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The Friedlander-Gordon-Miller conjecture is true

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Abstract

We complete the proof of the Friedlander, Gordon and Miller Conjecture that every finite abelian group whose Sylow 2-subgroup either is trivial or both non-trivial and non-cyclic is R-sequenceable. This settles a question of Ringel for abelian groups.

1 Introduction

In 1961 Gordon [2] defined a group G to be *sequenceable* when there exists a permutation

$$g_0, g_1, g_2, \dots, g_{n-1}$$

of its elements so that the sequence of partial products

$$g_0, g_0g_1, g_0g_1g_2, \dots, g_0g_1g_2\cdots g_{n-1}$$

are distinct. In that same paper he proved the following theorem.

1.1 Theorem. *A finite abelian group G is sequenceable if and only its Sylow 2-subgroup is non-trivial and cyclic.*

In 1974 Ringel [9] asked when there exists a permutation

$$g_1, g_2, \ldots, g_{n-1}$$

of the non-identity elements of a group such that the sequence

$$g_2 g_1^{-1}, g_3 g_2^{-1}, \ldots, g_{n-1} g_{n-2}^{-1}, g_1 g_{n-1}^{-1}$$

also is a permutation of the non-identity elements. A group G that admits such a permutation is called *R-sequenceable.* As a matter of fact, Paige [8] used this concept in 1951, but it was Ringel's problem that motivated the most important paper on this topic (discussed below).

We now provide a context which establishes the close connection between the two concepts. Given a group G and a subset S of G such that S does not contain the identity element of G, we define the *Cayley digraph* $\overrightarrow{Cay}(G;S)$ by letting its vertices be the elements of G and having an arc (g_1, g_2) if and only if $g_2 = g_1 s$ for some $s \in S$. One special such Cayley digraph in which we are particularly interested is when $S = G - \{1\}$, that is, the set S has everything in it other than the identity element. We use the special notation $\overrightarrow{K}(G)$ for this Cayley digraph.

It is easy to see that a fixed element $s \in S$ generates a subdigraph consisting of directed cycles whose lengths are all $|s|$, where $|s|$ denotes the order of s. Thus, we obtain a factorization of $\overrightarrow{Cay}(G;S)$ into $|S|$ directed 2-factors. We call this factorization the *Cayley factorization* of $\overrightarrow{Cay}(G;S)$ and denote it by $\overrightarrow{\mathcal{F}}(G;S)$.

If $\overrightarrow{D}$ is a subdigraph of $\overrightarrow{Cay}(G;S)$ with $|S|$ arcs, and $\overrightarrow{D}$ has exactly one arc from each directed 2-factor in $\overrightarrow{\mathcal{F}}(G;S)$, then we say that $\overrightarrow{D}$ is *orthogonal* to $\overrightarrow{\mathcal{F}}(G;S)$. In this language, the group G is sequenceable when $\overrightarrow{K}(G)$ admits an orthogonal Hamilton directed path, and G is R-sequenceable when $\overrightarrow{K}(G)$ admits an orthogonal directed cycle of length $|G| - 1$.

In spite of the similarity between these two concepts, they arose from quite different settings. Gordon was interested in row-complete Latin squares, whereas, Ringel was considering embeddings of complete graphs into orientable surfaces of positive genus.

We now say a few words about some notational conventions in this paper. We use (x, y) to denote an arc from x to y in a digraph, and xy to denote an edge joining x and y in a graph. Continuing in this vein, $(x_1, x_2, x_3, \ldots, x_n)$ denotes a directed path of length $n-1$, $(x_1, x_2, \ldots, x_n, x_1)$ denotes a directed cycle of length n, $x_1 x_2 \ldots x_n$ denotes a path of length $n-1$ in a graph and $x_1 x_2 \ldots x_n x_1$ denotes a cycle of length n in a graph. We use cyclic notation for permutations and in order to distinguish permutations from directed paths, we are careful with the exposition. Thus, as a permutation, $(1, 2, 3, 4)$ is the cyclic permutation mapping 4 to 1, and i to $i+1$ for $i = 1, 2, 3$.

For the rest of this paper, we consider only finite abelian groups and use additive notation with one exception. For the direct sum of a copies of the cyclic group Z_n, we write Z_n^a rather than aZ_n.

As mentioned above, Friedlander, Gordon and Miller [1] wrote the most significant paper on Ringel's problem. They conjectured that if G is a finite abelian group whose Sylow 2-subgroup is either trivial or both non-trivial and non-cyclic, then G is R-sequenceable. (In other words, the conjecture is saying that if G is not covered by Theorem 1.1, then it is R-sequenceable.) They established that the conjecture holds in many cases and introduced the following important strengthening of R-sequenceability. If $\overrightarrow{C} = (g_1, g_2, \ldots, g_{n-1}, g_1)$ is a directed cycle of length $n-1$ that is orthogonal to $\overrightarrow{K}(G)$, where G is an abelian group of order n, with the additional properties that 0 is the vertex missed by $\overrightarrow{C}$, and there exist three successive elements g_i, g_{i+1}, g_{i+2} on $\overrightarrow{C}$ such that $g_i + g_{i+2} = g_{i+1}$, then we say that G is *R^*-sequenceable.* We sometimes say that $g_1, g_2, \ldots, g_{n-1}$ is an *R^*-sequence.*

Friedlander, Gordon and Miller made considerable progress on the conjecture in [1], but did not solve it completely. Nevertheless, several of their results are important tools for the general conjecture. Some of the missing cases were settled in [4, 5, 11]. The proof of the conjecture is completed in this paper. We express the completion in the form of the following theorem that includes all finite abelian groups.

