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New families of strongly regular graphs

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Abstract

In this article we construct a series of new infinite families of strongly regular graphs with the same parameters as the point-graphs of non-singular quadrics in $\mathrm{PG}(n,2)$. We study these graphs, describing and counting their maximal cliques, and determining their automorphism groups.

1 Introduction

A strongly regular graph $\mathrm{srg}(v,k,\lambda,\mu)$, is a graph with v vertices such that each vertex lies on k edges; any two adjacent vertices have exactly λ common neighbours; and any two non-adjacent vertices have exactly μ common neighbours. We consider the strongly regular graphs constructed from a non-singular quadric $\mathcal{Q}_n$ in $\mathrm{PG}(n,q)$. The *point-graph* $\Gamma_{\mathcal{Q}_n}$ of $\mathcal{Q}_n$ has vertices corresponding to the points of $\mathcal{Q}_n$. Two vertices in $\Gamma_{\mathcal{Q}_n}$ are adjacent if the corresponding points of $\mathcal{Q}_n$ lie on a line contained in $\mathcal{Q}_n$. It is well known (see for example [3]) that $\Gamma_{\mathcal{Q}_n}$ is a strongly regular graph. In this article we let $q=2$, and construct from $\Gamma_{\mathcal{Q}_n}$ approximately $n/2$ new strongly regular graphs with the same parameters as $\Gamma_{\mathcal{Q}_n}$ (see Table 4 for a precise count).

This article proceeds as follows. Section 2 contains several preliminary results we need. Section 3 describes our construction of a series of infinite families of strongly regular graphs, the proof of the construction is given in Section 4. In Section 5, we classify and count the maximal cliques in the new graphs. In Section 6 we prove that our construction yields new families of strongly regular graphs. Finally, in Section 7, we determine the automorphism group of the new graphs.

In previous work, Kantor [8] constructed a strongly regular graph from $\Gamma_{\mathcal{Q}_n}$ with the same parameters in the case when $\mathcal{Q}_n$ contains a spread. Kantor conjects that his graph is not isomorphic to $\Gamma_{\mathcal{Q}_n}$. We show in Section 6.1 that the graph constructed by Kantor is not isomorphic to any of our new graphs. Abiad and Haemers [1] construct several strongly regular graphs from the symplectic graph over GF(2). The dual of these graphs have the same parameters as the point-graph of a non-singular parabolic quadric, so n is even. It is not known if these graphs are isomorphic to our examples with n even.

2 Background Results

In [5], Godsil and McKay take a graph Γ, and use a vertex partition to construct a new graph Γ' that has the same spectrum as Γ. It is well-known (see for example [4]) that if a graph Γ' has the same spectrum as a strongly regular graph Γ, then Γ' is also strongly regular with the same parameters as Γ. Specialising the Godsil-McKay construction to a partition of size two in a strongly regular graph gives the following result.

Result 2.1
1. A *Godsil-McKay partition* of a graph is a partition of the vertices into two sets $\{\mathcal{X}, \mathcal{Y}\}$ satisfying:
   I. The set $\mathcal{X}$ induces a regular subgraph.
   II. Each vertex in $\mathcal{Y}$ is adjacent to 0, $\frac{1}{2}|\mathcal{X}|$ or $|\mathcal{X}|$ vertices in $\mathcal{X}$.
2. *Godsil-McKay construction.* Let Γ be a strongly regular graph with Godsil-McKay partition $\{\mathcal{X}, \mathcal{Y}\}$. Construct the graph Γ' with the same points and edges as Γ, except: for each vertex R in $\mathcal{Y}$ with $\frac{1}{2}|\mathcal{X}|$ neighbours in $\mathcal{X}$, delete these $\frac{1}{2}|\mathcal{X}|$ edges and join R to the other $\frac{1}{2}|\mathcal{X}|$ vertices in $\mathcal{X}$. Then the graph Γ' is strongly regular with the same parameters as Γ.

Let $\mathcal{Q}_n$ be a non-singular quadric in PG(n,q). The *projective index* g of $\mathcal{Q}_n$ is the dimension of the largest subspace contained in $\mathcal{Q}_n$. A g-space contained in $\mathcal{Q}_n$ is called a *generator* of $\mathcal{Q}_n$. If $n = 2r$ is even, then a non-singular quadric is called a parabolic quadric, denoted $\mathcal{P}_{2r}$, which has projective index $g = r-1$. If $n = 2r+1$ is odd, then there are two types of non-singular quadrics: the elliptic quadric denoted $\mathcal{E}_{2r+1}$ has projective index $g = r-1$; and the hyperbolic quadric denoted $\mathcal{H}_{2r+1}$ has projective index $g = r$. The points and generators of $\mathcal{Q}_n$ also form a polar space of rank $g+1$. We repeatedly use the following two properties of quadrics and polar spaces, see [7, Chapter 22] for more information on quadrics, and [7, Section 26.1] for more information on polar spaces.

Result 2.2 *Let $\mathcal{Q}_n$ be a non-singular quadric in* PG(n,q) *and let* Π *be a k-space. If the quadric $\mathcal{Q}_n \cap \Pi$ contains a $(k-1)$-space, then $\mathcal{Q}_n \cap \Pi$ is either* Π*, or one or two $(k-1)$-spaces.*

Result 2.3 *Let $\mathcal{Q}_n$ be a non-singular quadric in* PG$(n, 2)$*, with projective index g. Let Σ be a generator of $\mathcal{Q}_n$, and X a point of $\mathcal{Q}_n$ not in Σ. Then there is a unique generator Π of $\mathcal{Q}_n$ that contains X and meets Σ in a $(g-1)$-space. Further, the points in Σ which lie on a line of $\mathcal{Q}_n$ through X are exactly the points in $\Sigma \cap \Pi$.*

3 Our construction

We begin with a small example to illustrate the general construction.

Example 3.1 Let ℓ be a line of the elliptic quadric $\mathcal{E} = \mathcal{E}_{2r+1}$ in PG$(2r+1, q)$. Partition the points of $\mathcal{E}$ into the following three types.

