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Superthrackles

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Dedication

The work in this paper was done as part of the second author's thesis [14] under the supervision of the kind and ever cheerful Dan Archdeacon. This paper is dedicated to Dan, who was always optimistic, enthusiastic, brilliant, generous and supportive.

Abstract

A drawing of a (simple) graph in the plane is a superthrackle if every pair of edges cross exactly once. This is a variant of a thrackle, in which pairs of non-adjacent edges cross exactly once but adjacent edges do not cross. We characterize the graphs that are superthrackleable. They are either *a)* bipartite and planar, or *b)* non-bipartite and projective-planar with every face an even walk. As a corollary, a graph is superthrackleable if and only if it can be drawn with all crossings at a single point, or drawn as a generalized thrackle.

1 Introduction

A *drawing* of a graph G on a surface S is a representation where vertices are points and edges are Jordan arcs connecting their endpoints while passing through no other vertex. We assume two distinct edges intersect in a finite number of points and that at each intersection the edges cross transversally. A drawing is *cellular* if each region in the complement $S - G$ is homeomorphic to an open disk. An *embedding* is a drawing with no crossings.

The usual goal when drawing a graph is to minimize the number of crossings. To this end we place three restrictions: *a)* no edge crosses itself, *b)* no two adjacent edges cross, and *c)* no two edges cross more than once. A drawing satisfying these three conditions is called *good*. The motivation is that any drawing of G that is not good can be modified to a good drawing with fewer crossings.

Consider the other extreme: good drawings that have the maximum possible number of crossings. A *thrackle* is a planar drawing of a graph in which every pair of non-adjacent edges cross. A graph is *thrackleable* if it can be drawn as a thrackle. In the late 1960's J.H. Conway conjectured that the number of edges in a thrackleable graph was at most the number of vertices. His conjecture remains unsolved. D.R. Woodall [15] showed that to prove Conway's conjecture it suffices to show that two even cycles meeting at a single vertex cannot be thrackled. He also gave a complete characterization of graphs that could be thrackled, assuming Conway's conjecture were true. For practice, the reader is invited to show that a 4-cycle cannot be thrackled, to find a thrackle drawing of a 5-cycle and to find a thrackle drawing of a 6-cycle; this latter task is more difficult.

Many variants of thrackles have been studied: outerplanar thrackles [6], generalized thrackles [4, 5, 9] (which we will return to later in this paper), tangled thrackles [10, 13] and spherical thrackles [3], for example. We study a different variation of thrackles, relaxing the condition that two adjacent edges do not cross. A *superthrackle* is a drawing in which every pair of edges cross, adjacent or non-adjacent. Again for practice, the reader is invited to find a superthrackle drawing of a 4-cycle. Lemma 5.1 shows that any thrackleable graph can be superthrackled. A superthrackle drawing of K_4 is shown at the end of Figure 2, where every pair of edges cross transversally at the common central point.

Our goal is to characterize superthrackleable graphs.

The *projective plane* P^2 is the non-orientable compact surface with a single crosscap. Equivalently, it is the sphere with antipodal points identified. It is commonly represented (and will be here) as a disk where it is understood that each boundary point x is identified with its antipodal point $-x$. For an example see Part (a) of Figure 2 where K_4 is embedded in P^2 using this representation.

We now state our Main Theorem.

Theorem 1.1 *A graph is superthrackleable if and only if it is either:*

1. *bipartite and planar, or*
2. *non-bipartite and projective planar with every face of even length.*

Any contractible simple cycle on the projective plane has a neighborhood isomorphic to a cylinder; it is 2-*sided*. Any noncontractible simple (also called *essential*) cycle has a neighborhood isomorphic to a Möbius strip; it is 1-*sided*. A graph embedded in the projective plane is a *parity embedding* when a simple cycle is one-sided if and only if it is of odd length.

An *even edge-subdivision* of a graph is one in which every edge is subdivided an even number of times. A *vertex subdivision* at v subdivides every edge incident with v precisely once. Two graphs G and G' are *parity homeomorphic* if there is a graph H which can be obtained from G and from G' by the operations of even edge-subdivision and vertex subdivision. Notice that a cycle's length modulo 2 is fixed by both vertex subdivision and even edge-subdivision, so if graphs G and G' are parity homeomorphic then corresponding cycles in the two graphs have the same length modulo 2.