1.2 Theorem. *If G is a finite abelian group, then the following hold:*

(1) *G is sequenceable if the Sylow 2-subgroup is cyclic and non-trivial; and*

(2) *G is R-sequenceable if the Sylow 2-subgroup either is trivial, or the Sylow 2-subgroup is non-trivial and non-cyclic.*

2 First Stage of Proof

Part (1) of Theorem 1.2 is covered by Theorem 1.1. So we move to part (2) which has a natural partition into two subcases. The first subcase is that G has even order with its Sylow 2-subgroup non-trivial and non-cyclic. The second subcase is that G has odd order, that is, the Sylow 2-subgroup is trivial. We consider the first subcase next beginning with some useful results from [1].

2.1 Lemma. *The cyclic group Z_n is R^*-sequenceable for all odd $n > 5$.*

2.2 Lemma. *Let G be an R^*-sequenceable abelian group and Z_n, $n > 1$, an odd order cyclic group. Then the following hold:*

(1) *If G has even order, then $G \oplus Z_n$ is R^*-sequenceable; and*

(2) *If G has odd order, then $G \oplus Z_n$ is R^*-sequenceable whenever 3 does not divide n.*

2.3 Lemma. *Elementary abelian groups are R-sequenceable.*

The next two results are from [4, 7], respectively.

2.4 Lemma. *If G is an even order abelian group and its Sylow 2-subgroup is neither Z_2^3 nor $Z_2 \oplus Z_4$, then G is R-sequenceable.*

2.5 Lemma. *If G is R^*-sequenceable, then $Z_2^3 \oplus G$ is R^*-sequenceable.*

We now establish a method for handling the missing even order abelian groups. This is inspired by Häggkvist's Lemma in [3]. Consider the cycle $u_0u_1u_2\dots u_ru_0$. The edge u_iu_j divides the cycle into two subpaths with common end vertices u_i and u_j. The *length* of the edge u_iu_j is the length of the shorter of the two paths unless both subpaths have the same length in which case the length of the edge is $(r+1)/2$.

The following lemma follows from Corollary 2 of [6] but the proof we give here is more straightforward. The proof for m odd may be found in [10].

2.6 Lemma. *If we label the vertices of K_n cyclically as $u_0, u_1, u_2, \dots, u_{n-1}$, where $n = 2m > 4$, then there is a Hamilton path whose first edge has length m and every other edge length is used twice.*

Proof. When m is odd, start a path with the edge u_0u_m which has length m. Continue with the edge u_mu_1 and then zig zag back and forth decreasing the length by one with each edge until finishing with the edge $u_{(m-1)/2}u_{(m+1)/2}$. We refer to this kind of path as a *zig-zag path.* At this point we have used one edge of each of the lengths $1, 2, 3, \dots, m$.

Next we add the edge $u_{(m+1)/2}u_{(3m-1)/2}$ which has length $m-1$. The unused vertices are u_{m+1}, u_{m+2} through $u_{(3m-3)/2}$, of which there are $(m-3)/2$ such vertices, and $u_{(3m+1)/2}, u_{(3m+3)/2}$ through u_{2m-1}, of which there are $(m-1)/2$ such vertices. We now continue with an increasing zig-zag path starting with the edge $u_{(3m-1)/2}u_{(3m+1)/2}$ and finishing with the edge $u_{m+1}u_{2m-1}$ of length $m-2$. The resulting path satisfies the conclusions of the lemma. Figure 1 shows the path for $m = 5$.

The solution when m is even is different in that we describe an iterative procedure for which we show that it results in a path with the desired properties. We require some notation. We denote the current path by P and say the *terminal vertex* of P is the end vertex distinct from u_0. The interval $\mathrm{I}[u_i, u_j]$, $i \le j$, denotes the set of vertices $\{u_i, u_{i+1}, \dots, u_j\}$.

Suppose P misses the α vertices $\mathrm{I}[u_{2m-\alpha}, u_{2m-1}]$. If, in addition, the remaining vertices missed by P are $\mathrm{I}[u_1, u_{\alpha-1}]$ and $u_{\alpha+1}$, the terminal vertex of P is u_α, and the edge lengths not used twice by P are $2, 3, \dots, 2\alpha+1$, then we say the P is R-*sided.* Note that the interval notation makes no sense when $\alpha = 1$. In this case, we treat the interval $[u_1, u_0]$ as empty so that P terminates at u_1 and the vertex u_2 is not on P.

The other possibility is that the remaining vertices missed by P are $\mathrm{I}[u_1, u_{\alpha+2}]$ and $u_{2m-\alpha-2}$, the terminal vertex of P is $u_{2m-\alpha-1}$, and the edge lengths not used twice by P are $2, 3, \dots, 2\alpha+4$. In this case we say the P is L-*sided.* The interval notation makes no sense here for $\alpha = 0$. So we treat the interval $[u_{2m}, u_{2m-1}]$ as empty and maintain the remaining conditions.