(i) points of $\mathcal{E}$ on ℓ,
(ii) points of $\mathcal{E}$ that are on a plane of $\mathcal{E}$ that contains ℓ,
(iii) the remaining points of $\mathcal{E}$.

Define a new graph Γ_1 with vertices the points of $\mathcal{E}$, and edges given in Table 1. Note that the last row of Table 1 describes the edges of Γ_1 that are different to the edges of the point-graph $\Gamma_{\mathcal{E}}$ of $\mathcal{E}$.

Table 1: Edges in Γ_1

Vertex pair	Vertex types	Vertex pair is an edge of Γ_1:
P, P'	P, P' are type (i)	always (as PP' is always a line of $\mathcal{E}$)
P, Q	P is type (i), Q is type (ii)	always (as PQ is always a line of $\mathcal{E}$)
Q, Q'	Q, Q' are type (ii)	when QQ' is a line of $\mathcal{E}$
P, R	P is type (i), R is type (iii)	when PR is a line of $\mathcal{E}$
R, R'	R, R' are type (iii)	when RR' is a line of $\mathcal{E}$
Q, R	Q is type (ii), R is type (iii)	when QR is a 2-secant of $\mathcal{E}$

It can be shown directly using geometric techniques that Γ_1 is regular if and only if $q = 2$, and that in this case Γ_1 is strongly regular with the same parameters as $\Gamma_{\mathcal{E}}$. This can also be proved using the Godsil-McKay construction as follows. Consider the partition $\{\mathcal{X}, \mathcal{Y}\}$ of $\Gamma_{\mathcal{E}}$ where $\mathcal{X}$ contains the vertices of type (ii), and $\mathcal{Y}$ contains the vertices of type (i) and (iii). Geometric techniques can be used to show that this partition satisfies the conditions of Result 2.1(1) if and only if $q = 2$. Note that the graph constructed in Result 2.1(2) from this partition is the graph Γ_1, hence Γ_1 is strongly regular when $q = 2$. □

We now give our general construction of a series of infinite families of strongly regular graphs. This construction generalises Example 3.1. First we define a partition of the vertices of the point-graph of $\mathcal{Q}_n$.

Definition 3.2 Let $\mathcal{Q}_n$ be a non-singular quadric in $\mathrm{PG}(n,q)$, and let Γ be the point-graph of $\mathcal{Q}_n$. Let s be an integer with $0 \le s < g$, where g is the projective index of $\mathcal{Q}_n$. Let α_s be an s-dimensional subspace contained in $\mathcal{Q}_n$. The points of $\mathcal{Q}_n$ (and so the vertices of Γ) can be partitioned into three types:

(i) points in α_s,
(ii) points of $\mathcal{Q}_n \backslash \alpha_s$ that lie in some $(s+1)$-dimensional subspace Π with $\alpha_s \subset \Pi \subset \mathcal{Q}_n$,
(iii) the remaining points of $\mathcal{Q}_n$.

Let $\mathcal{X}_s$ be the vertices of Γ of type (ii) and let $\mathcal{Y}_s$ be the vertices of Γ of type (i) and (iii).

Note that if $s = g$, then there are no points of type (ii), so we need $s < g$. We will show that the partition $\{\mathcal{X}_s, \mathcal{Y}_s\}$, $0 \le s < g$, is a Godsil-McKay partition if and only if $q = 2$. By [7, Theorem 26.6.6], the group fixing $\mathcal{Q}_n$ is transitive on the subspaces of dimension s contained in $\mathcal{Q}_n$. So for each s, $0 \le s < g$, we can use Result 2.1 to construct a unique strongly regular graph Γ_s from Γ. We state the main result here, and give the proof in Section 4.

Theorem 3.3 *In* $\mathrm{PG}(n,2)$*, let* $\mathcal{Q}_n$ *be a non-singular quadric of projective index* $g \ge 1$ *with point-graph* Γ*. For each integer* s*,* $0 \le s < g$*, let* Γ_s *be the graph obtained using the Godsil-McKay construction with the partition* $\{\mathcal{X}_s, \mathcal{Y}_s\}$ *defined in Definition* 3.2*. Then* Γ_s *is a strongly regular graph with the same parameters as* Γ*.*

We show in Section 6 that $\Gamma_0 \cong \Gamma$, and that for each n, Γ_s, $0 \le s < g$ are $g-1$ non-isomorphic graphs.

4 Proof of Theorem 3.3

Throughout this section, let $\mathcal{Q}_n$ be a non-singular quadric in $\mathrm{PG}(n,q)$ of projective index g, and let α_s be a subspace of dimension s, $0 \le s < g$, contained in $\mathcal{Q}_n$. Let Γ be the point-graph of $\mathcal{Q}_n$, and let $\{\mathcal{X}_s, \mathcal{Y}_s\}$ be the partition of the vertices of Γ (and so of the points of $\mathcal{Q}_n$) defined in Definition 3.2. We will show that $\{\mathcal{X}_s, \mathcal{Y}_s\}$ satisfies Conditions I and II of Result 2.1. First we count the points in $\mathcal{X}_s$.

Lemma 4.1 *1. If* $\mathcal{Q}_n = \mathcal{E}_{2r+1}$*, then* $|\mathcal{X}_s| = \dfrac{q^{s+1}(q^{r-s}+1)(q^{r-s-1}-1)}{(q-1)}$.

2. If $\mathcal{Q}_n = \mathcal{H}_{2r+1}$*, then* $|\mathcal{X}_s| = \dfrac{q^{s+1}(q^{r-s-1}+1)(q^{r-s}-1)}{(q-1)}$.

3. If $\mathcal{Q}_n = \mathcal{P}_{2r}$*, then* $|\mathcal{X}_s| = \dfrac{q^{s+1}(q^{r-s-1}+1)(q^{r-s-1}-1)}{(q-1)}$.

Proof We prove this in the case $\mathcal{Q}_n$ is $\mathcal{E} = \mathcal{E}_{2r+1}$, which has projective index $g = r-1$ and point-graph denoted $\Gamma_{\mathcal{E}}$. The cases when $\mathcal{Q}_n$ is $\mathcal{H}_{2r+1}$ and $\mathcal{P}_{2r}$ are proved in a very similar manner.

