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Existence of near resolvable $(v, k, k-1)$ BIBDs with $k \in \{9, 12, 16\}$

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Abstract

A necessary condition for existence of a $(v, k, k-1)$ near resolvable BIBD is $v \equiv 1 \pmod{k}$. In this paper, we update earlier known existence results when $k \in \{9, 12, 16\}$, and show this necessary condition is sufficient, except possibly for 26, 37 and 149 values of v for $k = 9, 12, 16$ respectively. Some new results for existence of $(9, 8)$-frames of type 9^t are also obtained; in particular, we show these exist for all $t \geq 139$.

1 Introduction

A (K, λ) group divisible design, ((K, λ)-GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a set of points, $\mathcal{G}$ is a partition of X into *groups*, and $\mathcal{B}$ is a collection of subsets of X (called blocks), each with size from K, such that any two distinct points from X appear in λ blocks if they lie in different groups from $\mathcal{G}$ and in no blocks if in the same group. The parameter λ is sometimes omitted if it equals 1, and if $K = \{k\}$, the notation (k, λ)-GDD instead of $(\{k\}, \lambda)$-GDD is more commonly used. Also, a (K, λ)-GDD is said to have type $g_1^{u_1} g_2^{u_2} \ldots, g_t^{u_t}$ if it has u_i groups of size g_i for $1 \leq i \leq t$. A (v, K, λ) PBD or pairwise balanced design, is a (K, λ)-GDD of type 1^v, and for k a positive integer, a (v, k, λ) BIBD is a (k, λ)-GDD of type 1^v. A transversal design, TD(k, v) is a $(k, 1)$-GDD of type v^k.

A (K, λ)-GDD $(X, \mathcal{G}, \mathcal{B})$ is called *resolvable* if its block set $\mathcal{B}$ can be partitioned into *parallel classes*, each containing every point of X exactly once. A (K, λ)-GDD is called a frame if its block set $\mathcal{B}$ can be partitioned into *holey parallel classes*, each containing every point from $X \backslash G_i$ once (for some group G_i) and no points from G_i. A (v, k, λ) BIBD is called *near resolvable*, or a (v, k, λ) NRB, if it is a (k, λ)-frame of type 1^v. (In some papers, the notation (v, k, λ) NRBIBD is used instead of (v, k, λ) NRB, but that notation will not be used in this paper.) Necessary conditions for existence of a (v, k, λ) NRB are $v \equiv 1 \pmod{k}$ and $\lambda \equiv 0 \pmod{(k-1)}$.

In this paper, we are mainly interested in $(v, k, k-1)$ NRBs. The notation NRB(k) is sometimes used to denote the set $\{v : \text{a } (v, k, k-1) \text{ NRB exists}\}$. It is known that for any k the set NRB(k) is PBD-closed, that is, if a $(v, T, 1)$ PBD exists and a $(t, k, k-1)$ NRB exists for all $t \in T$, then a $(v, k, k-1)$ NRB also exists.

Existence of $(v, k, k-1)$ NRBs has been studied by several researchers for certain values of k. For $k = 3$, existence was established by Hanani [21]. For $k = 4$, solutions exist when $v \in \{5, 9, 13, 17, 29\}$ since these values are prime powers (see for instance, [18, Lemma 2.6.1]). When $v > 17$, $v \equiv 1 \pmod 4$ and $v \notin \{29, 33\}$, solutions can be obtained from PBD-closure of NRB(4), since a $(v, \{5, 9, 13\}, 1)$ PBD exists for all such v [20]. When $v = 33$, various cyclic solutions are known; see for instance [6, page 128].

For $k = 5$, existence was solved for all but 26 cases in [23], and this list was reduced to 8 cases in [22]. These 8 cases ($v = 46$, 51, 116, 141, 201, 266, 296, and 351) were obtained in [16, Corollaries 3.7, 3.10] and [18, Section 4.3]. We point out that the constructions in [16, Lemmas 2.4, 2.5] for a $(5, 4)$-frame of type 5^7 and a $(26, 5, 4)$ NRB were not correct, but alternative constructions for these designs can be found in [3] and [4] respectively. Also, an alternative proof for existence of $(v, 5, 4)$ NRBs with $v \notin \{6, 26\}$ can be found in [2, 3].

In [18, Tables 4.5.1.3 and 4.6.12] and [16, Table 3] it was noted that for $k = 6, 7, 8$, $(v, k, k-1)$ NRBs exist for $v \equiv 1 \pmod k$ with at most 2 possible exceptions for $k = 6$, 14 possible exceptions for $k = 7$, and 17 possible exceptions for $k = 8$. For $k = 6$, much of the preliminary work was done in [22]. For $k = 6$, the 2 possible exceptions ($v = 55, 145$) were solved in [8], and for $k = 7$, all 14 possible exceptions were later removed in [1, 4]. For $k = 8$, all but 2 of the 17 unknown cases ($v = 385$ and 553) were removed in [9].

For $k = 10$, it is known [19] that no $(v, 10, 9)$ NRB can exist when $v = 21$. Later in [7], existence of $(v, 10, 9)$ NRBs was solved for all but at most 42 other values of v. In [15], existence of $(v, 12, 11)$ NRBs was solved with at most 83 possible exceptions. Also, for $k \in \{12, 16\}$, Furino [16] and Furino et al. [18, Table 4.2.11] noted that their generic search algorithm could produce $(v, k, k-1)$ NRBs for all but 53 values of v when $k = 12$ and 158 values of v when $k = 16$. However no list of possible exceptions was given here. Some similar (but usually larger) bounds are also given in [16, 18] for some other values of $k \leq 30$. For $k = 17$ however, the bound given needs to be revised, since it assumed existence of a $(290, 17, 16)$ NRB which is currently unknown.

In this paper, we shall investigate existence of $(9, 8)$-frames of type 9^t, and of $(v, k, k-1)$ NRBs with $k \in \{9, 12, 16\}$. For $k = 9$, the best known existence result for $(v, 9, 8)$ NRBs is a rather weak one (in Section 4.1.7 of [18], it was stated that these designs exist for $v \equiv 1 \pmod 9$, $v > 18145$), and even this bound was inaccurate, since it assumed existence of an $(82, 9, 8)$ NRB, which is currently unknown. More generally, the proof of existence of $(q^2+1, q, q-1)$ NRBs for q an odd prime power in [18, Theorem 2.6.30] and [16, Lemma 2.8] was incorrect. Our existence proof makes use of $(9, 8)$-frames of type 9^t for $10 \leq t \leq 13$ and also for several values of t that are odd prime powers. Using these frames, we are also able to obtain all $(9, 8)$-frames of type 9^t with $t \geq 139$. For $k = 12$, we start with the list of 83 possible exceptions in [15], and are able to reduce this list to 37 possible exceptions. For $k = 16$, we first look at $v \leq 10033$, the largest unknown case in [16] and [18]. We provide some detail on existence of these smaller designs, and then also give a recursive proof for the larger designs. We also provide a list of unknown cases, which wasn't given in [16] or [18]. However here, we are only able to slightly reduce the number of possible exceptions in [16] and [18] (from 158 to 149).

The rest of this paper is organised as follows. Section 2 gives some general construction methods for frames and $(v, k, k-1)$ NRBs; these are mostly of a recursive nature. Section 3 looks at a special type of design, namely a base factorisation, and gives one useful construction which enables us to obtain some $(12, 11)$-frames of types $12^t, 24^t$ and some $(16, 15)$-frames of type 16^t. Several $(v, k, k-1)$ NRBs with $k \in \{12, 16\}$ are obtainable from these frames. Section 4 gives some direct constructions for some small $(k, k-1)$-frames and $(v, k, k-1)$ NRBs with $k \in \{9, 12, 16\}$. Section 5 looks at existence of $(9, 8)$-frames of type 9^t, and constructs these for $t \geq 139$. Sections 6, 7 and 8 look at existence of $(v, k, k-1)$ NRBs with $k = 9, 12, 16$ respectively and reduce the numbers of unknown cases to 26, 37 and 149 respectively.