If P is R-sided with $\alpha \ge 3$, then extend P by adding the 3-path $u_\alpha u_{2m-\alpha}u_{\alpha+1}$ $u_{2m-\alpha+2}$. These new edges have lengths $2\alpha-1, 2\alpha, 2\alpha+1$ and the terminal vertex of the updated P is now $u_{2m-\alpha+1}$. Thus, P is now L-sided and α has decreased by 3.

On the other hand, if P is L-sided with $\alpha \geq 1$, then extend P by adding the 3-path $u_{2m-\alpha-1}u_{\alpha+2}u_{2m-\alpha-2}u_{\alpha}$. These new edges have lengths $2\alpha+1, 2\alpha+2, 2\alpha+3$ and the terminal vertex of the updated P is now u_{α}. Thus, P is now R-sided and α has not changed.

Construct the initial path P by starting with the edge u_0u_m. Then add an increasing zig-zag path starting with the 2-path $u_mu_{m-1}u_{m+1}$ and continue until finishing with the edge from $u_{(3m-2)/2}$ to $u_{m/2}$ of length $m-1$. Then add the 3-path $u_{m/2}u_{(3m+2)/2}u_{3m/2}u_{(m-4)/2}$ to complete the initial P. Note that P is an R-sided path with $\alpha = (m-4)/2$.

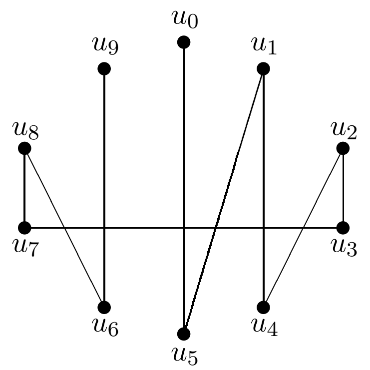

Figure 1

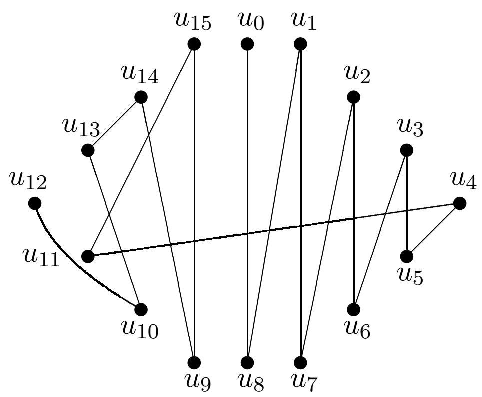

Figure 2

We now begin iterations of the procedure described above and may continue until we reach a path P that is L-sided with $\alpha = 0$, or R-sided with $\alpha \in \{1, 2\}$. If P is L-sided with $\alpha = 0$, then P terminates at u_{2m-1}, is missing the vertices u_1, u_2, u_{2m-2} and requires edges of lengths 2, 3, and 4. The completion $u_{2m-1}u_1u_{2m-2}u_2$ does the job.

If P is R-sided with $\alpha = 1$, then the terminal vertex is u_1, the missing vertices are u_{2m-1}, u_2, and the unused lengths are 2 and 3. The completion $u_1u_{2m-1}u_2$ works.

If P is R-sided with $\alpha = 2$, then the terminal vertex is u_2, the missing vertices are $u_{2m-2}, u_{2m-1}, u_1, u_3$, and the unused lengths are 2, 3, 4 and 5. There is no completion for this case. If this is the initial P, then $m = 8$ and Figure 2 gives a solution for $m = 8$. If this is not the initial P, then before the last iteration P was L-sided with $\alpha = 2$. So the vertices missed by P are $u_{2m-4}, u_{2m-2}, u_{2m-1}, \mathrm{I}[u_1, u_4]$, the terminal vertex is u_{2m-3}, and the missing lengths are 2 through 8. The completion that works is

$$u_{2m-3}u_3u_{2m-4}u_4u_{2m-1}u_1u_{2m-2}u_2.$$

This completes the proof. ■

Lemma 2.6 allows us to complete the even order case. Suppose the Sylow 2-subgroup of G is $Z_2 \oplus Z_4$. If this is the entire group G, then

$$(0,2), (1,3), (0,3), (1,1), (1,0), (1,2), (0,1)$$

is an R-sequence.

Write G as the direct sum of its Sylow subgroups. From the preceding paragraph we may assume that there is a summand of the form Z_q, where q is an odd prime power. So G is a direct sum of $Z_2 \oplus Z_4 \oplus Z_q \cong Z_2 \oplus Z_{4q}$ and an odd order abelian group.

Lemma 2.6 tells us that there is a path P (undirected) of length $4q - 1$ in K_{4q}, where we are thinking of this as a Cayley graph on Z_{4q}, such that an initial edge of P has length $2q$ (that is, joins 0 and $2q$) and all remaining edge lengths occur twice in P. Display the vertices of $Z_{4q} \oplus Z_2$ as a $2 \times 4q$ array with the obvious coordinate system from Z_2 and Z_{4q}.

Build an undirected cycle C of length $8q-1$ as follows. Join $(0,1)$ to both $(2q, 0)$ and $(2q, 1)$. Given two edges g_1g_2 and g_3g_4 of the same length in P, join $(g_1, 0)$ to $(g_2, 0)$ and $(g_1, 1)$ to $(g_2, 1)$, and join $(g_3, 0)$ to $(g_4, 1)$ and $(g_3, 1)$ to $(g_4, 0)$. Finally, if g is the terminal vertex of P distinct from 0, join $(g, 0)$ to $(g, 1)$.