By [7, Theorem 22.5.1], the number of subspaces of dimension s contained in $\mathcal{E}$ is

$$\frac{\Big((q^{r-s+1}+1)(q^{r-s+2}+1)\cdots(q^{r+1}+1)\Big)\times\Big((q^{r-s}-1)(q^{r-s+1}-1)\cdots(q^{r}-1)\Big)}{(q-1)(q^2-1)\cdots(q^{s+1}-1)}.$$

Moreover, replacing 's' by '$s+1$' in this equation gives the number of subspaces of dimension $s+1$ contained in $\mathcal{E}$. By [6, Theorem 3.1], the number of subspaces of dimension s in a subspace of dimension $s+1$ is $\big(q^{s+2}-1\big)/\big(q-1\big)$. By [7], the number of subspaces of dimension $s+1$ that contain α_s and are contained in $\mathcal{E}$ is a constant. To calculate it, we count ordered pairs (Π, Σ) where Π is an s-dimensional subspace contained in $\mathcal{E}$, Σ is an $(s+1)$-dimensional subspace contained in $\mathcal{E}$, and $\Pi \subset \Sigma$. This count gives the number of subspaces of dimension $s+1$ that contain α_s and are contained in $\mathcal{E}$ is

$$x = \frac{(q^{r-s}+1)(q^{r-s-1}-1)}{(q-1)}. \tag{1}$$

Each of these subspace contains q^{s+1} points that are not in α_s. Hence $|\mathcal{X}_s| = xq^{s+1}$ as required. □

We now show that $\{\mathcal{X}_s, \mathcal{Y}_s\}$ satisfies Condition I of Result 2.1.

Lemma 4.2 *Let Γ^* be the subgraph of Γ on the vertices in $\mathcal{X}_s$. Then Γ^* is a regular graph with degree k where:*

1. *if $\mathcal{Q}_n = \mathcal{E}_{2r+1}$, then* $k = (q^{s+1}-1) + \dfrac{q^{s+2}(q^{r-s-1}+1)(q^{r-s-2}-1)}{(q-1)}$;
2. *if $\mathcal{Q}_n = \mathcal{H}_{2r+1}$, then* $k = (q^{s+1}-1) + \dfrac{q^{s+2}(q^{r-s-2}+1)(q^{r-s-1}-1)}{(q-1)}$;
3. *if $\mathcal{Q}_n = \mathcal{P}_{2r}$, then* $k = (q^{s+1}-1) + \dfrac{q^{s+2}(q^{r-s-2}+1)(q^{r-s-2}-1)}{(q-1)}$.

Proof We prove this in the case $\mathcal{Q}_n$ is $\mathcal{E} = \mathcal{E}_{2r+1}$, which has projective index $g = r-1$ and point-graph denoted $\Gamma_{\mathcal{E}}$. The cases when $\mathcal{Q}_n$ is $\mathcal{H}_{2r+1}$ and $\mathcal{P}_{2r}$ are proved in a very similar manner.

Let Q be a vertex in $\mathcal{X}_s$, we need to count the number of vertices in $\mathcal{X}_s$ that are adjacent to Q. Recall that $\mathcal{X}_s$ consists of vertices of type (ii), so in $\mathrm{PG}(2r+1, q)$, Q is a point of the quadric $\mathcal{E}$, and the $(s+1)$-dimensional space $\Sigma = \langle Q, \alpha_s\rangle$ is contained in $\mathcal{E}$. A vertex Q' in $\mathcal{X}_s$ is adjacent to Q if the line QQ' is contained in $\mathcal{E}$. We partition the lines of $\mathcal{E}$ through Q into three families: $\mathcal{F}_1$ contains the lines of $\mathcal{E}$ through Q that lie in Σ; $\mathcal{F}_2$ contains the lines of $\mathcal{E}$ through Q (not in $\mathcal{F}_1$) that lie in an $(s+2)$-dimensional subspace that contains Σ and is contained in $\mathcal{E}$; and $\mathcal{F}_3$ contains the remaining lines of $\mathcal{E}$ through Q.

We first look at $\mathcal{F}_1$. The number of lines in $\mathcal{F}_1$ equals the number of lines through a point in an $(s+1)$-dimensional subspace, so by [6, Theorem 3.1],

$$|\mathcal{F}_1| = \frac{(q^{s+1}-1)}{(q-1)}. \tag{2}$$

Each of the lines in $\mathcal{F}_1$ contains the point Q and meets α_s in one point. So each line in $\mathcal{F}_1$ gives rise to $q-1$ vertices in $\mathcal{X}_s$ which are adjacent to Q in the graph Γ^*. In total, $\mathcal{F}_1$ contributes $(q-1) \times |\mathcal{F}_1| = (q^{s+1}-1)$ neighbours of Q in Γ^*.

Next we look at $\mathcal{F}_2$. Replacing 's' by '$s+1$' in (1) gives the number of subspace of dimension $s+2$ that contain the $(s+1)$-space $\Sigma = \langle Q, \alpha_s \rangle$ and are contained in $\mathcal{E}$ is $(q^{r-s-1}+1)(q^{r-s-2}-1)/(q-1)$. Similarly, (2) can be generalised to show that the number of lines through Q that lie in a subspace of dimension $s+2$, and do not lie in the $(s+1)$-space Σ is $\Big((q^{s+2}-1)/(q-1)\Big) - \Big((q^{s+1}-1)/(q-1)\Big) = q^{s+1}$. Hence

$$|\mathcal{F}_2| = q^{s+1} \times \frac{(q^{r-s-1}+1)(q^{r-s-2}-1)}{(q-1)}.$$

Each line in $\mathcal{F}_2$ contains one point of Σ, and the remaining q points correspond to q vertices that lie in $\mathcal{X}_s$ (and are not considered in $\mathcal{F}_1$). That is, each line in $\mathcal{F}_2$ contributes q neighbours to Q in the graph Γ^*. So in total, $\mathcal{F}_2$ contributes $q \times |\mathcal{F}_2| = q^{s+2}(q^{r-s-1}+1)(q^{r-s-2}-1)/(q-1)$ neighbours to Q in the graph Γ^*.