2 Construction Methods

In this section, we give a number of known general constructions for NRBs and frames that will be useful for obtaining several results in this paper. When not given here, proofs can be found in the references cited. In general, [18] is an excellent reference for constructions of these types.

Lemma 2.1 *[18, Lemma 2.6.1] If p is a prime power and $p \equiv 1 \pmod{k}$, then there exists a $(p, k, k-1)$ NRB obtainable by developing $(p-1)/k$ base blocks over GF(p).*

Lemma 2.2 *[18, Theorem 2.4.7] (Breaking up groups) Suppose there exists a (K, λ)-frame of type $g_1, g_2 \ldots, g_m$, and for each $i = 1, 2, \ldots, m$, a (K, λ)-frame of type $g^{g_i/g}e^1$ exists. (Note that if $e = 0$, then $g_i \in \{0, g\}$ is permissible here.) Then a (K, λ)-frame of type $g^t e^1$ exists for $t = \sum_{i=1}^{m}(g_i/g)$.*

In the special case that $\lambda = k - 1$, $e = 0$ or 1 and $g = 1$, Lemma 2.2 gives:

Lemma 2.3 *Suppose there exists a $(k, k-1)$-frame of type $g_1, g_2, \ldots, g_n$. Then if $e = 0$ or 1 and a $(g_i+e, k, k-1)$ NRB exists for each $i \in \{1, 2, \ldots, n\}$, a $(g+e, k, k-1)$ NRB exists for $g = \sum_{i=1}^{n} g_i$.*

Wilson's fundamental GDD construction can be modified to produce certain types of frames. The next lemma gives a general form of this construction:

Lemma 2.4 *[18, Corollary 2.4.3] Let $(X, \mathcal{G}, \mathcal{B})$ be a (master) $(K_1, 1)$-GDD and let w be a weight function on X such that $w(x)$ is a non-negative integer for all $x \in X$. Suppose for all $B \in \mathcal{B}$ there exists a (K_2, λ)-frame of type $(w(x) : x \in B)$. Then there exists a (K_2, λ)-frame of type $(\sum_{x \in G} w(x) : G \in \mathcal{G})$.*

In the previous lemma, if the input $(K_1, 1)$-GDD is a PBD, and all points are given constant weight g, then we obtain the following result:

Lemma 2.5 *[18, Corollary 2.4.4] If there exists a $(v, K, 1)$ PBD and a (K, λ)-frame of type g^s exists for each $s \in K$, then a (K, λ)-frame of type g^v exists.*

If further, we take $g = 1$ and $\lambda = k - 1$ in the previous lemma, the input frames are all $(s, k, k-1)$ NRBs (i.e. $(k, k-1)$-frames of type 1^s). In this case, the following PBD-closed result for $(v, k, k-1)$ NRBs is obtained:

Lemma 2.6 *If there exist a $(v, K, 1)$ PBD and an $(s, k, k-1)$ NRB for each $s \in K$, then there exists a $(v, k, k-1)$ NRB.*

There is also a useful construction which inflates a frame with a resolvable TD to obtain a larger frame:

Lemma 2.7 *[18, Corollary 2.4.6] If a (k, λ)-frame of type $h_1 h_2 \ldots h_m$ and a resolvable TD(k, g) both exist, then there exists a (k, λ)-frame of type $gh_1\, gh_2\, \ldots\, gh_m$. If further, a $(gh_i + 1, k, \lambda)$ NRB exists for each $i = 1, 2, \ldots, m$, then there exists a $(gh + 1, k, \lambda)$ NRB for $h = \sum_{i=1}^{m} h_i$.*

Lemma 2.8 *If k is a prime power, then there exists a $(k, 1)$-frame of type $(k-1)^{k+1}$.*

Proof: Deleting a point P (and the blocks containing it) from a resolvable $(k^2, k, 1)$ BIBD (an affine plane) gives a $(k, 1)$-GDD of type $(k-1)^{k+1}$. Its groups are $B_i \backslash \{P\}$ for each block B_i $(i = 1, 2, \ldots, k+1)$ containing P. This GDD is also a frame, since for each B_i, the blocks other than B_i in the parallel class (of the original BIBD) containing B_i form the required partial parallel class missing the group $B_i \backslash \{P\}$. ∎

Lemma 2.9 *For any positive integer k, a $(k, k-1)$-frame of type 1^{k+1} exists. If k is a prime power, a $(k, k-1)$-frame of type k^{k+1} also exists.*

Proof: A $(k, k-1)$-frame of type 1^{k+1} is obtained by taking all k-element subsets of a size $k+1$ set. For a $(k, k-1)$-frame of type k^{k+1}, apply Lemma 2.7 with $g = k$ to this frame. ∎

Lemma 2.10 *If $k+1$ is a prime power, then there exists a $(k, k-1)$-frame of type k^{k+2}.*

Proof: Deleteing one point and its blocks from an affine plane of order $k+1$ gives a $(k+1, 1)$-GDD of type k^{k+2}. We can now apply Lemma 2.4 to this GDD, giving all points weight 1, since a $(k, k-1)$-frame of type 1^{k+1} exists by Lemma 2.1. ∎

Lemma 2.11 *If k is a prime power, then there exists a cyclic $(k^2+k+1, , k, k-1)$ NRB.*

Proof: Since k is a prime power, there exists a symmetric $(k^2+k+1, k+1, 1)$ BIBD which is obtainable by developing a block $B = \{b_1, b_2, \ldots, b_{k+1}\}$ over $G = Z_{k^2+k+1}$. For each $x \in G$, let $B-x$ denote the block obtained by subtracting x from all elements of B. Now consider the $k+1$ blocks $C_l = (B - b_l)\backslash\{0\}$, $(l = 1, 2, \ldots, k+1)$. These $k+1$ blocks generate a cyclic $(k^2+k+1, k, k-1)$ BIBD: any non-zero element of G equals $b_i - b_j$ for exactly one pair (i, j), and if $x = b_i - b_j$ is any such nonzero element, then x appears once as a difference between two values in each C_l $(l = 1, 2, \ldots, k+1)$ except when $l = i$ or j. In addition, since the original BIBD was symmetric with index $\lambda = 1$, any two of its blocks intersect in exactly 1 point. Therefore no two blocks C_{l_1}, C_{l_2} $(l_1, l_2 \in \{1, 2, \ldots, k+1\}, l_1 \neq l_2)$ can contain any common points. Thus the $k(k+1)$ points in the blocks C_l $(l = 1, 2, \ldots, k+1)$ are all distinct, and these blocks form a partial parallel class missing the point 0. Also adding any $x \in G$ to these $k+1$ blocks gives a partial parallel class missing x. The $(k^2+k+1, k, k-1)$ BIBD obtained by developing blocks C_l $(l = 1, 2, \ldots, k+1)$ over G is thus an NRB, as required. ∎

Example 2.12 *Using $(0, 1, 4, 14, 16)$ as a base block for a cyclic $(21, 5, 1)$ BIBD in the previous lemma, we obtain the following base blocks for a cyclic $(21, 4, 3)$ NRB: $C_1 = (1, 4, 14, 16)$, $C_2 = (20, 3, 13, 15)$, $C_3 = (17, 18, 10, 12)$, $C_4 = (7, 8, 11, 2)$, $C_5 = (5, 6, 9, 19)$.*

Definition 2.13 *A (v, k, λ) DM (difference matrix) over an abelian group G of order v is a $k \times v\lambda$ array D with entries from G such that for each i, j, $1 \leq i < j \leq k$, the multiset $\{D_{i,l} - D_{j,l} : 1 \leq l \leq v\lambda\}$ contains each element of G λ times.*

In this paper we will require only a few difference matrices, and a few TDs obtainable from them. These DMs and TDs are all obtainable by the following theorem:

Lemma 2.14 *a. [5] If $v = p_1 p_2 \ldots p_n$ is a factorisation of v into prime powers with $p_1 \leq p_2 \leq \ldots \leq p_n$, then there exist a TD$(p_1+1, v)$ and a $(v, p_1, 1)$ DM over GF$(p_1) \times GF(p_2) \ldots \times GF(p_n)$.*

b. [14] If G is an abelian group of order v, k is a prime power and there exists a $(v, k, 1)$ BIBD obtainable by developing one or more blocks over G, then a TD$(k+1, v)$ and a $(v, k, 1)$ DM over G exist.

