_G| \leq 2|M_{\overline{G}}|$, and our lemma follows. ■

Observation 3.2. *If H admits a perfect matching containing M_G, then the conditions (a) and (b) in Lemma 3.2 force F_0 to be a perfect matching.*

Lemma 3.3. *Let G, M and H be defined as in Lemma 3.1. Further, assume that $(k, t, |V(G)|) \neq (0, 1, 4)$. Then:*

(1) M_G does not extend to a perfect matching in H.

(2) $|M_G| \leq k - k_1 - 2k_2 + 1$ and $|M_{\overline{G}}| \geq t + k_2 - 1$.

(3) $k_2 = 0$ and $|M_{\overline{G}}| = t - 1$.

(4) $t \geq 1$ and $|M_G| = k - k_1 + 1$.

Proof. Let F_0 be a matching in H containing M_G satisfying the conditions (a) and (b) defined as in Lemma 3.2.

(1) Assume, to the contrary, that M_G extends to a perfect matching in H. By Observation 3.2, F_0 is a perfect matching in H, and $|F_0| = |A| = |X| - k_1 \geq$

$2(k+t)-k_1$. On the other hand, by Lemma 3.2(2), $|F_0| \le |M_G|+2|M_{\overline{G}}| = 2(|M_G|+|M_{\overline{G}}|)-|M_G| = 2(k+t-k_1-k_2)-|M_G|$, we have $2(k+t-k_1-k_2)-|M_G| \ge 2(k+t)-k_1$, which implies $|M_G|+k_1+2k_2 \le 0$. Therefore, $k_1 = k_2 = 0$ and $M_G = \emptyset$. This means $H = G$ and $M = M_{\overline{G}}$. We also have $|F_0| = |A| = 2(k+t)$.

Let $G' = \overline{G} - V(M)$, $\overline{A'} = \overline{A} - V(M)$ and $\overline{B'} = \overline{B} - V(M)$. Since $G = H$ and $M = M_{\overline{G}}$, we can apply Lemma 3.1 to G and $F_0 \cup M$, and obtain $|\overline{A'}| \equiv |\overline{B'}| \equiv 1 \pmod 2$ and $E_{\overline{G}}(\overline{A'}, \overline{B'}) = \emptyset$. Since $\left|\overline{A'}\right|$ and $\left|\overline{B'}\right|$ are odd, $\overline{A'} \ne \emptyset$ and $\overline{B'} \ne \emptyset$. Take $\bar{x} \in \overline{A'}$ and $\bar{y} \in \overline{B'}$. Since $E_{\overline{G}}\left(\overline{A'}, \overline{B'}\right) = \emptyset$, $\bar{x}\bar{y} \notin E(\overline{G})$ and $xy \in E(G)$.

Assume the edge xy extends to a perfect matching F in G. Let $F' = (F - \{xy\}) \cup \{x\bar{x}, y\bar{y}\} \cup M$. Then F' is a matching in $G\overline{G}$ with $M \subset F'$ and $V(F') = V(F) \cup \{\bar{x}, \bar{y}\}$. However, this contradicts Lemma 3.1 since $\left|\overline{A} - V(F')\right| = \left|\overline{A'} - \{\bar{x}\}\right| \equiv 0 \pmod 2$. Therefore, the edge xy does not extend to a perfect matching in G. This implies that G is not 1-extendable, and hence $k = 0$.

At this stage, we know that G is a balanced bipartite graph of order exactly $4t$ and minimum degree at least $2t-1$. Then by Theorem 2.4, G is $(2t-2)$-extendable. Since G is not 1-extendable, this implies $2t-2 \le 0$, or $t \le 1$. On the other hand, we have $|V(G)| = 4t > 0$. Therefore, we have $t = 1$ and $|V(G)| = 4$. We now have $(k, t, |V(G)|) = (0, 1, 4)$, which contradicts the assumption and (1) follows..