A graph has a parity embedding in the projective plane if and only if its parity homeomorphs have a parity embedding. The two classes of graphs described in Theorem 1.1 are exactly those with projective-planar parity embeddings. Theorem 2.9 characterizes the graphs with a parity embedding on the projective plane. Together these give the following variant of Theorem 1.1.

Theorem 1.2 *A graph is superthrackleable if and only if:*

1. *it has a parity embedding in the projective plane, or equivalently*
2. *no subgraph is a parity homeomorph of one of the graphs of Figure 3 (considered as unsigned graphs with solid edges subdivided once).*

The remainder of the paper is devoted to the proof of Theorem 1.1 and studying its consequences. In Section 2 we give a basic lemma about superthrackle drawings, examine some small cases, and discuss signed graphs, signed embeddings, and their relation to parity embeddings. In Section 3 we show that the graphs in Theorem 1.1 can be superthrackled. Section 4 shows that these are the only superthrackleable graphs. Section 5 gives some consequences of our result, in particular relating superthrackles to other variations of thrackles. Concluding remarks are given in Section 6.

2 Basic concepts

In this section we give some elementary results and concepts. The following observation is central in our arguments.

Lemma 2.1 *If a graph G can be superthrackled, then any subgraph H of G can be superthrackled.*

Proof: Restrict the superthrackle drawing of G to a drawing of H; the result is also a superthrackle drawing. ■

We group the remaining background material in two subsections. Section 2.1 concerns graphs with only a few cycles, while Section 2.2 introduces signed graphs and signed embeddings.

2.1 Superthrackling graphs with few cycles

We turn our attention to when a graph with only a few cycles can be superthrackled.

Lemma 2.2 *Two disjoint odd cycles cannot be superthrackled.*

Proof: Let the disjoint odd cycles be C_1 and C_2 and suppose they can be superthrackled. Any superthrackle drawing of C_1 forms a planar graph where, considering the crossings as vertices, every vertex is of degree 2 or 4. We apply the well-known result that the faces of such a drawing can be properly 2-colored *black* and *white*. Let v be a vertex of C_2 lying in a black face of the superthrackle drawing of C_1. In the superthrackle drawing of C_2 every edge of C_1 crosses every edge of C_2. Each such crossing changes the color of the face in the drawing of C_1. Since both cycles are of odd length, the face color changes an odd number of times, contradicting the assumption that the cycle is closed. ■

We next consider a *wedge of cycles*, two cycles C_1 and C_2 that intersect in a single vertex v. In a drawing of a wedge of cycles we say that the cycles are *interlaced* if in a small neighborhood around v the two edges incident with C_1 are separated by the two edges incident with C_2.

Lemma 2.3 *Suppose that a wedge of two cycles is superthrackled. Then the cycles are interlaced if and only if they are both of odd length.*

Proof: Consider a superthrackle drawing of C_1. We again 2-color the faces of this drawing. The cycle C_2 is interlaced with C_1 at v if and only if its edge-ends incident with v occur in faces of two different colors. As we walk along C_2 each crossing of an edge of C_1 changes the color of the face, so the cycles are interlaced if and only if the number of such crossings is odd. This happens precisely when both cycles are odd. ■

A *θ-graph* is a graph with two vertices u, v and three internally vertex-disjoint paths A, B, C joining these vertices, or equivalently, a graph homeomorphic to $K_{2,3}$. Consider a drawing of a θ-graph. Each vertex has one of two possible orientations: either the edges in A, B, C appear in clockwise or anticlockwise order. The drawing is a *preserver* if the order on the incident edges in A, B, C is the same at both u, v, and is a *converter* otherwise. For example, the drawing of a θ-graph without crossings is a converter.

The following lemma is a variation of Lemma 2.3 from [9].

Lemma 2.4 *A superthrackle drawing of a θ-graph H is a converter if and only if at most one of the paths is of odd length.*

Proof: Let H be the θ-graph labelled as described where the paths A, B, C have lengths a, b, c respectively. We look at properties of a superthrackle drawing of H.