The following theorem indicates how some difference matrices with $\lambda = 1$ can be useful in constructing $(v, k, k-1)$ NRBs.

Theorem 2.15 *[22], [18, Theorem 2.5.5] Suppose there exist a* $(kn+1,\ km+1,\ 1)$ *DM over an abelian group,* G *of order* $kn+1$*, and a* $(kn+1, k, k-1)$ *NRB obtainable by developing one or more blocks over* G*. Suppose also,* $0 \le w \le n$*, and a* $(km, k, k-1)$ *resolvable BIBD, a* $(km+1, k, k-1)$ *NRB plus a* $(kw+1, k, k-1)$ *NRB all exist. Then a* $(km(kn+1)+kw+1, k, k-1)$ *NRB exists.*

3 Base Factorisations

In [13], Baker introduced a new type of design known as a base factorisation or $BF_\lambda(k, v)$, and in [17], Furino et al. used them to construct certain types of frames. More specifically, a $BF_\lambda(k, v)$ is a design consisting of a set $\mathcal{V}$ of v points and a collection $\mathcal{B}$ of blocks, each of size $k/2$ or k, satisfying the following extra properties:

a. $\mathcal{B}$ is partionable into a set $\mathcal{P}$ of parallel classes;

b. Each point in $\mathcal{V}$ appears in exactly λ blocks of size $k/2$ in $\mathcal{B}$;

c. For each pair (A, B) of distinct points in $\mathcal{V}$, if $\lambda_c(A, B)$ denotes the number of blocks of size c containing both A and B, then $\lambda_k(A, B) + 2\lambda_{k/2}(A, B) = \lambda$.

A base factorisation is called uniform if all blocks in each of its parallel classes in $\mathcal{P}$ have the same size. All base factorisations in this paper are uniform, and are obtained by Lemma 3.1 or Lemma 3.4.

Lemma 3.1 *Let* k *be even. If a resolvable* $(k, k/2, k/2-1)$ *BIBD exists, then so does a uniform* $BF_{k-1}(k, k)$*.*

Proof: Combine the blocks of the resolvable BIBD with one block of size k containing all the points. ∎

Corollary 3.2 *A uniform* $BF_{11}(12, 12)$ *and a uniform* $BF_{15}(16, 16)$ *both exist.*

Proof: This follows from the previous lemma, since a resolvable $(12, 6, 5)$ BIBD over $Z_{11} \cup \{\infty\}$ can be obtained by developing the blocks $\{\infty, 0, 1; 2, 4, 7\}$ and $\{3, 8, 10; 5, 6, 9\}$ (mod 11) and a resolvable $(16, 8, 7)$ BIBD over $Z_{15} \cup \{\infty\}$ can be obtained by developing the blocks $\{\infty, 0, 1, 4; 2, 5, 8, 10\}$, and $\{3, 7, 9, 14; 6, 11, 12, 13\}$ (mod 15). ∎

Remark 3.3 The two resolvable $(v, k, k-1)$ BIBDs given in the proof of Corollary 3.2 possess one extra property: their size k blocks can be partitioned into sub-blocks of size $k/2$ in such a way that the sub-blocks form a resolvable $(v, k/2, k/2-1)$ BIBD. (In those two examples, the sub-blocks of size $k/2$ are separated by semicolons.) In the language of papers such as [7, 12, 15], any resolvable $(v, k, k-1)$

BIBD with this property is known as a generalised whist design or $(k/2, k)$ GWhD(v). Whenever a resolvable $(k, k/2, k/2-1)$ BIBD is also a $(k/4, k/2)$ GWhD(k) for some $k \equiv 0 \pmod 4$, it can be used to construct a uniform $BF_{k-1}(k, 2k)$ as illustrated in the next lemma.

Lemma 3.4 *Let $k \equiv 0 \pmod 4$. If there exists a $(k/4, k/2)$ GWhD(k), then a uniform $BF_{k-1}(k, 2k)$ also exists.*

Proof: Let V_1 be the point set for the $(k/4, k/2)$ GWhD(k). The required $BF_{k-1}(k, 2k)$ will have point set $V = Z_2 \times V_1$. For each block C with sub-blocks C_1, C_2 in the $(k/4, k/2)$ GWhD(k) we construct the following block of size k: $(\{0\} \times C) \cup (\{1\} \times C)$. We also construct the following two blocks of size $k/2$: $(\{x\} \times C_1) \cup \{x+1\} \times C_2)$, $x = 0, 1$.

Finally there is a parallel class containing the following two blocks of size k: $(\{x\} \times V_1)$ $x = 0, 1$. This parallel class should be included twice.

Note that if B_1, B_2 are the two blocks in any parallel class of the $(k/4, k/2)$ GWhD, then there are four blocks of size $k/2$ in the $BF_{k-1}(k, 2k)$ that are associated with B_1 or B_2. These blocks will form a parallel class in our $BF_{k-1}(k, 2k)$. So will the two blocks of size k associated with B_1 or B_2. The final two (repeated) blocks of size k form a (repeated) parallel class.

Note that the $(k/4, k/2)$ GWhD(k) has replication number $r = k - 1$. Also, for each block B in the $(k/4, k/2)$ GWhD, and for each point $y \in B$, there is exactly one block of size $k/2$ in the base factorisation associated with B containing $(0, y)$ and one such block containing $(1, y)$. Therefore, since the $(k/4, k/2)$ GWhD(k) had replication number $k - 1$, each point appears in $\lambda = k - 1$ blocks of size $k/2$ in the base factorisation.

Finally we need to confirm that for any two distinct points A, B in the base factorisation, $\lambda_k(A, B) + 2\lambda_{k/2}(A, B) = k - 1$. First, if for some z_1, z_2 $(z_1 \neq z_2)$ we have $A = (x, z_1)$ and $B = (x, z_2)$ $(x \in \{0, 1\})$, then A and B appear together in $(k/2 - 1) + 2 = k/2 + 1$ blocks of size k and $k/4 - 1$ blocks of size $k/2$. If $z_1 = z_2$, two points $A = (0, z_1)$ and $B = (1, z_2)$ appear together in $k - 1$ blocks of size k and in zero blocks of size $k/2$; if $z_1 \neq z_2$, they appear together in $k/2 - 1$ blocks of size k and $(k/2 - 1) - (k/4 - 1) = k/4$ blocks of size $k/2$. Thus in all cases, $\lambda_k(A, B) + 2\lambda_{k/2}(A, B) = k - 1$ as required. ∎

Example 3.5 *There exists a $BF_{11}(12, 24)$.*

Proof: We apply Lemma 3.4 to the $(3, 6)$ GWhD(12) given in the proof of Corollary 3.2. Take the point set as $Z_2 \times (Z_{11} \cup \{\infty\})$. The first two blocks below form a parallel class of blocks of size 12 which should be developed (mod $(-, 11)$). Developing the last two blocks (mod $(2, -)$) produces a parallel class of blocks of size 6; developing these (mod $(-, 11)$) then produces 11 parallel classes. Finally there is a parallel class of size 12 blocks which should be included twice; it consists of the 2 blocks $\{x\} \times (Z_{11} \cup \{\infty\})$, $x = 0, 1$.

$$\{(0,\infty),\ (0,0),\ (0,1),\ (0,2),\ (0,4),\ (0,7),\ (1,\infty),\ (1,0),\ (1,1),\ (1,2),\ (1,4),\ (1,7)\}$$
$$\{(0,3),\ (0,8),\ (0,10),\ (0,5),\ (0,6),\ (0,9),\ (1,3),\ (1,8),\ (1,10),\ (1,5),\ (1,6),\ (1,9)\}$$
$$\{(0,\infty),\ (0,0),\ (0,1),\ (1,2),\ (1,4),\ (1,7)\}$$
$$\{(0,3),\ (0,8),\ (0,10),\ (1,5),\ (1,6),\ (1,9)\}$$