The preceding construction yields a cycle C (undirected) of length $8q - 1$. Note that the vertex $(0, 0)$ is not included in C. Also note that three successive vertices are $(2q, 0), (0, 1), (2q, 1)$ and $(2q, 1) + (2q, 0) = (0, 1)$. Hence, if we direct C in either direction to obtain a directed cycle, both directed cycles provide an R^*-sequence for $Z_2 \oplus Z_{4q}$. As the remaining summands in the direct sum of G have odd order, we may apply part (1) of Lemma 2.2 as many times as required to obtain that G is R^*-sequenceble.

If the Sylow 2-subgroup of G is Z_2^3, then Lemma 2.3 takes care of the case that $G \cong Z_2^3$, and Lemmas 2.1 and 2.5 take care of the case that there is a cyclic group of odd order bigger than 5 in the direct sum of Sylow p-subgroups. Also, if both Z_3 and Z_5 appear in the Sylow subgroups of G, then Lemma 2.1 tells us that Z_{15} is R^*-sequenceable. Lemma 2.5 then takes care of this situation.

So we are left with groups of the form $Z_2^3 \oplus Z_3^a$ and $Z_2^3 \oplus Z_5^b$, where $a, b > 0$. Following are R^*-sequences for $Z_2^3 \oplus Z_3 \cong Z_2^2 \oplus Z_6$ and $Z_2^2 \oplus Z_{10}$, respectively:

$$(0,0,1), (0,1,1), (0,1,0), (0,0,5), (1,0,0), (1,0,1), (0,0,4), (1,1,0),$$
$$(1,1,4), (1,0,5), (1,1,2), (1,1,5), (0,1,5), (1,0,2), (0,1,3), (1,1,1),$$
$$(1,0,3), (0,1,2), (0,1,4), (0,0,2), (1,0,4), (0,0,3), (1,1,3)$$

and

$$\begin{gathered}(0,0,1),(1,1,6),(1,1,5),(0,1,9),(0,1,5),(0,0,3),(1,1,3),(0,1,2),\\(1,1,4),(0,1,0),(1,0,3),(0,1,7),(0,0,6),(0,0,9),(1,0,9),(1,0,1),\\(0,1,3),(1,1,8),(1,0,4),(1,0,8),(0,1,6),(1,1,9),(1,1,7),(1,0,2),\\(0,1,1),(0,0,2),(0,1,4),(1,1,1),(0,0,7),(0,0,8),(0,1,8),(0,0,5),\\(1,1,2),(1,0,6),(0,0,4),(1,0,5),(1,0,0),(1,0,7),(1,1,0).\end{gathered}$$

We then use part (1) of Lemma 2.2 to obtain that G is R^*-sequenceable for both forms. This completes the proof of Theorem 1.2 when G has even order.

3 The Gadget

To complete the proof of Theorem 1.2 for groups of odd order, we first state the following corollary which is an easy consequence of Lemma 2.1 and Lemma 2.2.

3.1 Corollary. *If G is an odd order abelian group whose Sylow 3-subgroup either is trivial, or non-trivial and cyclic, or R^*-sequenceable, then G itself is R^*-sequenceable unless $G \cong Z_3$ or $G \cong Z_5$ both of which are R-sequenceable.*

The preceding corollary means that we need only show that abelian groups whose Sylow 3-subgroups are non-trivial and non-cyclic are R-sequenceable. The method we employ works, in fact, for all odd order groups and there is no gain in efficiency by restricting ourelves to those groups satisfying the preceding condition on the Sylow 3-subgroups. Thus, we present the general method.

We work with direct sums. Given the direct sum $G \oplus H$, we shall display the vertices as an $|H| \times |G|$ array, where the columns correspond to the elements of G and the rows correspond to elements of H. We develop some lemmas which prove to be very useful, but we need a definition first.

3.2 Definition. Let f be a permutation of H and let $g_1, g_2 \in G$. We define the *f-lift of the arc* (g_1, g_2) onto $\overrightarrow{K}(G \oplus H)$ to be the set of arcs $\{((g_1, h), (g_2, f(h))) : h \in H\}$. We denote this set of arcs by $\pi_f(g_1, g_2)$.

In spite of the fact we use functional notation for permutations, we compose permutations from left to right because we move through the arrays from left to right. This gives us the composition rule $(fg)(x) = g(f(x))$.

3.3 Lemma. *Let G and H be abelian groups. If $(g_1, g_2, \ldots, g_{r+1})$ is a directed path in $\overrightarrow{K}(G)$ of length r, and $f_1, f_2, \ldots, f_r$ are permutations of H, then the set of arcs*

$$\pi_{f_1}(g_1, g_2) \cup \pi_{f_2}(g_2, g_3) \cup \cdots \cup \pi_{f_r}(g_r, g_{r+1})$$

forms $n = |H|$ vertex-disjoint directed paths of length r in $\overrightarrow{K}(G \oplus H)$, where the last vertex of the directed path with initial vertex (g_1, h) is $(g_{r+1}, f_1 f_2 \cdots f_r(h))$.