Finally we look at $\mathcal{F}_3$. Let ℓ be a line in $\mathcal{F}_3$, so ℓ contains Q, but the $(s+2)$-space $\Pi = \langle \alpha_s, \ell \rangle$ is not contained in $\mathcal{E}$. Suppose that ℓ contains another point Q' that corresponds to a vertex in $\mathcal{X}_s$. Then $\Pi \cap \mathcal{E}$ contains the two distinct $(s+1)$-dimensional subspaces $\Sigma = \langle \alpha_s, Q \rangle$ and $\Sigma' = \langle \alpha_s, Q' \rangle$. As Π is not contained in $\mathcal{E}$, Π meets $\mathcal{E}$ in exactly the two $(s+1)$-spaces Σ and Σ'. Thus $\ell = QQ'$ is not a line of $\mathcal{E}$, and so ℓ contains exactly two points Q, Q' that are vertices of $\mathcal{X}_s$, moreover they are not adjacent in Γ^*. Thus $\mathcal{F}_3$ contributes 0 neighbours to Q in the graph Γ^*.

Summing the neighbours of Q in Γ^* obtained from the families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ gives the required result. Note that if $s = g-1$, so $s = r-2$, then $|\mathcal{F}_2| = 0$, and the degree of Γ^* is $q^{r-1}-1$. □

Now we look at Condition II of Result 2.1. Note that throughout the proofs in this article, we consistently use P, P' to denote points of type (i); Q, Q' to denote points of type (ii); and R, R' to denote points of type (iii).

Lemma 4.3 *The partition $\{\mathcal{X}_s, \mathcal{Y}_s\}$ satisfies Condition II of Result 2.1 if and only if $q = 2$.*

Proof We prove this in the case $\mathcal{Q}_n$ is $\mathcal{E} = \mathcal{E}_{2r+1}$, which has projective index $g = r-1$ and point-graph denoted $\Gamma_{\mathcal{E}}$. The cases when $\mathcal{Q}_n$ is $\mathcal{H}_{2r+1}$ and $\mathcal{P}_{2r}$ are proved in a very similar manner.

We need to show that in the graph $\Gamma_{\mathcal{E}}$, each vertex in $\mathcal{Y}_s$ is adjacent to 0, $\frac{1}{2}|\mathcal{X}_s|$ or $|\mathcal{X}_s|$ vertices in $\mathcal{X}_s$. There are two cases to consider since the vertices in $\mathcal{Y}_s$ are of type (i) or (iii). First consider a vertex P in $\mathcal{Y}_s$ of type (i). Let $Q \in \mathcal{X}_s$, so Q is a vertex of type (ii). Hence in $\mathrm{PG}(2r+1, q)$, $P \in \alpha_s$ and Q lies in an $(s+1)$-space Π with $\alpha_s \subset \Pi \subset \mathcal{E}$. Hence PQ is a line of $\mathcal{E}$, and so P and Q are adjacent vertices in $\Gamma_{\mathcal{E}}$. That is, each vertex of type (i) in $\mathcal{Y}_s$ is adjacent to each of the $|\mathcal{X}_s|$ vertices in $\mathcal{X}_s$.

Now consider a vertex R in $\mathcal{Y}_s$ of type (iii). We count the number of vertices Q in $\mathcal{X}_s$ for which RQ is a line of $\mathcal{E}$. We will show that this number is not 0 or $|\mathcal{X}_s|$, and further, is $\frac{1}{2}|\mathcal{X}_s|$ if and only if $q = 2$. Let Σ be a subspace of $\mathcal{E}$ of dimension $s+1$ that contains α_s. So $\Sigma \backslash \alpha_s$ consists of points of type (ii), hence $R \notin \Sigma$. Consider the $(s+2)$-space $\Pi = \langle \Sigma, R \rangle$. As $\alpha_s \subset \Sigma$, we have $\langle \alpha_s, R \rangle \subset \Pi$. As R is of type (iii), $\langle \alpha_s, R \rangle$ is not contained in $\mathcal{E}$. Hence Π is not contained in $\mathcal{E}$. So $\Pi \cap \mathcal{E}$ contains the $(s+1)$-space Σ and the point $R \notin \Sigma$. Hence by Result 2.2, $\Pi \cap \mathcal{E}$ is two distinct $(s+1)$-spaces. That is, $\Pi \cap \mathcal{E} = \{\Sigma, \Sigma'\}$ where Σ' is an $(s+1)$-space that contains R. As R is type (iii), Σ' does not contain α_s. Hence $\Sigma \cap \Sigma'$ is an s-space distinct from α_s, and so $\Sigma \cap \Sigma' \cap \alpha_s$ is a space of dimension $s-1$. Let Q be a point in $\Sigma \backslash \alpha_s$, so Q has type (ii). If $Q \in \Sigma \cap \Sigma'$, then as $Q, R \in \Sigma' \subset \mathcal{E}$, the line $m = QR$ is a line of $\mathcal{E}$. If $Q \notin \Sigma \cap \Sigma'$, then as $\Pi \cap \mathcal{E} = \{\Sigma, \Sigma'\}$, the line $m = QR$ is a 2-secant of $\mathcal{E}$. That is, Q is a neighbour of R in $\Gamma_{\mathcal{E}}$ if and only if $Q \in \Sigma \cap \Sigma'$.