The main significance of base factorisations for the results in this paper comes from the following theorem:

Theorem 3.6 *Suppose k is even and there exists a uniform base factorisation, $BF_{k-1}(k, v)$. Then if q is an odd prime power such that $q > k$, there exists a $(k, k-1)$-frame of type v^q. If further a $(v + 1, k, k - 1)$ NRB exists, a $(vq + 1, k, k - 1)$ NRB also exists.*

Proof: For the first part see [17, Lemma 3.8] or [18, Theorem 2.6.27]. For the second part, apply Lemma 2.3 with $e = 1$. ∎

Remark 3.7 *A $(4, 8)$ GWhD(16) exists (see Corollary 3.2 and Remark 3.3). It can be used to obtain a $BF_{15}(16, 32)$ and $(16, 15)$-frames of type 32^q for q an odd prime power ≥ 17. However, this does not help us obtain any new $(v, 16, 15)$ NRBs, since it is known that a $(33, 16, 15)$ NRB cannot exist [19].*

4 Some Direct Constructions

In this section, we construct a number of $(k, k - 1)$-frames of type u^h for various values of u and h. These constructions are usually of the following type. The points are taken to be elements of an abelian group G of size uh; the groups of size u are then taken as cosets of some size u subgroup U of G. An initial set of base blocks (which form a partial parallel class missing the group U) is then given. Other partial parallel classes are then obtained by developing the blocks in the initial one over G.

Lemma 4.1 *A $(v, 9, 8)$ NRB exists for $v \in \{10, 28, 46, 55\}$.*

Proof: For $v = 10$, the required design is obtained by taking a size 10 set and its ten 9-element subsets as blocks. See [4, Example 3.4] for $v = 28$, and [24] for $v = 46, 55$. ∎

Lemma 4.2 *There exist $(9, 8)$-frames of types 3^{31} and 2^{28}, a $(16, 5)$-frame of type 3^{17}, and a $(16, 15)$-frame of type 3^{49}.*

Proof: For a $(9, 8)$-frame of type 3^{31}, the point set is is Z_{93}, and groups are $\{y, y + 31, y + 62\}$ for $0 \leq y \leq 30$. Multiply the last 3 blocks below by $1, 25, 67$ and the first block by 1 only. The resulting 10 blocks form an initial partial parallel class missing the group $\{0, 31, 62\}$; other partial parallel classes are obtained by developing blocks in this initial one (mod 93).

$$\{3,75,15,9,39,45,11,89,86\} \quad \{6,26,34,41,49,55,56,59,82\}$$
$$\{12,17,19,35,44,51,63,70,71\} \quad \{1,18,24,37,52,72,74,84,85\}$$

For a $(9,8)$-frame of type 2^{28}, the point set is GF(4) $\times Z_{14}$, and groups are $\{(y,z),(y,z+7)\}$ for $y \in GF(4)$ and $z = 0,1,2,\ldots,13$. Let x be a primitive element in GF(4) satisfying $x^2 = x+1$, and multiply the two blocks below by $(1,1)$, $(x,9)$ and $(x+1,11)$. The resulting 6 blocks form a partial parallel class missing the group $\{(0,0),(0,7)\}$. Other partial parallel classes are obtained by developing this initial one over GF(4) $\times$ Z_{14}.

$$\{(0,\ 1),(0,\ 2),(1,\ 0),(1,\ 4),(1,\ 6),(1,\ 3),(x,2),(x,10),(x+1,1)\}$$
$$\{(0,13),(0,12),(1,5),(1,11),(1,13),(1,2),(x,7),(x,\ 9),(x+1,12)\}$$

For a $(16,5)$-frame of type 3^{17} (also given in [7]), the point set is $Z_3 \times Z_{17}$, and groups are $Z_3 \times \{x\}$ for $x \in Z_{17}$. Blocks are obtained by developing the following base block (mod $(3,17)$): $\{(0,1)$, $(0,2)$, $(0,4)$, $(0,8)$, $(0,16)$, $(0,15)$, $(0,13)$, $(0,9)$, $(1,3)$, $(1,12)$, $(1,14)$, $(1,5)$, $(2,6)$, $(2,7)$, $(2,11)$, $(2,10)\}$. The first partial parallel class (which misses the group $Z_3 \times \{0\}$) is obtained by adding $(0,0)$, $(1,0)$ and $(2,0)$ to the given base block and other partial parallel classses are obtained by developing this first one over Z_{17}.

Similarly, for a $(16,15)$-frame of type 3^{49}, the point set is $Z_3 \times GF(49)$, and groups are $Z_3 \times \{x\}$ for $x \in GF(49)$. Let z be a primitive element of GF(49) satisfying $z^2 = 3z+2$. A partial parallel class missing the group $Z_3 \times \{0\}$ is obtained by multiplying any one of the three blocks B_1, B_2 B_3 below by $(1,1)$, $(1,z^4)$ and $(1,z^8)$, and then adding $(0,0)$, $(1,0)$ and $(2,0)$ to the resulting three blocks. Other partial parallel classses are obtained by developing each of these three partial parallel classes over $GF(49)$.

$$\begin{aligned}
B_1 = &\ \{(0,z),\ (0,z^{13}),\ (0,z^{25}),\ (0,z^{37}),\ (0,z^2),\ (0,z^{14}),\ (0,z^{26}),\ (0,z^{38}),\\
&\ (1,z^7),\ (1,z^{19}),\ (1,z^{31}),\ (1,z^{43}),\ (2,z^8),\ (2,z^{20}),\ (2,z^{32}),\ (2,z^{44})\}\\
B_2 = &\ \{(0,z),\ (0,z^{13}),\ (0,z^{25}),\ (0,z^{37}),\ (0,z^{11}),\ (0,z^{23}),\ (0,z^{35}),\ (0,z^{47}),\\
&\ (1,z^2),\ (1,z^{14}),\ (1,z^{26}),\ (1,z^{38}),\ (2,z^4),\ (2,z^{16}),\ (2,z^{28}),\ (2,z^{40})\}\\
B_3 = &\ \{(0,z^2),(0,z^{14}),\ (0,z^{26}),\ (0,z^{38}),\ (0,z^3),\ (0,z^{15}),\ (0,z^{27}),\ (0,z^{39}),\\
&\ (1,z^8),\ (1,z^{20}),\ (1,z^{32}),\ (1,z^{44}),\ (2,z^9),\ (2,z^{21}),\ (2,z^{33}),\ (2,z^{45})\}
\end{aligned}$$

Lemma 4.3 *For $q \in \{10,11,12,13,17,25,49\}$, a $(9,8)$-frame of type 9^q exists.*

Proof: For $q = 10$, apply Lemma 2.9 with $k = 9$. For $q = 12$, the point set is GF(4) $\times Z_{27}$, and groups are translates of $\{0\} \times \{0,3,6,\ldots,24\}$. Let x be a primitive element in GF(4) satisfying $x^2 = x+1$, and multiply the first three blocks below by $(1,1)$, $(x,10)$ and $(x+1,19)$. Together with the last two blocks, this gives us 11 base blocks which form a partial parallel class missing the group $\{0\} \times \{0,3,6,\ldots,24\}$. Other partial parallel classes are obtained by developing these 11 base blocks over GF(4) $\times$ Z_{27}.