(2) By (1), F_0 is not a perfect matching in G. This implies $A \not\subset V(F_0)$. Let $a \in A - V(F_0)$. If $N_H(a) \not\subset V(F_0)$, then we can take $b \in N_H(a) - V(F_0)$ and let $F_1 = F_0 \cup \{ab\}$. Then $\overline{V(M_{\overline{G}})} \subset V(F_0) \subset V(F_1)$ and $|F_1| = |F_0| + 1$, which contradicts the condition (b). Hence we have $N_H(a) \subset V(F_0)$. Then $|F_0| \ge \deg_H a \ge \deg_G a - k_1 \ge k + 2t - 1 - k_1$. On the other hand, by Lemma 3.2(2), $|F_0| \le |M_G| + 2|M_{\overline{G}}| = |M_G| + |M_{\overline{G}}| + |M_{\overline{G}}| = k + t - k_1 - k_2 + |M_{\overline{G}}|$. Therefore, we have $|M_{\overline{G}}| + k + t - k_1 - k_2 \ge k + 2t - 1 - k_1$, which yields $|M_{\overline{G}}| \ge t + k_2 - 1$ and $|M_G| \le k + t - k_1 - k_2 - (t + k_2 - 1) = k - k_1 - 2k_2 + 1$. So (2) follows.

(3) Suppose $k_1 \le k$. Then, by Corollary 2.2, H is $(k - k_1)$-extendable. On the other hand, by (1), M_G does not extend to a perfect matching in H. This implies $|M_G| \ge k - k_1 + 1$. Then by (2), we have $k - k_1 + 1 \le |M_G| \le k - k_1 - 2k_2 + 1$, which implies $k_2 = 0$ and $|M_G| = k - k_1 + 1$. Then $|M_{\overline{G}}| = k + t - k_1 - k_2 - |M_G| = t - 1$.

Next, suppose $k_1 \ge k+1$. Then again by (2), $0 \le |M_G| \le k - k_1 - 2k_2 + 1 \le -2k_2$, which implies $k_2 = |M_G| = 0$ and $k_1 = k+1$. Then $|M_{\overline{G}}| = k+t-k_1-k_2-|M_G| = t-1$ and (3) follows.

(4) follows by (3) and the fact that $|M_G| = (k+t)-k_1-k_2-|M_{\overline{G}}|$. This proves (4) and completes the proof of our lemma. ■

Lemma 3.4. *Let G, M and H be defined as in Lemma 3.1 and let F_0 be defined as in Lemma 3.2. Then $|V(F_0)| = |V(H)| - 2$.*

Proof. Since M_G does not extend to a perfect matching in H by Lemma 3.3(1), it suffices to prove the existence of a matching F with (1) $M_G \subset F$, (2) $|V(F)| = |V(H)| - 2$ and (3) $\overline{V(M_{\overline{G}})} \subset V(F)$.

Assume, to the contrary, that H does not admit a matching satisfying (1)–(3). We first prove that H admits a matching satisfying (1) and (2). Suppose $k_1 \leq k$. Then H is $(k-k_1)$-extendable. By Lemma 3.3(4), we can take $ab \in M_G$ with $a \in A$ and $b \in B$. Then $|M_G - \{ab\}| = k - k_1$, and $M_G - \{ab\}$ extends to a perfect matching F_1 in H. If $ab \in F_1$, then F_1 is a perfect matching in H with $M_G \subset F_1$. This contradicts Lemma 3.3(1). Hence $ab \notin F_1$. Let $\{ab_1, a_1b\} \subset F_1$, and let $F_2 = \big(F_1 - \{ab_1, a_1b\}\big) \cup \{ab\}$. Then F_2 is a matching in H satisfying (1) and (2).

Next suppose $k_1 \geq k+1$. Since $|M_G| = k - k_1 + 1 \geq 0$, we have $k_1 = k+1$ and $M_G = \emptyset$. Since $t \geq 1$, we have $|X| \geq 2(k+t) \geq k+2$. Thus, we can apply Lemma 2.5 with $X' = V(M^X_{G\overline{G}}) \cap X$ and $Y' = Y_1$ and obtain a matching F_3 in H with $|A - V(F_3)| = |B - V(F_3)| = 1$. (Note $M^Y_{G\overline{G}} = \emptyset$ by Lemma 3.3(3).) Since $M_G = \emptyset$, F_3 satisfies both (1) and (2).

Choose a matching F in H that satisfies (1) and (2) so that $|\overline{V(M_{\overline{G}})} \cap V(F)|$ is as large as possible. Let $V(H) - V(F) = \{x, y\}$ with $x \in A$ and $y \in B$. By the assumption, F fails to satisfy (3). By symmetry, we may assume $x \in \overline{V(M_{\overline{G}})}$. If $xy \in E(H)$, then $F \cup \{xy\}$ is a perfect matching in H containing M_G, contradicting Lemma 3.3(1). Hence we may assume $xy \notin E(H)$.