Suppose $Q \in \Sigma \cap \Sigma'$, $Q \notin \alpha_s$, we characterise the points on the line $m = QR$. First suppose $m = QR$ contains a second point Q' of type (ii). So $\langle \alpha_s, Q' \rangle$ is an $(s+1)$-space contained in $\mathcal{E}$. Thus Π contains three distinct $(s+1)$-spaces of $\mathcal{E}$, namely $\Sigma, \Sigma', \langle \alpha_s, Q' \rangle$, contradicting Result 2.2. Thus m contains exactly one point of type (ii), namely Q, and the rest of the points on m are type (iii). Hence in the graph $\Gamma_{\mathcal{E}}$, the line m gives rise to one neighbour of R that lies in $\mathcal{X}_s$, namely Q. Thus each point of $\Sigma' \cap \Sigma$ not in α_s gives rise to exactly one vertex in $\mathcal{X}_s$ that is a neighbour of R. This is true for every $(s+1)$-space Σ with $\alpha_s \subset \Sigma \subset \mathcal{E}$. Moreover, each neighbour of R in $\mathcal{X}_s$ corresponds to a point of $\mathcal{E}$ that lies in exactly one such $(s+1)$-space, so arises exactly once in this way. Hence the number of neighbours of R that lie in $\mathcal{X}_s$ equals the number of points of $\mathcal{E} \backslash \alpha_s$ that lie in some $\Sigma \cap \Sigma'$ for $(s+1)$-spaces $\Sigma, \Sigma' \subset \mathcal{E}$ with $\alpha_s \subset \Sigma$, $\alpha_s \not\subset \Sigma'$ and $\Sigma \cap \Sigma'$ an s-space. We next count these points.

Firstly, the number of $(s+1)$-dimensional spaces that contain α_s and are contained in $\mathcal{E}$ is given in (1). Secondly, let Σ be an $(s+1)$-space containing α_s, and Σ' an $(s+1)$-space that meets Σ in an s-space not containing α_s. Then the number of points in $\Sigma \cap \Sigma'$ which are not in α_s is $\big((q^{s+1}-1)/(q-1)\big) - \big((q^s-1)/(q-1)\big) = q^s$. Hence in the graph $\Gamma_{\mathcal{E}}$, there are

$$y = \frac{q^s\ (q^{r-s}+1)(q^{r-s-1}-1)}{(q-1)}$$

vertices in $\mathcal{X}_s$ that are neighbours of R. To satisfy Condition II of Result 2.1, we need $y \in \{0,\ \frac{1}{2}|\mathcal{X}_s|,\ |\mathcal{X}_s|\}$. Now $y = 0$ if and only if $r - s - 1 = 0$, which does not occur as $s < g = r - 1$. Further, $|\mathcal{X}_s|$ is calculated in Lemma 4.1, and $y < |\mathcal{X}_s|$. Using Lemma 4.1, $y = |\mathcal{X}_s|/2$ if and only if $q = 2$.

Thus the vertices in $\mathcal{Y}_s$ of type (i) are adjacent to $|\mathcal{X}_s|$ of the vertices in $\mathcal{X}_s$. Further, the vertices in $\mathcal{Y}_s$ of type (iii) are not adjacent to 0 or all the vertices of $\mathcal{X}_s$, and are adjacent to $\frac{1}{2}|\mathcal{X}_s|$ of the vertices in $\mathcal{X}_s$ if and only if $q = 2$. That is, Condition II of Result 2.1 is satisfied in for the partition $\{\mathcal{X}_s, \mathcal{Y}_s\}$ of $\Gamma_{\mathcal{E}}$ if and only if $q = 2$. □

It is now straightforward to prove Theorem 3.3.

Proof of Theorem 3.3 Let $\mathcal{Q}_n$ be a non-singular quadric of $\mathrm{PG}(n, 2)$ with projective index g. Let s be an integer with $0 \le s < g$, let α_s be a s-space contained in $\mathcal{Q}_n$, and let $\{\mathcal{X}_s, \mathcal{Y}_s\}$ be the partition given in Definition 3.2. By Lemmas 4.2 and 4.3, the partition $\{\mathcal{X}_s, \mathcal{Y}_s\}$ satisfies Conditions I and II of Result 2.1(1). Hence we can use Result 2.1(2) to construct a graph Γ_s. Note that as the group fixing $\mathcal{Q}_n$ is transitive on the s-spaces of $\mathcal{Q}_n$, $0 \le s \le g$, different choices of the subspace α_s give rise to the same (up to isomorphism) graph. So for any s, $0 \le s < g$, the graph Γ_s is a strongly regular graph with the same parameters as Γ. □

Remark 4.4 As $0 \le s < g$, we have $g \ge 1$. This places a bound on n: when $\mathcal{Q}_n$ is a hyperbolic quadric, we need $n \ge 3$; when $\mathcal{Q}_n$ is a parabolic quadric, we need $n \ge 4$; and when $\mathcal{Q}_n$ is an elliptic quadric, we need $n \ge 5$.

It is useful to note that the proof of Lemma 4.3 gives a description of the edges in the graph Γ_s. That is, let P, P' be vertices of type (i), Q, Q' vertices of type (ii), and R, R' vertices of type (iii). Then $\{P, P'\}$, $\{P, Q\}$, $\{P, R\}$, $\{Q, Q'\}$, $\{R, R'\}$ are edges of Γ_s if PP', PQ, PR, QQ', RR' are lines of $\mathcal{Q}_n$ respectively; and $\{Q, R\}$ is an edge of Γ_s if QR is a 2-secant of $\mathcal{Q}_n$. In summary, we have:

Corollary 4.5 *Let Γ_s, $0 \le s < g$ be the graph constructed in Theorem 3.3. The adjacencies in Γ_s are the same as those given in Table 1.*

Remark 4.6 We note that if $q \ne 2$, then geometric techniques similar to those used here show that the graph Γ_s with $s > 0$ is *not* regular.

5 Maximal cliques of Γ_s

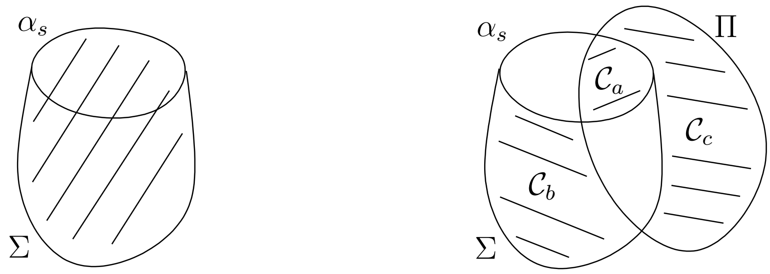
In Section 5.1, we classify the maximal cliques in the graph Γ_s, and in Section 5.2, we count them.