$\{(0,1),\ (0,2),\ (1,0),\ (1,1),\ (1,5),\ (x,4),\ (x,8),\ (x+1,15),\ (x+1,25)\}$
$\{(0,7),\ (0,23),\ (1,6),\ (1,4),\ (1,26),\ (x,11),\ (x,12),\ (x+1,5),\ (x+1,24)\}$
$\{(0,13),\ (0,26),\ (1,9),\ (1,19),\ (1,8),\ (x,19),\ (x,21),\ (x+1,4),\ (x+1,20)\}$
$\{(1,2),\ (1,3),\ (1,25),\ (x,20),\ (x,3),\ (x,7),\ (x+1,11),\ (x+1,3),\ (x+1,16)\}$
$\{(1,14),\ (1,16),\ (1,18),\ (x,5),\ (x,25),\ (x,18),\ (x+1,23),\ (x+1,7),\ (x+1,18)\}$

For $q = 11, 13, 17, 25, 49$, the required designs have point set $GF(9) \times GF(q)$, and groups are $GF(9) \times \{y\}$ for $y \in GF(q)$. Let x and z be primitive elements in GF(9) and GF(q) respectively, with x satisfying $x^2 = x + 1$. In each case, two initial base blocks are given. Multiplying these blocks by $(1, z^{2t})$ for $0 \le t \le (q-3)/2$ gives $q-1$ base blocks which form a partial parallel class missing the group $GF(9) \times \{0\}$. Other partial parallel classes are obtained by developing these $q-1$ blocks over $GF(9) \times GF(q)$. For $q = 25$ and 49, z is taken to be a primitive element of GF(q) satisfying $z^2 = z + 3$ and $z^2 = 3z + 2$ respectively.

n	Base blocks
11	$\{(0,1),(1,3),(1,7),(2,8),(2,9),(x^2,2),(x^2,4),(x^6,5),(x^6,10)\}$
	$\{(0,10),(x,4),(x,8),(2x,2),(2x,3),(x^3,7),(x^3,9),(x^7,1),(x^7,6)\}$
13	$\{(0,1),(1,4),(1,8),(2,7),(2,10),(x^2,6),(x^2,12),(x^6,3),(x^6,11)\}$
	$\{(0,7),(x,2),(x,4),(2x,5),(2x,10),(x^3,3),(x^3,6),(x^7,8),(x^7,12)\}$
17	$\{(0,1),(1,3),(1,13),(2,9),(2,10),(x^2,7),(x^2,16),(x^6,2),(x^6,5)\}$
	$\{(0,7),(x,4),(x,6),(2x,2),(2x,12),(x^3,10),(x^3,15),(x^7,1),(x^7,14)\}$
25	$\{(0,1),(1,z^7),(1,z^{20}),(2,z^{22}),(2,z^{23}),(x^2,z^2),(x^2,z^{11}),(x^6,z^{17}),(x^6,z^{18})\}$
	$\{(0,z),(x,z^8),(x,z^{21}),(2x,z^{23}),(2x,1),(x^3,z^3),(x^3,z^{12}),(x^7,z^{18}),(x^7,z^{19})\}$
49	$\{(0,1),(1,z^2),(1,z^3),(2,z^7),(2,z^8),(x^2,z^{10}),(x^2,z^{11}),(x^6,z^{13}),(x^6,z^{14})\}$
	$\{(0,z),(x,z^3),(x,z^4),(2x,z^8),(2x,z^9),(x^3,z^{11}),(x^3,z^{12}),(x^7,z^{14}),(x^7,z^{15})\}$

We point out that for $q = 11, 13, 17, 25, 49$, respectively, the second block is obtained by multiplying the first one by $(x, 10)$, $(x, 7)$, $(x, 7)$, (x, z) or (x, z). This feature simplified the search procedure.

5 $(9, 8)$-frames of type 9^t

In this section, we will show that $(9, 8)$-frames of type 9^t exist for all $t \ge 139$. Our main construction tool will be Lemma 2.4, but first we deal with the known direct constructions when t is an odd prime or prime power.

Lemma 5.1 *There exists a* $(9, 8)$*-frame of type* 9^q *if one of the following three conditions is satisfied:* (1) q *is an odd prime with* $11 \le q \le 433$*,* (2) $q \ge 11$ *and* $q \equiv 3 \pmod 4$ *is a prime power or* (3) $q \in \{25, 49\}$*.*

Proof: For $q = 11, 13, 17, 25, 49$, see Lemma 4.3. The remaining cases were obtained by Furino et al. [18] in their Theorems 2.6.19, 2.6.21 and Table 2.6.23. (We mention that in [4], it was noted that the proof in [18, Theorem 2.6.19] for existence of $(k, k-1)$-frames of type k^q with k, q odd prime powers, $q \equiv 3 \pmod 4$ and $q > k$ only works when $q > 2k$. An improved proof, which works when k, q are odd prime powers, $q \equiv 3 \pmod 4$ and $q > k$ (except when $q = k + 2$ and $k \equiv 5$, 9 or 13 (mod 16)) is given in [4].) ∎

Lemma 5.2 *Suppose a TD$(13, m)$ exists,* $0 \leq u_i \leq m$ *for* $i = 1, 2, 3$, *and a* $(9, 8)$*-frame of type* 9^x *exists for* $x = m, u_1, u_2, u_3$. *Then a* $(9, 8)$*-frame of type* 9^t *exists for* $t = 10m + u_1 + u_2 + u_3$.

Proof: By truncating three groups of a TD$(13, m)$ to sizes u_1, u_2, u_3, we obtain a $\{10, 11, 12, 13\}$-GDD of type $m^{10}u_1^1u_2^1u_3^1$. The required frame can now be obtained by Lemma 2.4, giving weight 9 to all points, since $(9, 8)$-frames of type 9^y exist for $y \in \{10, 11, 12, 13\}$ by Lemma 4.3. ∎

Lemma 5.3 *A* $(9, 8)$*-frame of type* 9^t *exists for* $139 \leq t \leq 1613$.

Proof: When $t \notin \{139, 146, 147, 148, 149, 158, 159, 186, 206, 246\}$, and t does not lie in the range $[170, 179]$, apply Lemma 5.2, using the values of m indicated in Table 1. Values of u_i $(i = 1, 2, 3)$ are not included, but in each case $u_i \in S = \{0,\ 1,\ 10,\ 11,\ 12,\ 13,\ 17,\ 19,\ 23,\ 25,\ 27,\ 28,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\ 49,\ 53,\ 55,\ 59,\ 61,\ 64,\ 67,\ 71,\ 73,\ 79,\ 83\}$ and $u_i \leq m$. For $x \in \{m, u_1, u_2, u_3\}$, a $(9, 8)$-frame of type 9^x exists by Lemma 4.3 when $x \in \{10, 11, 12, 13\}$ and by Lemma 5.1 when x is an odd prime ≥ 11 or $x \in \{25, 49\}$. For $x \in \{28, 46, 55, 64\}$ it can be obtained by Lemma 2.7, inflating an $(x, 9, 8)$ NRB (i.e. a $(9, 8)$-frame of type 1^x) with a resolvable TD$(9, 9)$.

For $t \in \{146, 186, 206, 246\}$, truncate four groups of a TD$(14, m)$ (for $m = 13, 17, 19, 23$) to sizes $13, 1, 1, 1$ in such a way that the 3 points in truncated groups of size 1 do not all lie in one block. This gives a $\{10, 11, 12, 13\}$-GDD of type $m^{10}1^313^1$. Now apply Lemma 2.4 to this GDD, giving weight 9 to all points. For $t \in \{139, 149, 179\}$, the result follows from Lemma 5.1.

Table 1: Constructions for most $(9, 8)$-frames of type 9^t for $t \in [139, 1613]$ using Lemma 5.3.

m	Range for t		m	Range for t
13	$[140, 169] \setminus ([146, 149] \cup [158, 159])$		49	$[507, 606]$
17	$[180, 199] \setminus \{186\}$		59	$[607, 726]$
19	$[200, 239] \setminus \{206\}$		71	$[727, 846]$
23	$[240, 286] \setminus \{246\}$		83	$[847, 1026]$
27	$[287, 326]$		101	$[1027, 1146]$
31	$[327, 386]$		113	$[1147, 1326]$
37	$[387, 446]$		131	$[1327, 1406]$
43	$[447, 506]$		139	$[1407, 1613]$

For $170 \leq t \leq 178$, there exists a projective plane, PG$(2, q)$ for $q = 17$ with an oval (i.e. a set of $q + 1 = 18$ points, no three of which lie in any block). Deleting one point, P, and its blocks gives a TD$(18, 17)$. In 8 groups, delete all points except $t - 170$ points from an oval containing P. Blocks in the resulting GDD will have size 10, 11 or 12, depending on whether they contain 0, 1 or 2 oval points. This GDD is a $\{10, 11, 12\}$-GDD of type $17^{10}1^{t-170}$, to which we can again apply Lemma 2.4, giving all points weight 9.