Let $B_1 = N_H(x) - V(M_G)$ and $A_1 = \{u \in A\colon uv \in F, v \in B_1\}$. Since $y \notin B_1$, $|A_1| = |B_1| \geq \deg_H x - |M_G| \geq \deg_G x - k_1 - |M_G| \geq k + 2t - 1 - k_1 - (k - k_1 + 1) = 2t - 2$. On the other hand, by Lemma 3.3(3), $|\overline{V(M_{\overline{G}})}| = |V(M_{\overline{G}})| = 2|M_{\overline{G}}| = 2(t-1)$. Therefore, $A_1 \cup \{x\} \not\subset \overline{V(M_{\overline{G}})}$. Since $x \in \overline{V(M_{\overline{G}})}$, there exists a vertex $x' \in A_1 - \overline{V(M_{\overline{G}})}$.

Let $x'y' \in F$ and $F' = \big(F - \{x'y'\}\big) \cup \{xy'\}$. Then F' is a matching in H with $M_G \subset F'$ and $V(F') = V(H) - \{x', y\}$. Moreover, $\overline{V(M_{\overline{G}})} \cap V(F') = \big(\overline{V(M_{\overline{G}})} \cap V(F)\big) \cup \{x\}$. This contradicts the maximality of $\big|\overline{V(M_{\overline{G}})} \cap V(F)\big|$. Therefore, our lemma follows. ■

We are now ready to establish our main theorem. Since P_4 is the only bipartite graph of order four and extendability zero, Theorem 1.1 can be restated in the following way.

Theorem 3.5. *Let k and t be nonnegative integers, and let G be a k-extendable bipartite graph of order at least $4(k+t)$ and minimum degree at least $k+2t-1$. If $(k, t, |V(G)|) \neq (0, 1, 4)$, then $G\overline{G}$ is $(k+t)$-extendable.*

Proof. Assume $G\overline{G}$ is not $(k+t)$-extendable. Then there exists a matching M of size $k+t$ in $G\overline{G}$ which does not extend to a perfect matching in $G\overline{G}$. Let X and Y be the partite sets of G. Since G is 0-extendable, $|X| = |Y| \geq 2(k+t)$.

Now let $H = G - (V(M_{G\overline{G}}) \cup Y_1)$ be the graph defined in Lemma 3.1 with the partite sets $A = X - V(M^X_{G\overline{G}})$ and $B = Y - \big(V(M^Y_{G\overline{G}}) \cup Y_1\big)$. Further, let F_0 be the matching defined in Lemma 3.2.

By Lemma 3.4, we can set $A - V(F_0) = \{a\}$ and $B - V(F_0) = \{b\}$. Let $F = F_0 \cup \{a\bar{a}, b\bar{b}\} \cup M_{\overline{G}}$. Then F is a matching in $H\overline{H}$ with $M_G \cup M_{\overline{G}} \subset F$ and $V(H) \subset V(F)$.

Let $\overline{A_1} = \overline{A} - V(F)$ and $\overline{B_1} = \overline{B} - V(F)$. By Lemma 3.1, $|\overline{A_1}| \equiv |\overline{B_1}| \equiv 1 \pmod 2$ and $E_{\overline{H}}(\overline{A_1}, \overline{B_1}) = \emptyset$. Moreover, $V(e) \cap \overline{V(M_{\overline{G}})} \neq \emptyset$ for each $e \in F_0 - M_G$.

By Lemmas 3.2(2) and 3.3(3), $|F_0| \leq |M_G| + 2|M_{\overline{G}}| = |M_G| + |M_{\overline{G}}| + |M_{\overline{G}}| = (k + t - k_1 - k_2) + t - 1 = k + 2t - k_1 - 1$. On the other hand, by Lemma 3.4, $|F_0| = |A| - 1 = |X| - k_1 - 1 \geq 2(k+t) - k_1 - 1$. Therefore, $2k + 2t - k_1 - 1 \leq k + 2t - k_1 - 1$, which implies $k = 0$. Moreover, we have $|X| = 2(k+t) = 2t$, $|A| = 2t - k_1$, $|M_G| = k - k_1 + 1 = 1 - k_1$ and $\delta(G) \geq 2t - 1$. In particular, $k_1 \leq 1$.