First consider the induced superthrackle drawing D of the cycle $A \cup B$. As in Lemma 2.2 we consider crossings as vertices and properly 2-color the faces of D black and white. Walking along A from u to v, each crossing changes the black face between being on the right or left. The total number of such crossings is $ab + 2\binom{a}{2}$ (each of the a edges of A crosses the b edges of B, and each crossing of two distinct edges in A is counted once for each involved edge). Hence walking from u to v the faces to the right of the first and last edges of A are the same color if and only if $ab + 2\binom{a}{2}$ is even.

We add the path C. In our supposed superthrackle drawing C crosses edges of $A \cup B$ exactly $c(a + b)$ times. Each such crossing changes the color of the resident face in D, so the ends of C are in faces of $A \cup B$ with the same color if and only if $c(a + b)$ is even.

The key observation is that when $ab + 2\binom{a}{2}$ is even, the drawing is a converter if and only if the ends of C are in faces of the same color, and when $ab + 2\binom{a}{2}$ is odd, the drawing is a converter if and only if the ends of C are in faces of different colors. Hence the drawing is a converter if and only if $ab + 2\binom{a}{2} \equiv c(a + b) \pmod 2$. Equivalently, the drawing is a converter if and only if $ab + ac + bc \equiv 0 \pmod 2$. A simple calculation shows that the equation holds if and only if at most one of a, b, c is odd, giving our desired result ■

We use Lemma 2.4 to determine the clockwise/anticlockwise orientation of an ordered triple of edge-ends incident with a vertex. To this end we need an easy way to describe a θ-graph. Our notation is most easily shown by example. Consider $K_{3,3}$ where the vertex parts are a, b, c versus x, y, z. Consider a θ-graph with degree three vertices a, x and paths $A = [a, x]$, $B = [a, y, b, x]$, and $C = [a, z, c, x]$. Denote this θ-graph by $\langle ax, aybx, azcx \rangle$. Our example is a preserver in a superthrackle drawing since every path is of odd length.

Lemma 2.5 *Let H be a θ-subgraph with degree 3 vertices u, v. Then any subdivision H' of H obtained by even subdivisions and vertex subdivisions at $w \neq u, v$ is a preserver if and only if H is. A vertex subdivision at one of u, v changes preservers to converters, while vertex subdivisions at both leaves its status unchanged.*

Proof: A vertex subdivision changes the parity of the paths A, B, C precisely when it is at vertex u, v. An even subdivision does not change the parity of any path. ■

2.2 Signed graphs and their embeddings

A *signed graph* $G^{\pm}$ is a graph G together with a sign (plus or minus) associated with each edge. The *sign of a cycle* is the product of the signs on its edges, that is, a cycle is signed plus if and only if it has an even number of edges signed minus. Recall that a *vertex switch* at a vertex v changes the signs on every edge incident with v.

A vertex switch does not change the sign of any cycle. Two signatures are *switching equivalent* if there is a sequence of vertex switches changing one into the other.

A *signed subdivision* of a signed graph is formed by replacing each edge uv with a path joining u and v with new interior vertices such that the product of the signs on these edges is the same as the sign on uv. Two signed graphs are *switching homeomorphs* if they have signed subdivisions that are switching equivalent.

A *signed embedding* of $G^{\pm}$ is an embedding of G such that any cycle C of G is 1-sided if and only if it is negative. A signed graph is *projective-planar* if it has a signed embedding on the projective plane.

Lemma 2.6 *If $G^{\pm}$ has a signed embedding on a surface S, then any switching homeomorph $H^{\pm}$ has a signed embedding on S.*

We need the following theorem of Zaslavsky [17].

Theorem 2.7 *A signed graph is projective-planar if and only if it contains no subgraph that is switching homeomorphic to one of the graphs of Figure 3 (considered as signed graphs with dotted edges negative).*

Recall that a parity embedding is one in which a cycle is 1-sided if and only if it is of odd length. Form the signed graph G^{-} from a given unsigned G by making every edge negative. A cycle in G^{-} is negative if and only if it has odd length. This immediately yields:

Lemma 2.8 *A signed embedding of G^{-} is a parity embedding of G.*

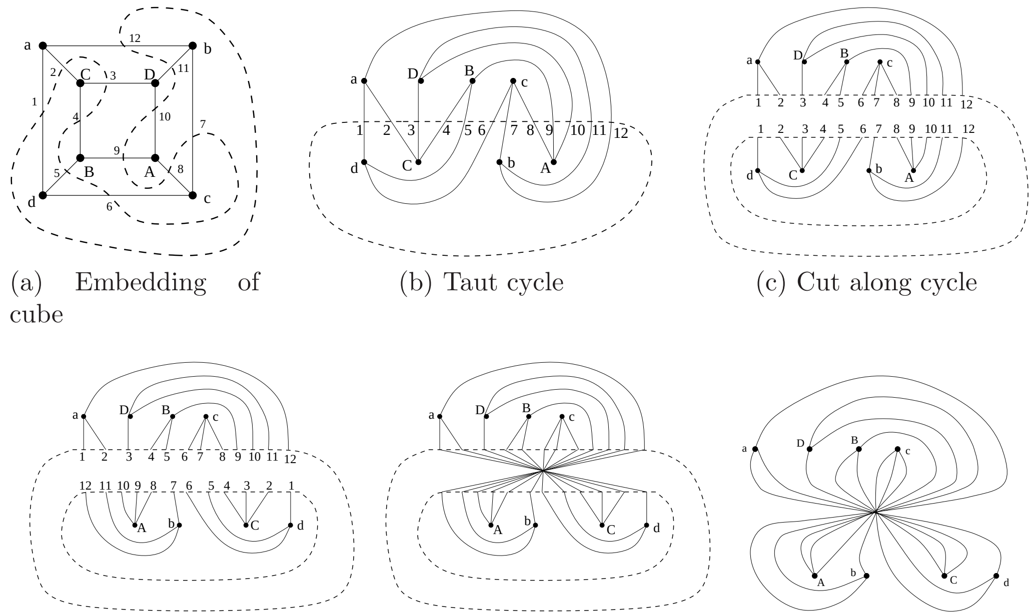
We restate Theorem 2.7 for parity embeddings.

Theorem 2.9 *A graph has a parity embedding in the projective plane if and only if it does not contain a subgraph that is parity homeomorphic to one of the graphs of Figure 3 (considered as unsigned graphs with solid edges subdivided once).*

3 When you can superthrackle

In this section we prove one direction of Theorem 1.1: namely, we show that the graphs described therein can be superthrackled. We deal with the two classes individually.

Proposition 3.1 *Let G be a planar bipartite graph. Then G is superthrackeable. Moreover, it can be drawn so that all crossings occur at a common point.*



(a) Embedding of cube (b) Taut cycle (c) Cut along cycle

(d) Flip inside (e) Add crossings (f) Superthrackled

Figure 1: Embedding of bipartite graph, the cube, on the sphere with contractible cycle giving a superthrackle on the sphere.

Proof: The proof uses a "cut-and-paste" construction illustrated in Figure 1. In this figure the underlying graph is the cube.

Consider a planar embedding of the bipartite G. Our first goal is a simple closed curve C in the plane that crosses each edge of G exactly once. We form an embedded bipartite graph G' containing G by adding edges to G until every face has size at most 4. A simple closed curve C' crossing every edge of G' exactly once does the same for G, so renaming if necessary we assume every face of G is of size 4.

We consider the embedded dual G^* of G. Every vertex of G^* is of degree 4, so G^* is Eulerian. Belyi [2] showed that every embedded Eulerian graph (in any surface) has a *non-self-intersecting* Eulerian circuit, that is, an Eulerian circuit such that every pair of consecutive edges are on the same face. We replace each degree 4 vertex of G^* by two degree 2 vertices paired with the edges as they appear in the Eulerian circuit. This splits the Eulerian circuit into our desired C.