For $t \in \{147, 148\}$, start with a PG(2, 13) on 183 points, delete all 27 points from 2 blocks B_1, B_2 intersecting in a given point, Q, and delete 9 or 8 other points that lie in some oval containing Q. Similarly, for $t \in \{158, 159\}$, start with a PG(2, 13), and delete all 14 points in one block plus 11 or 10 other points from an oval. For all these values of t, this gives a $(t, \{10, 11, 12, 13\}, 1)$ PBD, to which we apply Lemma 2.5 with $g = 9$. ∎

Lemma 5.4 *A $(9, 8)$-frame of type 9^t exists for all $t \geq 1613$.*

Proof: Our proof uses induction on t together with Lemma 5.2. Any integer $t \geq 1613$ can be written as $10m + (u = u_1 + u_2 + u_3)$ where m is any one of nine consecutive odd integers ≥ 139, $17 \leq (u = u_1 + u_2 + u_3) \leq 223$, and u_1, u_2, u_3 all belong to the set S given in Lemma 5.3. Since $139 \leq m < t$, a $(9, 8)$-frame of type 9^m exists by assumption, and since u_1, u_2, u_3 belong to S, a $(9, 8)$-frame of type 9^x exists for each $x \in \{u_1, u_2, u_3\}$. At most three of the nine consecutive odd values for m are divisible by 3, two by 5, two by 7 and one by 11, thus at least one of these values of m will not be divisible by any of $3, 5, 7, 11$. Hence a TD$(13, m)$ exists for this m, and the result follows from Lemma 5.2. ∎

Combining Lemmas 5.3 and 5.4, we therefore have:

Lemma 5.5 *A $(9, 8)$-frame of type 9^t exists for all $t \geq 139$.*

6 $(v, 9, 8)$ Near Resolvable BIBDs

In this section we construct $(9t + 1, 9, 8)$ NRBs for all but 26 positive integers t.

We start with the four known direct constructions from Lemma 4.1, repeated in the next lemma for convenience.

Lemma 6.1 *There exists a $(9t + 1, 9, 8)$ NRB for $t \in \{1, 3, 5, 6\}$.*

Lemma 6.2 *There exists a $(9t + 1, 9, 8)$ NRB for $t \in$ {2, 4, 7, 8, 12, 14, 18, 22, 30, 32, 34, 42, 44, 48, 54, 58, 60, 82, 84, 90, 92, 98, 102, 104}.*

Proof: For these values, we can apply Lemma 2.1 since $9t + 1$ is a prime power.

Lemma 6.3 *There exists a $(9t+1, 9, 8)$ NRB if t is in one of the following intervals:* $[19, 21]$, $[37, 41]$, $[64, 71]$, $[73, 81]$, $[109, 111]$.

Proof: For these values of t, apply Lemma 2.15 with $km = 9$ and $kn+1 \in \{19, 37, 64, 73, 109\}$. In each case $9n + 1$ is a prime power, hence the required conditions on n in Lemma 2.15 are satisfied by Lemmas 2.1 and 2.14.

Lemma 6.4 *There exists a* $(9t+1, 9, 8)$ *NRB for* $t \in \{11, 13, 17, 23, 25, 27, 29, 31, 39, 43, 47, 49, 53, 59, 61, 83, 89, 97, 101, 103, 107\}$.

Proof: For these values of t, t is an odd prime power, and a $(9, 8)$-frame of type 9^t exists by Lemma 5.1. Now apply Lemma 2.3 with $e = 1$ to this frame, forming a $(10, 9, 8)$ NRB on each group plus one extra point.

Lemma 6.5 *There exists a* $(9t+1, 9, 8)$ *NRB for* $t \in \{10, 28, 46, 55, 56, 57, 91, 93, 100\}$.

Proof: For these values of t, $9t$ is of the form ghu where a $(9, 8)$-frame of type h^u and a resolvable TD$(9, g)$ both exist. See Table 2. Apply Lemma 2.7 to obtain a $(9, 8)$-frame of type $(gh)^u$, then fill in the groups of this frame with one extra point, using Lemma 2.3 and a $(gh + 1, 9, 8)$ NRB.

The required $(9, 8)$-frame of type h^u exists by Lemma 4.2 for $t \in \{56, 93\}$, by Lemma 4.1 for $t \in \{10, 28, 46, 55\}$ and by Lemma 2.1 for $t = 57$. For $t \in \{91, 100\}$ it is obtainable by applying Lemma 2.3 with $e = 1$ to $(9, 8)$-frames of types 9^{10} and 9^{11}, which exist by Lemma 4.3. ∎

Table 2: Values of h, u and g for each t in Lemma 6.5.

t	(h, u)	g	t	(h, u)	g	t	(h, u)	g
10	(1,10)	9	55	(1,55)	9	91	(1,91)	9
28	(1,28)	9	56	(2,28)	9	93	(3,31)	9
46	(1,46)	9	57	(1,19)	27	100	(1,100)	9

So far we have shown that $(9t + 1, 9, 8)$ NRBs exist for all but 26 values of $t \leq 111$. These 26 unknown values are given later in Table 3. We now deal with the case $t > 111$.

Lemma 6.6 *A* $(9t+1, 9, 8)$ *NRB exists for all* $t > 111$.

Proof: If $111 < t \leq 138$, form a $(K, 1)$-GDD with $K = \{10, 11, 12, 13\}$ on t points with group sizes from $T = \{0, 1, 2, \ldots, 8\} \cup \{10, 11, 12, 13\}$. This GDD can be obtained by truncating two groups of a TD$(12, 11)$ when $111 < t \leq 129$, or one group of a TD$(11, 13)$ when $130 \leq t \leq 138$.

Since $(9, 8)$-frames of type 9^x exist for all $x \in K$, we can apply Lemma 2.4 to this GDD, giving weight 9 to all points. This gives a $(9, 8)$-frame with group sizes in $U = \{9, 18, 27, \ldots, 72\} \cup \{90, 99, 108, 117\}$. Finally apply Lemma 2.3 with $e = 1$ to this frame.

For $t \geq 139$, a $(9, 8)$-frame of type 9^t exists by Lemma 5.5, and the result follows by applying Lemma 2.3 with $e = 1$ to this frame. ∎

Combining all results of this section, we now have the following existence result for $(v, 9, 8)$ NRBs:

Lemma 6.7 *A* $(9t+1, 9, 8)$ *NRB exists for all positive integers* t*, except possibly the* 26 *values in Table 3.*

Table 3: Values of t for which no $(9t + 1, 9, 8)$ NRB is known.

9	15	16	24	26	33	35	36	45
50	51	52	62	63	72	85	86	87
88	94	95	96	99	105	106	108	

7 $(v, 12, 11)$ Near Resolvable BIBDs

In [15], Costa et al. found $(12t + 1, 12, 11)$ NRBs for all but 83 values of t. These values of t are listed below in Table 4; we take this table as our starting point.

Table 4: Values of t for which no $(12t + 1, 12, 11)$ NRB was known in [15]:

7	11	12	17	18	21	22	25	32	41
42	43	46	47	49	54	57	58	60	67
68	72	81	82	87	88	90	92	95	106
107	116	120	128	130	132	136	137	142	143
144	153	164	192	204	205	215	228	240	258
267	270	272	273	274	284	325	330	331	334
384	499	513	522	524	526	527	529	532	534
536	537	538	552	584	599	654	655	659	672
731	991	1175							

Lemma 7.1 *There exist* $(12t + 1, 12, 11)$ *NRBs for* $t \in \{17, 25, 41, 43, 47, 49, 67, 81, 107, 137, 599\}$.