If $(g_1, g_2, \ldots, g_r, g_1)$ is a directed cycle in $\overrightarrow{K}(G)$ of length r, and $f_1, f_2, \ldots, f_r$ are permutations of H, then the set of arcs

$$\pi_{f_1}(g_1, g_2) \cup \pi_{f_2}(g_2, g_3) \cup \cdots \cup \pi_{f_r}(g_r, g_1)$$

forms vertex-disjoint directed cycles. The number of directed cycles equals the number of cycles in the disjoint cycle decomposition of $f_1 f_2 \cdots f_r$.

PROOF. It is easy to see that $\pi_f(g_1, g_2)$ for any permutation f of H generates an orientation of a perfect matching between vertices whose first coordinate is g_1 and vertices whose first coordinate is g_2 so that every arc is oriented from g_1 to g_2. It then follows directly that we obtain n vertex-disjoint directed paths as claimed.

If we consider the directed path starting at (g_1, h), it is straightforward to see that its terminal vertex is $(g_{r+1}, f_1 f_2 f_3 \cdots f_r(h))$.

The argument for a directed cycle in $\overrightarrow{K}(G)$ is essentially the same except that π_{f_r} generates an arc from vertices in $G \oplus H$ whose first coordinate is g_r to vertices whose first coordinate is g_1. It is then easy to see that a cycle of length t in the disjoint cycle decomposition of $f_1 f_2 f_3 \cdots f_r$ generates a directed cycle of length rt in $G \oplus H$. The rest of the lemma now follows. ■

Lemma 3.3 gives us a way of controlling arcs in $\overrightarrow{K}(G \oplus H)$. But we really would like the arcs in the projection of an arc of $\overrightarrow{K}(G)$ to be generated by distinct elements of $G \oplus H$. This leads naturally to a known type of permutation. A permutation $f : H \to H$ is an *orthomorphism* if the function $g(x) = f(x) - x$ also is a permutation. The next lemma tells us that orthomorphisms are precisely what we need.

3.4 Lemma. *Let G and H be abelian groups. If f is an orthomorphism of H, then the arcs of $\pi_f(g_i, g_j)$ in $\overrightarrow{K}(G \oplus H)$ are generated by the group elements $(g_j - g_i, h)$ as h runs through H.*

PROOF. This follows immediately from the definition of an orthomorphism. ■

There are some special orthomorphisms we use. Let $|H|$ be odd and define the permutation T_0 on H by $T_0(h) = -h$ for $h \in H$. It is easy to see that T_0 is an orthomorphism because H contains no involutions. We extend this particular orthomorphism to T_a, $a \in H$, by defining $T_a(h) = 2a - h$. It is straightforward to check that T_a also is an orthomorphism. An important feature of these particular orthomorphisms is the following. When $H \cong Z_n$, n odd, then the composition

$$T_0 T_1 = h + 2 = (0, 2, \ldots, n-1, 1, 3, \ldots, n-2), \tag{1}$$

that is, the product is an n-cycle.

If G is an R^*-sequenceable abelian group of order m, then we have a directed cycle of length $m-1$ that misses the vertex 0 and has three successive vertices a, b, c for which $a + c = b$. Label the vertices of the directed cycle in succession as $g_1, g_2, \ldots, g_{m-1}$ so that $a = g_1, b = g_2, c = g_3$. The *canonical labelling* of the group $G \oplus H$ has the columns labelled so that the leftmost column is labelled g_1, the next column is labelled 0, and the remaining columns are labelled g_2 through g_{m-1} from left to right in that order.

We want to prove that $G \oplus H$ is R^*-sequenceable whenever possible. It is natural to work with lifts of arcs of the directed cycle in $\overrightarrow{K}(G)$, but this directed cycle misses the vertex 0 so that we need to get the vertices of the column labelled 0 involved. We now describe how to do so.

3.5 Definition. Suppose that G is an abelian group with non-zero elements g_1, g_2, g_3 satisfying $g_1 + g_3 = g_2$. Consider $G \oplus H$ with H abelian of odd order $n \geq 3$. The lifts $\pi_{T_0}(g_1, g_2) \cup \pi_{T_0}(g_2, g_3)$ consist of n vertex-disjoint directed paths of length 2 using all the vertices of columns g_1, g_2, g_3, and whose arcs are generated by $(g_2 - g_1, h)$ and $(g_3 - g_2, h)$ as h runs through H.

Now for each pair $h, -h$ of additive inverses, replace the pair of directed 2-paths

$$((g_1, h), (g_2, -h), (g_3, h))\text{and } ((g_1, -h), (g_2, h), (g_3, -h)), h \neq 0,$$

by the directed 3-paths

$$((g_1, h), (0, -h), (0, h), (g_3, -h)) \text{ and } ((g_1, -h), (g_2, h), (g_2, -h), (g_3, h).$$

The directed 2-path $((g_1, 0), (g_2, 0), (g_3, 0))$ is left unaltered. The new collection of directed paths is called the *gadget on columns* $g_1, 0, g_2, g_3$.