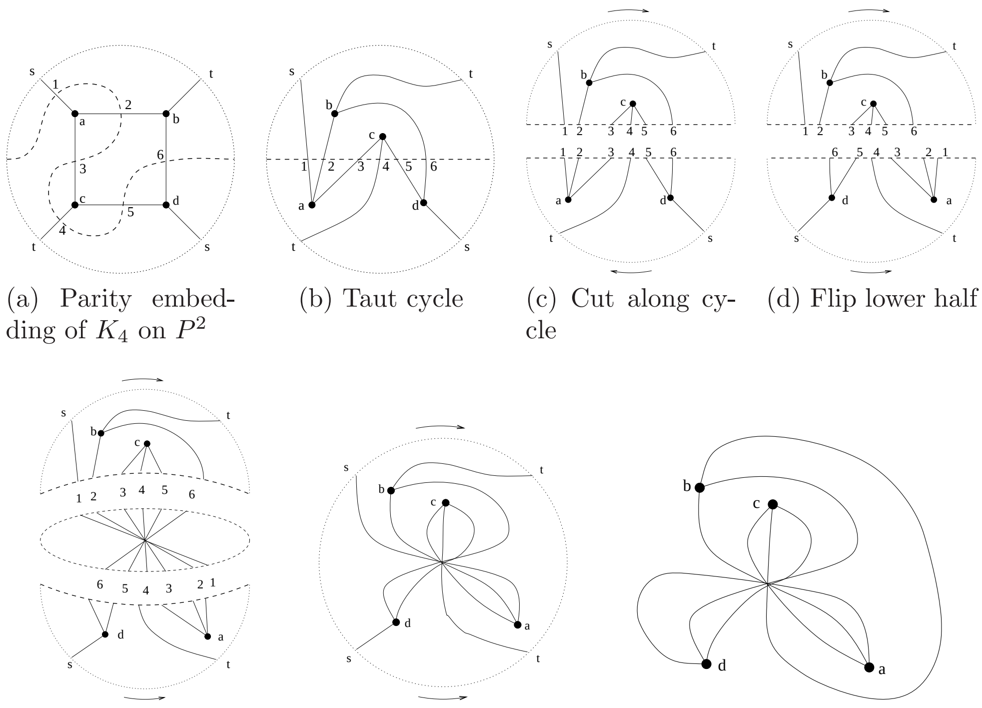
We next cut along the curve C. This disconnects the plane into two pieces, the *inside* and the *outside*. Along the boundary of both pieces C intersects the edges in order $e_1, \ldots, e_m$. We flip the inside region (reversing its orientation) so that it now intersects the edges in order $e_m, \ldots, e_1$. We next reconnect the corresponding edge ends. When doing so note that every edge now crosses every other edge exactly once. More strongly, we can do so such that there is a common point of intersection.

The drawing resulting from this construction is a superthrackle of G as desired. ∎

We next show the second class of graphs in Theorem 1.1 can be superthrackled.

Proposition 3.2 *Let G be a non-bipartite graph embedded in the projective plane so that every face is of even size. Then G is superthrackeable. Moreover, it can be drawn so that all crossings occur at a common point.*

Proof: The proof again uses a "cut-and-paste" construction, this time illustrated in Figure 2 where the underlying graph is K_4.



(a) Parity embedding of K_4 on P^2 (b) Taut cycle (c) Cut along cycle (d) Flip lower half

(e) Add crossing disk (f) Sew in (g) Superthrackle drawing

Figure 2: Parity embedding of non-bipartite K_4 on P^2 with an essential cycle giving a superthrackle on the sphere.

Consider a projective-planar embedding of G such that every face is of even size. Our goal this time is an essential (i.e., 1-sided) closed curve C in the projective plane that crosses each edge of G exactly once. As in Proposition 3.1 we add edges as necessary to form G' with every face of size at most 4, and so without loss of generality every face of G is of size 4. We find our non-self-intersecting Eulerian circuit again using Belyi's Theorem [2]. As before, we can split this to our desired curve C.

We must show that C is essential. If it isn't, then it is contractible and divides the projective plane P^2 into two pieces. Any crossing of an edge in G with C changes between these pieces, so every cycle of G is even. This contradicts the assumption that G is non-bipartite.

We next take an ϵ-neighborhood of C, which is a Möbius strip as C is 1-sided. Cut this out, sew in a disk and reconnect the edges through this disk. Any two edges now cross, and again this can be done with a single point of intersection.

The drawing resulting from this construction is a superthrackle of G on the sphere as desired. ■

4 When you can't superthrackle

In this section we show that some graphs cannot be superthrackled. This is exactly the set of graphs not in the two classes of Theorem 1.1.

Proposition 4.1 *Suppose that G contains a subgraph that is parity homeomorphic of one of the graphs shown in Figure 3 (considered as unsigned graphs with solid edges subdivided once). Then G is not superthrackleable.*

Proof: By Lemma 2.1 it suffices to show that no parity homeomorph of a graph in Figure 3 can be superthrackled. We examine these eight graphs in turn. In each case we use Lemma 2.2, 2.3, and/or 2.4.