Proof: For these values, t is an odd prime power ≥ 13. Therefore, since a uniform $BF_{11}(12, 12)$ exists by Corollary 3.2, a $(12, 11)$-frame of type 12^t and a $(12t+1, 12, 11)$ NRB exist by Theorem 3.6. ∎

Lemma 7.2 *There exist* $(12t + 1, 12, 11)$ *NRBs for* $t \in \{46, 54, 58, 82, 106, 142, 274, 526\}$.

Proof: For these values, $t/2$ is an odd prime power ≥ 13. By Example 3.5, there exists a uniform $BF_{11}(12, 24)$; therefore, by Theorem 3.6, there exist a $(12, 11)$-frame of type $24^{t/2}$ and a $(12t + 1, 12, 11)$ NRB. ∎

Lemma 7.3 *There exist* $(12t + 1, 12, 11)$ *NRBs for* $t \in \{272, 325, 330, 331, 334, 536\}$.

Proof: For $t = 272$, start with a TD$(17, 16)$, and for $t = 536$, truncate one group of a TD$(14, 41)$ to size 3, giving a $\{13, 14\}$-GDD of type $41^{13}3^1$. Inflating these GDDs using Lemma 2.4 and $(12, 11)$-frames of types 12^{17}, 12^{13} and 12^{14} (which exist by Theorem 3.6 or Lemma 2.10) gives $(12, 11)$-frames of types 192^{17} and $492^{13}36^1$. Fill in the groups of these frames with one extra point, using Lemma 2.3. Similarly, when $t \in \{325, 330, 331, 334\}$ a $\{13, 14\}$-GDD of type $25^{13}u^1$ (with $u = t - 325$) can be obtained by truncating one group of a TD$(14, 25)$ to size u. Again apply Lemma 2.4, giving all points in this GDD weight 12, and fill in all groups of these frames with one extra point, using Lemma 2.3. ∎

Lemma 7.4 *There exist* $(12t + 1, 12, 11)$ *NRBs for* $t \in \{273, 513, 527, 1175\}$.

Proof: For $t = 273$, we have a $(273, 17, 1)$ BIBD (a projective plane of order 16). For $t = 513, 527, 1175$, we have a $(513, \{19, 27\}, 1)$ PBD, a $(527, \{17, 31\}, 1)$ PBD, and a $(1175, \{25, 47\}, 1)$ PBD by forming a block on each group of TD$(19, 27)$, TD$(17, 31)$, and TD$(25, 47)$ respectively. Since $(12, 11)$-frames of types 12^x exist for each $x \in \{17, 19, 25, 27, 31, 47\}$ (by Theorem 3.6), we can apply Lemma 2.5 with $g = 12$ to these frames to obtain $(12, 11)$-frames of type 12^t, ($t \in \{273, 513, 527, 1175\}$). The groups of these frames can be filled by Lemma 2.3, using one extra point. ∎

Lemma 7.5 *There exist* $(12t + 1, 12, 11)$ *NRBs for* $t \in \{130, 258, 384, 499, 532, 534, 537, 538, 552, 584, 654, 655, 659, 672, 731, 991\}$.

Proof: For these values, apply Lemma 2.15, writing $12t+1 = 12m(12n+1)+(12w+1)$, using the values of $12m$, $12n + 1$ and w in Table 5. In each case, the required conditions on n in Lemma 2.15 are s atisfied by Lemmas 2.1 and 2.14. ∎

Using the results of this section, we can now give the following updated result for existence of $(12t + 1, 12, 11)$ NRBs:

Lemma 7.6 *A* $(12t+1, 12, 11)$ *NRB exists for any positive integer* t, *except possibly for the* 37 *values in Table 6.*

Table 5: Values of $12m$, $12n+1$ and w for each t in Lemma 7.5.

$12m$	$12n+1$	w	t
12	121	9	130
12	241	17	258
12	361	23	384
24	241	17	499
12	529	$0, 3, 5, 8, 9, 23$	$529, 532, 534, 537, 538, 552$
12	541	43	584
12	625	$29, 30, 34, 47$	$654, 655, 659, 672$
24	361	9	731
12	961	30	991

Table 6: Values of t for which a $(12t+1, 12, 11)$ NRB is unknown.

7	11	12	18	21	22	32	42	57	60
68	72	87	88	90	92	95	116	120	128
132	136	143	144	153	164	192	204	205	215
228	240	267	270	284	522	524			

8 $(v, 16, 15)$ Near Resolvable BIBDs

In [16, Table III] and [18, Section 4.1.7], Furino et. al. noted that their generic search algorithm could produce $(16t+1, 16, 15)$ NRBs for all but 158 values of t, the largest of which was 627. However no explicit list of unknown values was provided. In this section, we start by looking at $(16t+1, 16, 15)$ NRBs with $t \leq 628$, and provide information showing that these exist except possibly for 149 values of t. We then provide a recursive existence result for $t \geq 629$. There is one known non-existence result, which was mentioned earlier in Remark 3.7.

Lemma 8.1 *There does not exist a* $(16t+1, 16, 15)$ *NRB for* $t = 2$.

Lemma 8.2 *There exists a* $(16t+1, 16, 15)$ *NRB for each* $t \in \{$1, 3, 5, 6, 7, 12, 15, 16, 21, 22, 28, 33, 36, 39, 40, 42, 48, 55, 58, 60, 63, 72, 75, 76, 78, 85, 88, 93, 105, 106, 111, 117, 126, 130, 132, 133, 135, 138, 142, 150, 162, 166, 168, 172, 175, 177, 190, 195, 201, 207, 208, 210, 216, 226, 231, 235, 237, 250, 261, 265, 267, 268, 282, 291, 300, 312, 315, 322, 366, 382, 390, 588, 603$\}$.

Proof: For these values, we can apply Lemma 2.1, since $16t+1$ is a prime power.

Lemma 8.3 *There exists a* $(16t+1, 16, 15)$ *NRB for each* $t \in \{$19, 23, 25, 27, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 101, 107, 109, 121, 125, 127, 131, 137, 139, 149, 151, 157, 163, 167, 169, 173, 179, 181, 191, 197, 211, 223, 227, 229, 233, 239, 243, 251, 271, 277, 281, 283, 293, 311, 313, 317, 367, 373, 379, 383, 389, 601, 607, 613, 617, 619}.

Proof: For these values of t, t is an odd prime power > 16. Therefore, since a $BF_{15}(16, 16)$ exists by Corollary 3.2, a $(16, 15)$-frame of type 16^t and a $(16t+1, 16, 15)$ NRB exist by Theorem 3.6. ∎

Lemma 8.4 *There exists a* $(16t + 1, 16, 15)$ *NRB for each* t *in Table 7.*

Proof: These designs are obtainable by Lemma 2.15 with the values of $16m$, $16n + 1$ and w given in Table 7. When $16n+1 \neq 273$, the additive group G used is GF$(16n+1)$, and when when $16n + 1 = 273$, we take $G = Z_{273}$. The required conditions on n in Lemma 2.15 are all satisfied by Lemma 2.1 and Lemma 2.14 when $16n + 1 \neq 273$, or by Lemma 2.11 and Lemma 2.14 when $16n + 1 = 273$. ∎

Table 7: Values of $16m$, $16n + 1$ and w for each t in Lemma 8.4.