5.1 Description of Maximal Cliques of Γ_s

Throughout this section, let $\mathcal{Q}_n$ be a non-singular quadric of $\mathrm{PG}(n, 2)$ of projective index g with point-graph Γ. For s an integer with $0 \le s < g$, let α_s be an s-space of $\mathcal{Q}_n$. Let Γ_s be the graph described in Theorem 3.3.

We first describe the maximal cliques of the point-graph Γ of $\mathcal{Q}_n$. The largest subspaces contained in $\mathcal{Q}_n$ are the generators, which have dimension g, and so contain $2^{g+1}-1$ points. Further, any subspace of $\mathcal{Q}_n$ is contained in a generator of $\mathcal{Q}_n$. Hence the maximal cliques of Γ have $2^{g+1} - 1$ vertices and correspond to generators of $\mathcal{Q}_n$.

We want to study maximal cliques in Γ_s, we begin by studying cliques of Γ_s of size $2^{g+1}-1$, then show that these are maximal. We define a *g-clique* of Γ_s to be a clique of size $2^{g+1}-1$. The next lemma describes two types of g-cliques of Γ_s, we show later that these are the maximal cliques of Γ_s. The first type corresponds to generators of $\mathcal{Q}_n$ containing α_s, and so corresponds to maximal cliques of the original graph Γ. Figure 1 illustrates the two types of g-cliques described in Lemma 5.1.



Type A Type B

Figure 1: g-cliques of Γ_s

Lemma 5.1 *Let Γ_s, $0 \le s < g$, be the graph constructed as in Theorem 3.3.*

A. Let Σ be a generator of $\mathcal{Q}_n$ that contains α_s, then the points of Σ form a g-clique of Γ_s.

B. Let Π, Σ be two generators of $\mathcal{Q}_n$ such that: Σ contains α_s; Π does not contain α_s; and Π, Σ meet in a $(g-1)$-dimensional space. Let $\mathcal{C}_a$ be the 2^s-1 points of $\alpha_s \cap \Pi$; $\mathcal{C}_b$ be the 2^g-2^s points of Σ that are not in α_s or Π; and $\mathcal{C}_c$ be the 2^g points of $\Pi\backslash\Sigma$, see Figure 1. Then the points in $\mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_c$ form a g-clique of the graph Γ_s.

Proof For part A, let Σ be a generator of $\mathcal{Q}_n$ that contains α_s. Let $\mathcal{C}$ be the set of vertices of Γ_s that correspond to the points of Σ. As $\mathcal{C}$ consists of vertices of type (i) and (ii) only, two vertices of $\mathcal{C}$ are adjacent if the corresponding two points lie on a line of $\mathcal{Q}_n$. As Σ is contained in $\mathcal{Q}_n$, every pair of distinct points in Σ lie in a line of $\mathcal{Q}_n$. Hence every pair of distinct vertices in $\mathcal{C}$ are adjacent, so $\mathcal{C}$ is a clique. Further, Σ contains $2^{g+1}-1$ points, so $|\mathcal{C}| = 2^{g+1}-1$. Thus $\mathcal{C}$ is a g-clique of Γ_s.

We now consider the set $\mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_c$ described in part B. By construction, the three sets $\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c$ are pairwise disjoint, $\mathcal{C}_a$ consists of points of type (i), $\mathcal{C}_b$ consists of points of type (ii), and $\mathcal{C}_c$ contains no points of type (i). Suppose $\mathcal{C}_c$ contained a point Q of type (ii), so $\langle \alpha_s, Q\rangle$ is an $(s+1)$-space of $\mathcal{Q}_n$. By construction, $\langle \alpha_s, Q\rangle$ is not contained in Π or Σ, so contains a point X not in Σ or Π. So the $(g+1)$-space $\langle \Pi, \Sigma\rangle$ meets $\mathcal{Q}_n$ in at least Π, Σ, X, contradicting Result 2.2. Hence $\mathcal{C}_c$ consists of points of type (iii). Note that straightforward counting shows that the number of points in $\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c$ is as stated in the theorem, and $|\mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_c| = 2^{g+1}-1$.

We need to show that any pair of vertices in the set corresponding to $\mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_c$ are adjacent. Recall Corollary 4.5 shows that the adjacencies in Γ_s are as described in Table 1. Let $P, P' \in \mathcal{C}_a$, $Q, Q' \in \mathcal{C}_b$, $R, R' \in \mathcal{C}_c$ be distinct points. (Note that the argument below is easily adjusted to work if $\mathcal{C}_a$ or $\mathcal{C}_b$ has size 1.) As P, P' have type (i), Q, Q' have type (ii) and R, R' have type (iii), the following pairs of points lie in a subspace of $\mathcal{Q}_n$, and so lie on a line of $\mathcal{Q}_n$: $P, P' \in \alpha_s \subset \mathcal{Q}_n$, $Q, Q' \in \Sigma \subset \mathcal{Q}_n$, $P, Q \in \Sigma \subset \mathcal{Q}_n$, $P, R \in \Pi \subset \mathcal{Q}_n$, $R, R' \in \Pi \subset \mathcal{Q}_n$. Hence the corresponding pairs of vertices are all adjacent in Γ_s.

To complete the proof that $\mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_c$ corresponds to a g-clique of Γ_s, we need to show that Q, R are adjacent in Γ_s, so by Table 1, we need to show that QR is a 2-secant of $\mathcal{Q}_n$. The line QR lies in the $(g+1)$-space $\langle \Pi, \Sigma \rangle$, which meets $\mathcal{Q}_n$ in exactly Π and Σ. As $Q \in \Sigma \backslash \Pi$ and $R \in \Pi \backslash \Sigma$, the line QR is not contained in $\mathcal{Q}_n$, so it is a 2-secant of $\mathcal{Q}_n$. Hence QR is an edge of Γ_s. That is, $\mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_c$ is a set of $2^{g+1} - 1$ vertices of Γ_s such that any two vertices are adjacent, and so it is a g-clique of Γ_s. □

We will show that the only maximal cliques in Γ_s are the g-cliques of Class A and B. We need some preliminary lemmas. Note that the g-cliques of Class A contain no points of type (iii), we begin by showing that the converse also holds.