If $k_1 = 1$, then $M_G = \emptyset$, $|X| = 2t$ and $|A| = 2t - 1$. Moreover, $\delta(G) \geq 2t - 1$ and $\delta(H) \geq 2t - 2$. By applying Theorem 2.4 to G, we see that H is $(2t-3)$-extendable. On the other hand, by Lemma 3.3(1), H does not contain a perfect matching. These imply $2t - 3 < 0$. Since $t \geq 1$ by Lemma 3.3(4), we have $t = 1$ and $M_{\overline{G}} = \emptyset$. We now have $M_G = M_{\overline{G}} = M^Y_{G\overline{G}} = \emptyset$ and hence $M = M^X_{G\overline{G}}$. Then $\{u\bar{u} \colon u \in V(G)\}$ is a perfect matching containing M. This is a contradiction. Therefore, we have $k_1 = 0$.

At this stage we have $k_1 = k_2 = k = 0$, $H = G$, $|M_G| = 1$ and $\delta(G) \geq 2t - 1$. By Theorem 2.4, G is $(2t-2)$-extendable. On the other hand, by Lemma 3.3(1), M_G does not extend to a perfect matching in G. Thus we have $2t - 2 \leq 0$, which implies $t = 1$ and $|V(G)| = 4$. Now we have $(k, t, |V(G)|) = (0, 1, 4)$. This is a contradiction, and the theorem follows. ■

4 Sharpness

In this section, we discuss the sharpness of Theorem 1.1. First we consider the condition on minimum degree.

Plummer [7] proved the following theorem.

Theorem 4.1 ([7]). *A connected k-extendable graph is $(k+1)$-connected.*

This theorem implies that the minimum degree of a connected k-extendable graph is at least $k + 1$. Therefore, as long as connected graphs are concerned, the assumption on minimum degree in Theorem 1.1 follows from the k-extendability if $t \leq 1$. Therefore, when we discuss the sharpness, we may assume $t \geq 2$. In this range, we will prove that the condition is best-possible.

For nonnegative integers a_0, a_1, a_2, a_3, a_4 and a_5, let A_0, A_1, A_2, A_3, A_4 and A_5 be pairwise disjoint independent sets of vertices with $|A_i| = a_i$ $(0 \leq i \leq 5)$. Then define $G(a_0, a_1, a_2, a_3, a_4, a_5)$ by

$$V\big(G(a_0, a_1, a_2, a_3, a_4, a_5)\big) = \bigcup_{i=0}^{5} A_i \text{ and}$$
$$E\big(G(a_0, a_1, a_2, a_3, a_4, a_5)\big) = \{uv \colon u \in A_i, v \in A_{i+1}, 0 \leq i \leq 5\},$$

where the suffices are taken modulo 6. Note that $G(a_0, a_1, a_2, a_3, a_4, a_5)$ is a bipartite graph with partite sets $A_0 \cup A_2 \cup A_4$ and $A_1 \cup A_3 \cup A_5$. Note also that we allow a_i to be zero. For example, $G(0, 1, 1, 1, 1, 0) = G(1, 1, 1, 1, 0, 0) \simeq P_4$.

Lemma 4.2. *For nonnegative integers a_0, a_1, a_2, a_3, a_4 with $a_0 + a_2 + a_4 = a_1 + a_3$, $a_1 \geq a_0$ and $a_3 \geq a_4$, $G = G(a_0, a_1, a_2, a_3, a_4, 0)$ admits a perfect matching.*

Proof. Since $a_1 \geq a_0$, there exists a matching F' between A_0 and A_1 covering all vertices of A_0. Similarly, since $a_3 \geq a_4$, there exists a matching F'' between A_3 and A_4 covering all vertices of A_4. By our hypothesis $a_2 = (a_1 - a_0) + (a_3 - a_4)$, the remaining graph $G - V(F' \cup F'')$ contains a perfect matching F. Hence, $F \cup F' \cup F''$ is a perfect matching of G as required. ■