$16m$	$16n + 1$	w	t
16	17	$0, 1$	$17, 18$
16	49	$0, 1, 3$	$49, 50, 52$
16	81	$0, 1, 3, 5$	$81, 82, 84, 86$
16	97	$0, 1, 3, 5, 6$	$97, 98, 100, 102, 103$
16	113	$0, 1, 3, 5, 6, 7$	$113, 114, 116, 118, 119, 120$
16	193	$0, 1, 3, 5, 6, 7, 12$	$193, 194, 196, 198, 199, 200, 205$
16	241	$0, 1, 3, 5, 6, 7, 12, 15$	$241, 242, 244, 246, 247, 248, 253, 256$
16	257	$0, 1, 3, 5, 6, 7, 12, 15$	$257, 258, 260, 262, 263, 264, 269, 272$
16	273	$0, 1, 3, 5, 6, 7, 12, 15$	$273, 274, 276, 278, 279, 280, 285, 288$
16	289	$0, 1, 3, 5, 6, 7, 12, 15,$ $16, 17, 18$	$289, 290, 292, 294, 295, 296, 301, 304,$ $305, 306, 307$
16	353	$12, 15, 16, 17, 18, 19, 21, 22$	$365, 368, 369, 370, 371, 372, 374, 375$
16	577	$12, 15, 18, 25, 27, 29$	$589, 592, 595, 602, 604, 606$
16	593	$0, 1, 3, 5, 6, 7, 12, 15,$ $16, 17, 18, 19, 21, 22, 23, 25,$ $27, 28, 29, 31$	$593, 594, 596, 598, 599, 600, 605, 608,$ $609, 610, 611, 612, 614, 615, 616, 618,$ $620, 621, 622, 624$
16	625	$0, 1, 3$	$625, 626, 628$

Lemma 8.5 *Suppose* $0 \leq u_i \leq m$ *for* $i = 1, 2$, *a* $TD(19, m)$ *exists, and there exist* $(16z + 1, 16, 15)$ *NRBs for* $z = m, u_1, u_2$. *Then a* $(16t + 1, 16, 15)$ *NRB exists for* $t = 17m + (u = u_1 + u_2)$.

Proof: By truncating two groups of a TD(19, m) to sizes u_1, u_2, we obtain a {17,18,19}-GDD of type $m^{17}u_1^1u_2^1$. There exist $(16, 15)$-frames of type 16^x for $x = 17, 18, 19$; these can be obtained by Theorem 3.6 when $x \in \{17, 19\}$ and by Lemma 2.10 when $x = 18$. Hence we can apply Lemma 2.4, giving all points weight 16 to obtain a $(16, 15)$-frame of type $(16m)^{17}(16u_1)^1(16u_2)^1$. Now apply Lemma 2.3 with $e = 1$. ∎

Lemma 8.6 *There exists a* $(16t + 1, 16, 15)$ *NRB for all* t *in the intervals* $[323, 361]$ *and* $[391, 587]$*.*

Proof: This follows from Lemma 8.5 with $m \in \{19, 23, 25, 27, 29, 31\}$. Table 8 gives suitable values of m for each t. ∎

Table 8: Values of m for each t in Lemma 8.6.

m	Range for t	m	Range for t	m	Range for t
19	$[323, 361]$	23	$[391, 437]$	25	$[438, 473]$
27	$[474, 507]$	29	$[508, 541]$	31	$[542, 587]$

Lemma 8.7 *A* $(16t + 1, 16, 15)$ *NRB exists for each* $t \in \{51, 147, 204, 255, 297, 591\}$*.*

Proof: For $t = 297$ and 591, observe that $16{\cdot}297+1 = 49{\cdot}97$, and $16{\cdot}591+1 = 49{\cdot}193$. Therefore for these values of t, we can apply Lemma 2.6, since it is possible to obtain a $(16t + 1, \{49, 97, 193\}, 1)$ PBD by forming a block on each group of a TD(49, 97) or TD(49, 193).

For $t = 255$, start with a $(16, 15)$-frame of type 15^{17} which exists by Lemma 2.8, and for $t = 51, 147, 204$, start with $(16, 15)$-frames of types 3^{17}, 3^{49}, 3^{17} respectively; these exist by Lemma 4.2. Applying Lemma 2.7 with $g = 16$, 16, 16 and 64 gives $(16, 15)$-frames of types 240^{17}, 48^{17}, 48^{49} and 192^{17}. Now apply Lemma 2.3 with $e = 1$ to these frames. ∎

So far in this section we have proved the following result:

Lemma 8.8 *A* $(16t + 1, 16, 15)$ *NRB does not exist for* $t = 2$*, but exists if* t *is a positive integer* ≤ 628 *other than the* 149 *values in Table 9.*

In the remainder of this section, we shall show the restriction $t \leq 628$ in the previous lemma is not necessary. The main tool for dealing with the larger values of t is Lemma 8.5.

Lemma 8.9 *A* $(16t + 1, 16, 15)$ *NRB exists for all* $t \geq 629$*.*

Table 9: Values of t for which a $(16t+1, 16, 15)$ NRB is unknown.

2	4	8	9	10	11	13	14	20	24	26	30	32	34	35
38	44	45	46	54	56	57	62	64	65	66	68	69	70	74
77	80	87	90	91	92	94	95	96	99	104	108	110	112	115
122	123	124	128	129	134	136	140	141	143	144	145	146	148	152
153	154	155	156	158	159	160	161	164	165	170	171	174	176	178
180	182	183	184	185	186	187	188	189	192	202	203	206	209	212
213	214	215	217	218	219	220	221	222	224	225	228	230	232	234
236	238	240	245	249	252	254	259	266	270	275	284	286	287	298
299	302	303	308	309	310	314	316	318	319	320	321	362	363	364
376	377	378	380	381	384	385	386	387	388	590	597	623	627	

When $17 \cdot 37 = 629 \le t \le 17 \cdot 631 + 612 = 11329$, these NRBs are obtainable by Lemma 8.5 with m an odd prime power in the range $[37, 631]$, or more specifically, $m \in \{37, 41, 43, 47, 49, 53, 59, 61, 67, 73, 79, 83, 89, 97, 107, 113, 121, 131, 139, 149, 157, 163, 173, 181, 191, 199, 211, 227, 241, 257, 277, 293, 313, 331, 353, 373, 397, 421, 449, 467, 491, 509, 523, 547, 571, 593, 619, 631\}$. More specifically, the following ranges for t are covered for each m: $[17m, 17m+68]$ when $37 \le m \le 49$, $[17m, 17m+102]$ when $53 \le m \le 83$, $[17m, 17m+170]$ when $89 \le m \le 191$, $[17m, 17m+374]$ when $199 \le m \le 353$, and $[17m, 17m+612]$ when $373 \le m \le 631$.

If $t \ge 17 \cdot 631 + 612 = 11329$, we can use induction on t together with Lemma 8.5. Here, t can be written as $17(n+30)+x = 17(n+2y)+(510-34y+x)$ where $n > 629$ is an odd integer, $0 \le x \le 33$ and $0 \le y \le 14$. The integer $u = 510 - 34y + x$ lies in the range $[0, 543]$ and can always be written as $u_1 + u_2$ where u_1, u_2 are integers in the range $[0, 373]$, and u_1, u_2 aren't in Table 9 (i.e. $(16u_i + 1, 16, 15)$ NRBs exist for $i = 1, 2$). In addition, of the 15 consecutive odd integers $n+2y$ $(y = 0, 1, \dots, 14)$ there are at most 5 divisible by 3, 2 more by 5, 2 more by 7, 2 more by 11, 2 more by 13 and 1 more by 17. Therefore at least one of these 15 integers $n+2y$ is not divisible by any prime ≤ 17, and if $n+2y^*$ is one such value, a $\mathrm{TD}(19, n+2y^*)$ exists by Lemma 2.14. A $(16(n+2y^*)+1, 16, 15)$ NRB exists by assumption, as $629 \le n+2y^* < t$. In all cases, a $(16g+1, 16, 15)$ NRB exists for $g \in \{n+2y^*, u_1, u_2\}$, hence a $(16t+1, 16, 15)$ NRB can be obtained by Lemma 8.5 with $m = n+2y^*$ and $u = 510 - 34y^* + x$. ∎

Summarising the results of this section, we now have the following existence result for $(16t+1, 16, 15)$ NRBs:

Lemma 8.10 *A $(16t+1, 16, 15)$ NRB does not exist for $t = 2$, but exists if t is a positive integer other than the 149 values given earlier in Table 9.*

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