Lemma 5.2 *Let $\mathcal{C}$ be a g-clique of Γ_s, $0 \le s < g$, that contains no vertices of type* (iii)*, then $\mathcal{C}$ is a g-clique of Class A.*

Proof Let $\mathcal{C}$ be a g-clique of Γ_s, $0 \le s < g$, that contains no vertices of type (iii). Suppose $\mathcal{C}$ is not contained in a generator of $\mathcal{Q}_n$. We consider the number of points of $\mathcal{C}$ in each generator of $\mathcal{Q}_n$. Let Σ be a generator of $\mathcal{Q}_n$ that contains the maximum number of points of $\mathcal{C}$. As $\mathcal{C}$ is not contained in Σ, there is a point A of $\mathcal{C}$ that is not in Σ. By Result 2.3, there is a unique generator Π of $\mathcal{Q}_n$ that contains A and meets Σ in a $(g-1)$-space. Further, the points of Σ that lie on a line of $\mathcal{Q}_n$ through A are exactly the points of $\Sigma \cap \Pi$. As $\mathcal{C}$ contains no points of type (iii), edges in $\mathcal{C}$ correspond to lines of $\mathcal{Q}_n$. In Γ_s, each vertex in $\mathcal{C}$ is adjacent to the vertex A, so in $\mathrm{PG}(n, 2)$, the points of $\mathcal{C} \cap \Sigma$ lie in $\Sigma \cap \Pi$. Hence $|\Pi \cap \mathcal{C}| \ge |\Sigma \cap \mathcal{C}| + 1$, which contradicts the choice of Σ being the generator with the largest intersection with $\mathcal{C}$. Hence $\mathcal{C}$ is contained in a generator of $\mathcal{Q}_n$. As $|\mathcal{C}| = 2^{g+1} - 1$, the vertices of $\mathcal{C}$ correspond exactly to the points of this generator, and so $\mathcal{C}$ is a Class A g-clique.

□

Lemma 5.3 *Every generator of $\mathcal{Q}_n$ contains at least one point of type* (ii)*.*

Proof Let $\mathcal{Q}_n$ be a non-singular quadric of projective index g and let Π be a generator of $\mathcal{Q}_n$. There are two cases to consider. Firstly, if Π contains α_s, then Π contains only points of type (i) and (ii). Hence, as $s < g$, Π contains at least one point of type (ii). Next consider the case where Π meets α_s in a subspace α_t of dimension t, with $-1 \le t \le s-1$. Let P_1 be a point of $\alpha_s \backslash \alpha_t$. As $P_1 \notin \Pi$, by Result 2.3 there exists a unique generator Σ_1 of $\mathcal{Q}_n$ that contains P_1 and meets Π in a $(g-1)$-space.

Moreover, if $Y \in \alpha_t$, then $P_1Y \subset \alpha_s$ and so is a line of $\mathcal{Q}_n$, hence by Result 2.3, $\alpha_t \subset \Sigma_1$, and so $\alpha_t = \Pi \cap \Sigma_1 \cap \alpha_s$. Further, if X is a point of $\Pi \cap \Sigma_1$ not in α_s and $Y \in \langle \alpha_t, P_1 \rangle$, then the line XY lies in Σ_1 and so is a line of $\mathcal{Q}_n$.

If $\alpha_s \cap \Sigma_1 \neq \alpha_s$, we repeat this process. Let P_2 be a point of α_s not in Σ_1. By Result 2.3 there is a generator Σ_2 of $\mathcal{Q}_n$ that contains P_2 and meets Σ_1 in a $(g-1)$-space. Moreover, if $Y \in \langle \alpha_t, P_1 \rangle$, then $P_2Y \subset \alpha_s$, and so is a line of $\mathcal{Q}_n$, hence by Result 2.3, $\langle \alpha_t, P_1 \rangle \subset \alpha_s \subset \Sigma_2$. So $\langle \alpha_t, P_1, P_2 \rangle \subset \Sigma_2$, and $\alpha_t = \Pi \cap \Sigma_1 \cap \Sigma_2 \cap \alpha_s$. Note that $\Pi \cap \Sigma_1 \cap \Sigma_2$ has dimension at least $g-2$. Further, if X is a point of $\Pi \cap \Sigma_1 \cap \Sigma_2$ not in α_s, and $Y \in \langle \alpha_t, P_1, P_2 \rangle$, then XY lies in Σ_2 and so is a line of $\mathcal{Q}_n$.

Repeat this process a total of $k \leq s-t$ times, until $\langle \alpha_t, P_1, \ldots, P_k \rangle = \alpha_s$. Let $H = \Pi \cap \Sigma_1 \cap \cdots \cap \Sigma_k$, so H has dimension $d \geq g-k \geq g-(s-t)$, $H \cap \alpha_s = \alpha_t$, and $\alpha_s = \langle \alpha_t, P_1, \ldots, P_k \rangle \subset \Sigma_k$. Note that $\dim H - \dim \alpha_t = d - t \geq g - (s-t) - t = g - s > 0$, so $H \backslash \alpha_t$ is non-empty. Let X be a point of H not in α_s, and let $Y \in \alpha_s$. So $X, Y \in \Sigma_k$, hence XY is a line of $\mathcal{Q}_n$. That is, $\langle X, \alpha_s \rangle$ is an $(s+1)$-space of $\mathcal{Q}_n$ and hence X is a type (ii) point. As $X \in H \subset \Pi$, Π contains at least one point of type (ii) as required. □

We now show that there are only two types of g-cliques in Γ_s, namely those of Class A and B described in Lemma 5.1.