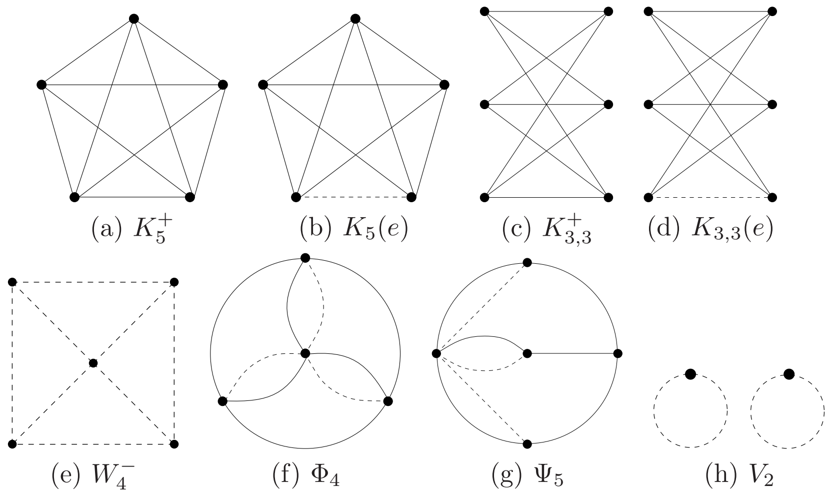


Figure 3: The obstruction set. As signed graphs, solid edges are positive and dotted edges are negative. As unsigned graphs each solid edge is subdivided once.

Case 1, V_2 : Any graph parity homeomorphic to V_2 is the disjoint union of two odd cycles. By Lemma 2.2 it cannot be superthrackled.

Case 2, $K_{3,3}^+$: Consider $K_{3,3}$ with vertex bipartition $1, 2, 3$ versus a, b, c. We first use Lemma 2.4 and in particular the notation for θ-graphs introduced immediately after that lemma to show that $K_{3,3}$ cannot be superthrackled. By way of contradiction suppose that we have a superthrackle drawing and without loss of generality suppose that the edges $(a1, a2, a3)$ appear in clockwise order around vertex a.

We first consider the θ-graph $\langle a1, a2b1, a3c1\rangle$. Every path is of odd length, so by Lemma 2.4 this is a preserver. Since $(a1, a2, a3)$ appear in clockwise order around a, $(a1, b1, c1)$ appear in clockwise order around 1. Next we consider the θ-graph $\langle a1, a2c1, a3b1\rangle$. Again, this is a preserver, and $(a1, a2, a3)$ in clockwise order about

a implies that $(a1, c1, b1)$ are clockwise around 1. This gives our desired contradiction since the two cyclic orders of edges $\{a1, b1, c1\}$ are different.

Any vertex subdivision at v changes the length of any path incident to v. A θ-graph changes from being a converter and preserver if and only if it has v as a degree 3 vertex. In either case we still get the same contradiction of two different cyclic orders. The same holds for even subdivisions, so no graph switching equivalent to $K_{3,3}$ is superthrackleable. The graph $K_{3,3}^+$ is parity homeomorphic to $K_{3,3}$ by vertex switches at a, b, c, completing this case.

Case 3, $K_{3,3}(e)$: The argument is the same as that of the previous case. We first start with the switching equivalent graph $K_{3,3}$ with a single edge subdivided. The two θ-graphs involved in the argument have at most one path of even length, and so by Lemma 2.4 are still preservers leading to the same contradiction. The argument for parity homeomorphs follows mutatis mutandis. Since $K_{3,3}(e)$ is a parity homeomorph of the graph of $K_{3,3}$ with a single edge subdivided (again by vertex switches at a, b, c) it, along with its other parity homeomorphs, are not superthrackleable.

Case 4, K_5^+ : Let the degree 4 vertices be $0, 1, 2, 3, 4$, and again by contradiction suppose that we have a superthrackle drawing. Each path from i to j is of even length, so K_5^+ is bipartite. Without loss of generality at vertex 0 the edges appear in clockwise cyclic order $(01, 02, 03, 04)$. Hence the cycle through $(0, 1, 3)$ is interlaced with the cycle through cycle $(0, 2, 4)$, contradicting Lemma 2.3 since both cycles are of even length. Any parity homeomorph of K_5^+ is also bipartite and so the same argument holds.