Lemma 4.3. *If nonnegative integers a_0, a_1, a_2, a_3, a_4 and a_5 satisfy $a_0 + a_2 + a_4 = a_1 + a_3 + a_5$ and $a_{i-1} + a_{i+1} \geq a_i$ for each i, $0 \leq i \leq 5$, then $G(a_0, a_1, a_2, a_3, a_4, a_5)$ admits a perfect matching.*

Proof. We proceed by induction on $a_0 + a_2 + a_4$. If $a_0 + a_2 + a_4 \leq 3$, then it is easy to show that $G(a_0, a_1, a_2, a_3, a_4, a_5)$ has a perfect matching by applying the hypothesis that $a_{i-1} + a_{i+1} \geq a_i$ for each i, $0 \leq i \leq 5$.

Suppose $a_0 + a_2 + a_4 \geq 4$. If $a_{i-1} + a_{i+1} = a_i$ holds for each i, $0 \leq i \leq 5$, we can take the sum of these six equalities and obtain $2\sum_{i=0}^{5} a_i = \sum_{i=0}^{5} a_i$, which yields $\sum_{i=0}^{5} a_i = 0$. This is a contradiction since $a_0 + a_2 + a_4 \geq 4$. Therefore, $a_{i-1} + a_{i+1} > a_i$ for some i, $0 \leq i \leq 5$. By symmetry, we may assume $a_0 + a_2 > a_1$. Then since $a_0 + a_2 + a_4 = a_1 + a_3 + a_5$, we have $a_3 + a_5 > a_4$.

We first consider when $a_5 = 0$. Since $a_3 + a_5 > a_4$ and $a_1 + a_5 \geq a_0$ it follows that $a_3 > a_4$ and $a_1 \geq a_0$. By Lemma 4.2, $G(a_0, a_1, a_2, a_3, a_4, 0)$ has a perfect matching. We now consider the case $a_0 = 0$. Let $G' = G(0, a_1, a_2, a_3, a_4, a_5) = G(a_1, a_2, a_3, a_4, a_5, 0)$. Since $a_0 + a_2 > a_1$ and $a_0 + a_4 \geq a_5$, it follows that $a_2 > a_1$ and $a_4 \geq a_5$. Again by Lemma 4.2, G' has a perfect matching.

We now assume that $a_0 \geq 1$ and $a_5 \geq 1$. Take $u \in A_5$ and $v \in A_0$, and let $G'' = G(a_0, a_1, a_2, a_3, a_4, a_5) - \{u, v\}$. Then $G'' = G(a_0 - 1, a_1, a_2, a_3, a_4, a_5 - 1)$. Since $a_3 + a_5 > a_4$, $a_3 + (a_5 - 1) \geq a_4$. Also, since $a_0 + a_2 > a_1$, $(a_0 - 1) + a_2 \geq a_1$. Furthermore, since $a_1 + a_5 \geq a_0$ and $a_0 + a_4 \geq a_5$, $a_1 + (a_5 - 1) \geq a_0 - 1$ and $a_4 + (a_0 - 1) \geq a_5 - 1$. Hence, G'' contains a perfect matching F by the induction hypothesis. Now $F \cup \{uv\}$ is a perfect matching in $G(a_0, a_1, a_2, a_3, a_4, a_5)$. ■

Theorem 4.4. *Let k, t, a and b be integers with $k \geq 0$, $t \geq 2$, $a \geq k + 2t$ and $b \geq 2k + 1$. Let $G = G(k + 1, 2t - 2, 2t - 3, a, a + b - k, b)$. Then:*

(1) G is a k-extendable bipartite graph,

(2) $\delta(G) = k + 2t - 2$, and

(3) $G\overline{G}$ is not $(k + t)$-extendable.

Proof. Let A_0, A_1, A_2, A_3, A_4 and A_5 be the independent sets of order $k + 1$, $2t - 2$, $2t - 3$, a, $a + b - k$ and b, respectively. Note that since $a + b - k \geq 2k + 2t + 1 > 0$, G is well-defined. Note also that G is a balanced bipartite graph.

(1) Let M be a matching of size k in G. We will show that M extends to a perfect matching in G. Let $H = G - V(M)$. It suffices to prove that H admits a perfect matching. Note that since G is a balanced bipartite graph, H is also a balanced bipartite graph.