3.6 Lemma. *The arcs of the gadget on columns $g_1, 0, g_2, g_3$ are generated by the elements $(g_2 - g_1, h), (g_3 - g_2, h), (0, h')$ for all $h \in H$ and all $h' \neq 0$ in H. Moreover, the terminal vertex of the directed path whose initial vertex is (g_1, h) is $(g_3, -h)$.*

Proof. The new arc $((g_1, h), (0, -h))$ of the gadget is generated by the group element $(g_3 - g_2, -2h)$ because $g_1 + g_3 = g_2$. Similarly, the arc $((0, h), (g_3, -h))$ is generated by the group element $(g_2 - g_1, -2h)$. The two vertical arcs $((0, -h), (0, h))$ and $((g_2, h), (g_2, -h))$ are generated by the group elements $(0, 2h)$ and $(0, -2h)$. Finally, the arc $((g_1, -h), (g_2, h))$ is generated by $(g_2 - g_1, 2h)$, and the arc $((g_2, -h), (g_3, h))$ is generated by $(g_3 - g_2, 2h)$. Hence, the claims about which group elements generate the arcs of the gadget follow.

It is easy to see that the directed path beginning at (g_1, h) terminates at $(g_3, -h)$ for all $h \in Z_n$. ■

The next lemma is the basis for establishing Theorem 1.2 when G has odd order.

3.7 Lemma. *Let G be an R^*-sequenceable abelian group of order m. If H is an odd order abelian group for which there are orthomorphisms $f_1, f_2, \ldots, f_t$ of H such that $T_0 f_1 f_2 \cdots f_t$ is an $|H|$-cycle and $m - t - 3 \geq 0$ is even, then $G \oplus H$ is R^*-sequenceable.*

PROOF. We use the canonical labelling of $G \oplus H$. The first four columns of the array correspond to the group elements $g_1, 0, g_2, g_3$ in that order, where $g_1 + g_3 = g_2$. Employ the gadget on these first four columns. Because of Lemma 3.6, it follows that if for each remaining (g_i, g_{i+1}) and (g_{r-1}, g_1), we employ a lift arising from an orthomorphism of H, the arcs will have been generated by all elements of $G \oplus H$ other than $(0, 0)$. Moreover, the vertex $(0, 0)$ is isolated and the vertices$(g_1, 0), (g_2, 0), (g_3, 0)$ occur in succession. Because $(g_1, 0)+(g_3, 0) = (g_2, 0)$, if the arcs form a single directed cycle, then $G \oplus H$ is R^*-sequenceable.

From Lemma 3.6, the permutation from column g_1 to column g_3 is T_0. We then successively employ the orthomorphisms $f_1, f_2, \ldots, f_t$ for the following lifts. By hypothesis, the product $T_0 f_1 f_2 \cdots f_t$ is a cycle of length $|H|$.

There are $m - (t + 3)$ further lifts to be employed. If $m - (t + 3) = 0$, we already have an $|H|$-cycle and we are done. If $m - (t + 3) > 0$, then it is even and we use T_0 for each subsequent lift. The product of an even number of T_0 permutations is the identity as T_0 is an involution. Thus, the final product is a cycle of length $|H|$ completing the proof. ■

This method of lifts brings to the fore why the prime 3 is a nagging problem. For $a \in Z_n$ satisfying $\gcd(n, a) = 1$, let M_a denote the permutation of Z_n defined by $M_a(x) = ax$. When 3 does not divide n, it is straightforward to check that both M_2 and $M_{(n-1)/2}$ are orthomorphisms. Note that $M_2 M_{(n-1)/2} = T_0$. Then $T_0 M_2 M_{(n-1)/2} T_0 T_1 = T_0 T_1$ is an n-cycle and Lemma 2.2 applies for $m \geq 7$. When 3 divides n, unfortunately, $M_{(n-1)/2}$ is not an orthomorphism forcing us to find special arguments for the prime 3. This is what we now examine.

3.8 Corollary. *If G is an R^*-sequenceable abelian group of odd order, then $G \oplus Z_{3^e}$ is R^*-sequenceable for $e \geq 2$.*

PROOF. It is easy to verify that the permutations $f_0 = T_0, f_1 = M_2$, and $f_2 = (0, 1)(2, 6, 3, 5, 8, 4)(7)$ satisfy $f_0 f_1 f_2 = (0, 1, 7, 2, 8, 6, 3, 5, 4)$ for $e = 2$. This means that $G \oplus Z_9$ is R^*-sequenceable when G is R^*- sequenceable according to Lemma 3.7.

For $e = 3$, let $f_0 = T_0$. Let

$$f_1 = (0, 26, 3, 8, 19, 7, 10, 16, 5, 24, 17, 12, 20, 14, 4, 22, 23, 25, 11, 18, 1, 13, 9, 6, 15, 2)(21)$$

and $f_2 =$

$$(0, 22, 21, 13, 11)(1, 6, 7)(2, 8, 15, 5, 23, 10, 19, 4, 24, 20, 3, 16, 18, 26, 14, 25)(9, 12)(17).$$

Again it is easy to verify that the permutation $f_0 f_1 f_2$ is a 27-cycle as required. Lemma 3.7 then implies that $G \oplus Z_{27}$ is R^*-sequenceable when G is R^*-sequenceable.

We now want to show that $G \oplus Z_{3^e}$ is R^*-sequenceable, when G is R^*-sequenceable, for all $e \geq 2$ and we proceed by induction on e having established the result for $e = 2, 3$. Consider $e \geq 4$. Let N be the subgroup of Z_{3^e} of order 3^{e-2} so that Z_{3^e}/N

is isomorphic to Z_9. Use $0, 1, \ldots, 8$ as the coset representatives and let $\overline{x}$ correspond to the element $N + x$ in the quotient group of order 9.