Case 5, $K_5(e)$: The argument is the same as that of the previous case. Here we subdivide every edge except for 01. The cycles described are interlaced with one odd and one even, again contradicting Lemma 2.3.

Case 6, W_4^- : Let v be the vertex of degree 4 and let (1234) be the cycle in $W_4^- - v$. Triangles $(v, 1, 2)$ and $(v, 3, 4)$ are both odd, so by Lemma 2.3 around v the edge $v1$ is opposite to $v2$. Similarly, triangles $(v, 1, 4)$ and $(v, 2, 3)$ are interlaced, so $v1$ is opposite to $v4$. But $v1$ can be opposite to only one of these two edges, a contradiction. The same argument holds for any parity homeomorph since the cycles involved are all still odd.

Case 7, Φ_4 : Consider the graph where each solid edge is subdivided once. Let v be the vertex of degree 6, let a, b, c be the vertices of degree four, and let $1, 2, 3, 4$ be the degree 2 vertices on the subdivided edges av, ac, ab, bc respectively.

By way of contradiction suppose this graph can be superthrackled and consider the rotation of edges incident with vertex a. Edges $a1, av$ cannot be interlaced with $a2, a3$ since the cycles $(a, 1, v)$ and $(a, 2, c, 4, b, 3)$ are not both odd. Edges $av, a2$ cannot be interlaced with edges $a1, a3$ since the cycles $(a, 3, b, v, 1)$ and $(a, 2, c, v)$ are not both odd. By symmetry, edges $av, a3$ cannot be interlaced with $a1, a2$. We have our desired contradiction. Again, the argument holds for any parity homeomorph.

Case 8, Ψ_5 : We suppose that every solid edge is subdivided exactly once. Let u be the degree 6 vertex, v the nonadjacent degree 3 vertex, a, b, c the remaining degree 3 vertices, and label $1, 2, 3$ so that $(u, a, 1)$, $(u, b, 2)$, and $(u, c, 3)$ are all triangles. We let va, vb, vc denote the paths of length two from v to a, b, c respectively. Suppose by way of contradiction that we have a superthrackle drawing and consider the rotation around u.

Assume without loss of generality that at vertex v we have edges (va, vb, vc) in clockwise order. We use a sequence of θ-graphs to find the complete clockwise rotation at u. Preservers $\langle vau, vb2u, vcu\rangle$, and $\langle va1u, vbu, vcu\rangle$ force clockwise rotations of $(ua, u2, uc)$ and $(u1, ub, uc)$ respectively at u. Converters $\langle va1u, vb2u, vcu\rangle$, and $\langle vau, vb2u, vc3u\rangle$ force clockwise rotations of $(u1, uc, u2)$ and $(ua, u3, u2)$ at u. This gives the complete clockwise rotation of $(u3, u2, u1, ub, uc, ua)$ at u. Preserver $\langle vau, vbu, vc3u\rangle$ forces the clockwise rotation of $(ua, ub, u3)$ at u which contradicts the rotation at u we have already found. The same argument holds for any parity homeomorph of Ψ_5 mutatis mutandis.

Having examined the eight graphs in turn, the proposition is shown. ■

We now summarize our results as together providing a proof of our main theorems, Theorems 1.1 and 1.2.

Proof: (of Theorems 1.1 and 1.2) By Propositions 3.1 and 3.2 the two classes of graphs can be superthrackled. For the converse, by Theorem 2.9 any graph G that does not have a parity embedding in the projective plane must contain a parity homeomorph of one of the eight graphs of Figure 3. By Proposition 4.1 no such graph can be superthrackled. ■

5 Consequences of our Main Theorem

We begin by relating thrackles and superthrackles.

Lemma 5.1 *If a simple graph G can be drawn as a thrackle, then it can be drawn as a superthrackle.*

Proof: Suppose that we have a thrackle drawing of G. Around each vertex we replace a small neighborhood of the drawing with a new neighborhood as shown in Figure 4. Since G has no loops or parallel edges, this introduces a single crossing between every pair of adjacent edges, so the resulting drawing is a superthrackle. ■

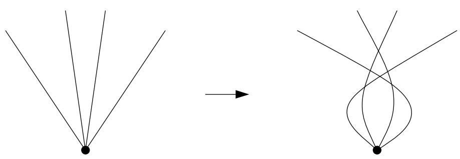


Figure 4: Twisting edges about a vertex

Lovász, Pach and Szegedy [9] showed that any thrackleable bipartite graph is planar. They also showed that the number of edges e in a thrackle drawing of a graph of order v has $e \leq 2v - 3$. Cairns and Nikolayevsky [4] improved this bound to $e \leq 3(v-1)/2$ and Fulek and Pach [7] further improved it to $e \leq \frac{167}{117}v < 1.428v$. In Xu's thesis [16], the bound is shown to be $e \leq 1.4v$.