Let $k_i = |\{uv \in M : u \in A_i, v \in A_{i+1}\}|$ $(0 \leq i \leq 5)$, where suffices are taken modulo 6. Note that $H = G(p_0, p_1, p_2, p_3, p_4, p_5)$, where $p_0 = k+1-k_5-k_0$, $p_1 = 2t-2-k_0-k_1$, $p_2 = 2t-3-k_1-k_2$, $p_3 = a-k_2-k_3$, $p_4 = a+b-k-k_3-k_4$, $p_5 = b-k_4-k_5$. Note also that, $p_i \geq 0$ for each i, $0 \leq i \leq 5$ and $p_0 + p_2 + p_4 = p_1 + p_3 + p_5$ since H is a balanced bipartite graph. By using the fact that $k = k_0 + k_1 + k_2 + k_3 + k_4 + k_5$ and our hypothesis that $a \geq k + 2t$ and $b \geq 2k + 1$, it is not difficult to show that $p_{i-1} + p_{i+1} \geq p_i$ for each i, $0 \leq i \leq 5$. By Lemma 4.3, H has a perfect matching. Hence, (1) follows.

(2) By using the hypothesis that $a \geq k + 2t$ and $b \geq 2k + 1$, it is easy to see that $|A_{i-1}| + |A_{i+1}| \geq k + 2t - 2$ for each i, $0 \leq i \leq 5$. By the definition of G, $\delta(G) = min\{|A_{i-1}| + |A_{i+1}| : 0 \leq i \leq 5\} = k + 2t - 2$ as required.

(3) Since $b \geq k + 1$, we can take a set F_1 of $k + 1$ independent edges joining A_0 and A_5. Since $\overline{A_1}$ induces a complete graph of order $2(t-1)$ in $G\overline{G}$, we can take a perfect matching F_2 of size $t-1$ in $\overline{A_1}$. Let $F = F_1 \cup F_2$. Then F is a matching of size $k+t$ in $G\overline{G}$. Let $G' = G\overline{G} - V(F)$. Then $N_{G'}(A_1) = A_2$. Since $|A_2| = 2t-3 < 2t-2 = |A_1|$, G' does not admit a perfect matching. Therefore, F does not extend to a perfect matching in $G\overline{G}$. ∎

By Theorem 4.4, there exist infinitely many k-extendable bipartite graphs G of minimum degree $k + 2t - 2$ such that $G\overline{G}$ is not $(k + t)$-extendable.

Next, we consider the condition on the order. If $(k, t) \in \{(0, 0), (1, 0)\}$, then $2k + 2 \geq 4(k + t)$. Since the order of a k-extendable graph is at least $2k + 2$, if (k, t) takes either value, the assumption $|V(G)| \geq 4(k + t)$ in Theorem 1.1 immediately follows from the k-extendability of G. Therefore, we may assume $(k, t) \notin \{(0, 0), (1, 0)\}$.

Proposition 4.5. *Let k and t be integers with $(k, t) \notin \{(0, 0), (1, 0)\}$ and let G be the balanced complete bipartite graph $K_{2k+2t-1,2k+2t-1}$. Then (1) G is k-extendable, (2) $\delta(G) \geq k + 2t - 1$, but (3) $G\overline{G}$ is not $(k + t)$-extendable.*

Proof. Since $(k, t) \notin \{(0, 0), (1, 0)\}$, $2t + 2k - 1 > k$. Hence G is k-extendable. Moreover, $\delta(G) = 2k + 2t - 1 \geq k + 2t - 1$. Therefore, (1) and (2) follows.

Let X and Y be the partite sets of G and take $x_0 \in X$. Then $|X - \{x_0\}| = 2k + 2t - 2$ and the subgraph of $G\overline{G}$ induced by $\overline{X - \{x_0\}}$ is a complete graph of order $2k + 2t - 2$, which admits a perfect matching F_0 of size $k + t - 1$. Choose $y_0 \in Y$ and let $F = F_0 \cup \{x_0y_0\}$. Then F is a perfect matching of size $k + t$ in $G\overline{G}$. However, $\overline{x_0}$ is an isolated vertex in $G\overline{G} - V(F)$. Hence F does not extend to a perfect matching in $G\overline{G}$.

∎

Proposition 4.5 shows that we cannot relax the assumption on the order in Theorem 1.1.

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