From above we know there are three orthomorphisms $\overline{f_0}, \overline{f_1}, \overline{f_2}$ of Z_{3^e}/N so that $\overline{f_0 f_1 f_2} = (\overline{0}, \overline{1}, \overline{7}, \overline{2}, \overline{8}, \overline{6}, \overline{3}, \overline{5}, \overline{4})$, and $\overline{f_0}(\overline{0}) = \overline{f_1}(\overline{0}) = \overline{0}$ and $\overline{f_2}(\overline{0}) = \overline{1}$. Suppose that $\overline{f_i}(\overline{x}) = \overline{y}$. Then let α be any orthomorphism of N. Define the *α-lift* action of f_i on $N + x$ by letting $f_i(n + x) = \alpha(n) + y$, $n \in N$. It is easy to see that f_i acting on the coset $N + x$ picks up all elements of the form $N + (y - x)$ via $f_i(n + x) - (n + x)$. Thus, f_i is an orthomorphism of Z_{3^e} if the action on each coset is defined via the lift of an orthomorphism of N as just described.

We now define f_0, f_1, f_2 to ensure that $f_0 f_1 f_2$ is a cycle of length 3^e. Let $\alpha_0, \alpha_1, \alpha_2$ be orthomorphisms of N such that $\alpha_0 \alpha_1 \alpha_2$ is a cycle of length 3^{e-2} on N by induction. We have that $\overline{f_0}$ maps $\overline{0}$ to itself. We use the lift of the orthomorphism α_0 on N to define f_0 on N. Continuing, we know that $\overline{f_1}$ also maps $\overline{0}$ to $\overline{0}$. We use the lift of α_1 to define f_1 acting on N. Finally, to get the action of f_2 on N, use the lift of α_2 to define the action of f_2 mapping N to $N + 1$. For all other lifts, use T_0 on N.

We claim that $f_0 f_1 f_2$ is a cycle of length 3^e. To see this, first note that $f_0 f_1 f_2$ acts as $[\overline{0}, \overline{1}, \overline{7}, \overline{2}, \overline{8}, \overline{6}, \overline{3}, \overline{5}, \overline{4}]$ on the cosets. Because $\alpha_0 \alpha_1 \alpha_2$ is a cycle of length 3^{e-2} on N and we use the lifts of these three orthomorphisms to give the action of f_0, f_1, f_2 on N, we see that if $\alpha_0 \alpha_1 \alpha_2(n_1) = n_2$, then $f_0 f_1 f_2(n_1) = n_2 + 1$. All remaining lifts use T_0 and there are an even number of them so that $f_0 f_1 f_2$ is a full cycle of length $9 \cdot 3^{e-2} = 3^e$ as required. ■

If every summand in the Sylow 3-subgroup has order at least 9, then any summand is R^*-sequenceable by Lemma 2.1. Repeated applications of Corollary 3.8 yield that the Sylow 3-subgroup is R^*-sequenceable. Corollary 3.1 then implies that G is R^*-sequenceable.

When exactly one summand in the Sylow 3-subgroup is Z_3, we require a lemma. Two useful items for the proof are given first.

The following are R^*-sequences for $Z_3 \oplus Z_9$ and $Z_3 \oplus Z_{27}$, respectively:

$$(2,0),(2,3),(0,3),(0,5),(1,2),(2,4),(1,1),(1,8),(0,1),(1,4),(1,0),(2,1),(2,2),$$
$$(2,8),(1,6),(0,2),(1,7),(0,8),(1,3),(0,7),(1,5),(0,4),(2,7),(0,6),(2,6),(2,5)$$

and

$$(0,1),(0,26),(0,25),(1,24),(2,10),(1,11),(1,25),(0,11),(2,16),(0,8),$$
$$(2,26),(1,15),(0,14),(2,4),(1,23),(0,23),(2,20),(1,8),(2,15),(1,0),$$
$$(0,10),(0,17),(1,19),(2,14),(0,19),(1,20),(1,13),(1,7),(1,18),(1,3),$$
$$(2,13),(2,17),(2,7),(0,22),(2,25),(1,6),(0,20),(2,0),(2,8),(2,5),$$
$$(0,2),(1,10),(2,1),(2,3),(0,7),(2,18),(2,9),(0,18),(1,14),(0,12),$$
$$(1,26),(1,2),(0,6),(2,12),(2,22),(1,4),(2,2),(2,21),(0,21),(2,23),$$
$$(0,16),(1,22),(2,11),(2,24),(2,6),(1,1),(0,24),(1,9),(0,3),(0,4),$$
$$(0,9),(0,15),(1,5),(1,21),(1,17),(1,12),(0,5),(1,16),(2,19),(0,13).$$

3.9 Lemma. *The group $G = Z_3 \oplus Z_{3^e}$, $e \geq 2$, is R^*-sequenceable.*

Proof. The statement is true for $e = 2, 3$ because R^*-sequences are given above. We proceed by induction on e and let $e > 3$. Let N be the cyclic subgroup of order 3^{e-2}. The quotient group G/N is isomorphic to $Z_3 \oplus Z_9$. Let the coset representatives be $\{(i, j) : 0 \leq i \leq 2, 0 \leq j \leq 8\}$ and let $\overline{(i, j)}$ denote the element $N + (i, j)$ of the quotient group.