Lemma 5.2 *If G can be superthrackled, then $e \leq 2v - 2$.*

Proof: A planar bipartite graph has at most $2v - 4$ edges and a cellular projective planar graph with all faces of size at least 4 has $e \leq 2v - 2$. These are the superthrackleable graphs. ∎

Non-bipartite projective planar quadrangulations show that this bound cannot be improved. By Lemma 5.1 this comes close to the bound of [9] for thrackles.

Two edges *meet at a point* if it is either a crossing point or a common endpoint. A *generalized thrackle* is a drawing in which every two edges meet at an odd number of points. In a generalized thrackle of a simple graph non-adjacent edges cross an odd number of times, while adjacent edges cross an even number of times.

Lemma 5.3 *Any superthrackable graph is also a generalized thrackle.*

Proof: Consider a superthrackle drawing of G. As in Lemma 5.1 add a twist so that now adjacent edges cross twice. The result is a generalized thrackle drawing. ∎

The converse is by no means obvious. However, Cairns and Nikolayevsky [5] showed the following.

Theorem 5.4 *A graph is generalized thrackleable on the sphere if and only if it has a parity embedding on the projective plane.*

This is exactly the same as our characterization of superthrackable graphs, so the classes of graphs are the same.

One is tempted to prove generalized thrackleable graphs are superthrackleable by removing two crossings at a time between pairs of edges. This can be done in some circumstances, see [12]. It is false in general since the odd crossing number is not equal to the crossing number, see [11] for definitions and details.

Notice, if a graph G can be drawn as a superthrackle, then by Lemma 5.3 it can also be drawn as a generalized thrackle, which, by Theorem 5.4 implies it has a parity embedding on the projective plane. Theorem 2.9 then implies that it does not contain any subgraph which is parity homeomorphic to any of the graphs in Figure 3. This provides us with an alternate proof of our Proposition 4.1.

We say a graph has a *1-point superthrackle* if it has a superthrackle drawing in which all edge crossings occur at a common point. Any 1-point superthrackle drawing is a superthrackle drawing. On the other hand, Propositions 3.1 and 3.2 show that any superthrackleable graph has a 1-point superthrackle. We combine the observations in this section with Theorem 1.1.

Theorem 5.5 *Let G be a graph. The following are equivalent:*

1. *G is a superthrackle;*
2. *G is a 1-point superthrackle;*
3. *G is a generalized thrackle;*
4. *G has a parity embedding in the projective plane;*
5. *G has no subgraph parity homeomorphic to any graph in Figure 3 (considered as unsigned graphs with solid edges subdivided once).*

6 Conclusion

Of course, it would be nice if the techniques of this paper would help prove Conway's original Thrackle Conjecture. We have tried without success.

The proofs in Section 4 involve when drawings of θ-graphs are converters or preservers. The method is similar to a characterization of planarity given in [1]. The arguments could be phrased in terms of balance in an associated signed triple graph.

A generalization is to consider thrackle drawings on other surfaces, that is, to draw a graph on some surface such that every edge crosses every other nonadjacent edge exactly once. Every graph can be thrackled when drawn on a surface of sufficiently large genus [8]. The *thrackle genus* of G is the smallest genus of an orientable surface on which G has a thrackle drawing.

Question 6.1 *Can we find a characterization of graphs with a given thrackle genus? Is there a "forbidden subgraph" characterization?*

One could ask more specifically:

Question 6.2 *When can a graph be drawn as a thrackle on the projective plane? On the torus?*

Similar questions could be asked about the *superthrackle genus*. The second author's dissertation [14] has some results about superthrackling graphs on a surface. Cairns and Nikolayevsky [5] examined generalized thrackles on a surface with results again tantalizingly similar to [14].

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