Display the elements of G as a $3^{e-2} \times 9$ array where the columns are cosets of the cyclic subgroup of order 3^{e-2} and they are written left to right in the order of the R^*-sequence for $Z_3 \oplus Z_9$ given above, where column $\overline{(0, 0)}$ is inserted between $\overline{(2, 0)}$ and $\overline{(2, 3)}$.

Even though the columns now correspond to cosets of $Z_{3^{e-2}}$ rather than the group itself—as they did earlier when we defined the lift of an arc onto the array for a direct sum—it should be clear how we define a lift now. Namely, if there is an arc from $\overline{(i, j)}$ to $\overline{(i', j')}$ in $\overrightarrow{K}(G/N)$ and f is a permutation of N, then for each $(i, j) + n \in \overline{(i, j)}$, we have an arc to $(i', j') + f(n)$. We then use the same notation π_f for the lift.

We then use π_{T_0} as the lift for the arcs from $\overline{(2, 0)}$ to $\overline{(2, 3)}$, and from $\overline{(2, 3)}$ to $\overline{(0, 3)}$. Note that one of the directed paths is $(2, 0), (2, 3)(0, 3)$ and this sequence of three vertices satisfies $(2, 0) + (0, 3) = (2, 3)$. So if we end up with a directed cycle of length $3^{e+1} - 1$, we have that $Z_3 \oplus Z_{3^e}$ is R^*-sequenceable.

It is now clear that if we carry out the obvious gadget operation, we end up with directed paths of length 3, except for the unaltered directed path, whose initial and terminal vertices behave like π_{T_0} from column $\overline{(2, 0)}$ to column $\overline{(0, 3)}$. In the proof of Corollary 3.8, we show that for all $e > 1$ there are two orthomorphisms f_1, f_2 such that $T_0 f_1 f_2$ is a cycle of length of length 3^e. So we use these two orthomorphisms for the next two lifts of arcs along the R^*-sequence for $Z_3 \oplus Z_9$. We then use T_0 for all subsequent lifts and this leads to a directed cycle of length $3^{e+1} - 1$ as required. ∎

We continue now with the subcase that the Sylow 3-subgroup has exactly one Z_3 term in the direct sum. The Sylow 3-subgroup is not cyclic so that Lemma 3.9 and repeated applications of Corollary 3.8 imply that the Sylow 3-subgroup is R^*-sequenceable. Lemma 3.1 then implies that G is R^*-sequenceable.

If there are two or more Z_3 terms in the direct sum for the Sylow 3-subgroup, there is a useful fact we exploit. Let

$$f_1 = \Big((0, 0), (2, 0), (0, 2), (1, 2), (1, 0), (0, 1)\Big)\Big((1, 1), (2, 2)\Big)\Big((2, 1)\Big)$$

and

$$f_2 = \Big((0, 0), (1, 0), (1, 1), (0, 2), (2, 2), (0, 1)\Big)\Big((1, 2), (2, 0)\Big)\Big((2, 1)\Big)$$

be two permutations of $Z_3 \oplus Z_3$. It is easy to check that both are orthomorphisms and that $T_0 f_1 f_2$ is a 9-cycle.

We then conclude that $Z_3 \oplus Z_3 \oplus G$ is R^*-sequenceable when G is R^*-sequenceable and has odd order from Lemma 3.7. So consider the Sylow 3-subgroup H itself. If

H has a summand Z whose order is at least 9, then both Z and $Z_3 \oplus Z$ are R^*-sequenceable by Lemma 2.1 or Lemma 3.9. Then H is R^*-sequenceable by starting with Z if there are an even number of Z_3 terms in the direct sum, or starting with $Z_3 \oplus Z$ if there are an odd number, and using the preceding fact. Therefore, H is R^*-sequenceable and Lemma 3.1 implies that G is R^*-sequenceable.

The preceding paragraph means we are left with the subcase that the Sylow 3-subgroup is Z_3^a for some $a \geq 2$. If this is all of G, then G is R-sequenceable by Lemma 2.3. So we may assume that there is a non-trivial Sylow p-subgroup for some prime $p > 3$. If $p > 5$, then we may repeatedly apply Lemmas 2.1, 2.2, and the above fact to obtain that G is R^*-sequenceable.

The same process works for $p = 5$ except $Z_3 \oplus Z_3 \oplus Z_5$. Following is an R^*-sequence for this group which completes the proof of Theorem 1.2.

$$
\begin{aligned}
&(0,0,1),(0,2,2),(0,2,1),(1,1,0),(0,2,3),(0,1,1),(0,1,2),(1,2,3),(0,0,2),\\
&(2,1,2),(0,0,4),(0,1,0),(1,0,3),(2,0,0),(2,1,3),(2,0,3),(0,1,3),(1,2,1),\\
&(2,2,1),(1,1,2),(2,1,0),(1,0,2),(1,0,0),(2,0,4),(1,1,1),(2,2,0),(2,2,2),\\
&(2,0,1),(2,2,3),(0,1,4),(2,1,1),(1,2,2),(0,2,0),(2,1,4),(1,1,4),(1,2,4),\\
&\qquad (1,1,3),(0,0,3),(1,0,4),(2,2,4),(1,2,0),(2,0,2),(1,0,1),(0,2,4).
\end{aligned}
$$

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