Lemma 5.4 *Let $\mathcal{C}$ be a g-clique in Γ_s, $0 \leq s < g$, then $\mathcal{C}$ is a g-clique of Class A or B.*

Proof Let $\mathcal{C}$ be a g-clique of Γ_s and denote the subsets of vertices of $\mathcal{C}$ of type (i), (ii), (iii) by $\mathcal{C}_{\rm i}$, $\mathcal{C}_{\rm ii}$, $\mathcal{C}_{\rm iii}$ respectively. If $\mathcal{C}_{\rm iii} = \emptyset$, then by Lemma 5.2, $\mathcal{C}$ corresponds to a generator of $\mathcal{Q}_n$ containing α_s, and so is of Class A. So suppose $\mathcal{C}_{\rm iii} \neq \emptyset$.

We begin by constructing two generators of $\mathcal{Q}_n$ whose union contains the g-clique $\mathcal{C}$. Firstly, as $\mathcal{C}$ is a clique of Γ_s, the subset $\mathcal{C}_{\rm i} \cup \mathcal{C}_{\rm iii}$ is also a clique, so any two vertices of $\mathcal{C}_{\rm i} \cup \mathcal{C}_{\rm iii}$ are adjacent in Γ_s. As $\mathcal{C}_{\rm i} \cup \mathcal{C}_{\rm iii}$ contains only vertices of type (i) and (iii), in $\mathrm{PG}(n, 2)$, any two points of $\mathcal{C}_{\rm i} \cup \mathcal{C}_{\rm iii}$ lie on a line of $\mathcal{Q}_n$. Hence $\mathcal{C}_{\rm i} \cup \mathcal{C}_{\rm iii}$ is contained in a subspace of $\mathcal{Q}_n$ and so by [7, Theorem 22.4.1] is contained in a generator Π of $\mathcal{Q}_n$. Secondly, consider the set of points $\alpha_s \cup \mathcal{C}_{\rm ii}$ in $\mathcal{Q}_n$. Let $Q \in \mathcal{C}_{\rm ii}$, so Q has type (ii), and $\langle Q, \alpha_s \rangle$ is contained in $\mathcal{Q}_n$. Hence $\alpha_s \cup \mathcal{C}_{\rm ii}$ is contained in a subspace of $\mathcal{Q}_n$ and so is contained in a generator Σ of $\mathcal{Q}_n$. So we have $\mathcal{C} \subset \Pi \cup \Sigma$. To show that $\mathcal{C}$ is a clique of Class B, we need to show that $\Pi \cap \Sigma$ has dimension $g-1$.

We first show that $\mathcal{C}_{\rm ii}$ is not empty. Suppose $\mathcal{C}_{\rm ii} = \emptyset$, then $\mathcal{C} = \mathcal{C}_{\rm i} \cup \mathcal{C}_{\rm iii}$ is contained in the g-space Π. As $|\mathcal{C}| = 2^{g+1} - 1$, we have $\mathcal{C} = \mathcal{C}_{\rm i} \cup \mathcal{C}_{\rm iii} = \Pi$. However, by Lemma 5.3, Π contains at least one point of type (ii), a contradiction. Thus $\mathcal{C}_{\rm ii} \neq \emptyset$.

As $\mathcal{C}_{\rm ii}, \mathcal{C}_{\rm iii}$ are not empty, let $Q \in \mathcal{C}_{\rm ii}$ and $R \in \mathcal{C}_{\rm iii}$. As Q, R lie in a clique of Γ_s, they are adjacent in Γ_s. Hence by Corollary 4.5, QR is a 2-secant of $\mathcal{Q}_n$. As $Q \in \mathcal{C}_{\rm ii} \subset \Sigma \subset \mathcal{Q}_n$ and QR is a 2-secant, we have $R \notin \Sigma$. Similarly $R \in \mathcal{C}_{\rm iii} \subset \Pi \subset \mathcal{Q}_n$ and QR a 2-secant implies $Q \notin \Pi$. In summary, we have

$$\mathcal{C} \subset \Sigma \cup \Pi; \quad \mathcal{C}_{\rm i} \subset \alpha_s \cap \Pi \cap \Sigma; \quad \mathcal{C}_{\rm ii} \subset \Sigma \backslash \Pi; \quad \mathcal{C}_{\rm iii} \subset \Pi \backslash \Sigma.$$

Next we determine the size of $\mathcal{C}_{\rm i}$, $\mathcal{C}_{\rm ii}$ and $\mathcal{C}_{\rm iii}$. As $\mathcal{C}_{\rm iii} \neq \emptyset$, there is a point $R \in \mathcal{C}_{\rm iii}$, so $R \notin \Sigma$. By Result 2.3, there is a unique generator Π_1 of $\mathcal{Q}_n$ that contains R and meets Σ in a $(g-1)$-space denoted $H = \Sigma \cap \Pi_1$. There are two cases to consider as $H \cap \alpha_s$ has dimension s or $s-1$. If H contained α_s, then $\langle R, \alpha_s\rangle \subset \Pi_1$ would be a subspace of $\mathcal{Q}_n$, which implies that R is type (ii), a contradiction. Thus $H \cap \alpha_s$ is an $(s-1)$-space. If $P \in \mathcal{C}_{\rm i}$, then $P, R \in \mathcal{C}$, so P, R are adjacent in Γ_s and so PR is a line of $\mathcal{Q}_n$. Thus $P \in H$, and so $P \in H \cap \alpha_s$. Thus $\mathcal{C}_{\rm i} \subseteq H \cap \alpha_s$, and so $|\mathcal{C}_{\rm i}| \leq |H \cap \alpha_s| = 2^s - 1$. By the construction of H, each point in $H\backslash\alpha_s$ lies on a line of $\mathcal{Q}_n$ with R, and each point of $\Sigma\backslash(H \cup \alpha_s)$ lies on a 2-secant of $\mathcal{Q}_n$ with R. So the type (ii) points of $\mathcal{C}$ are contained in $\Sigma\backslash(H \cup \alpha_s)$. That is, $|\mathcal{C}_{\rm ii}| \leq |\Sigma\backslash(H \cup \alpha_s)| = (2^{g+1} - 1) - \big((2^g - 1) + 2^s\big) = 2^g - 2^s$.

As $\mathcal{C}_{\rm ii} \neq \emptyset$, there is a point $Q \in \mathcal{C}_{\rm ii}$, so $Q \in \Sigma\backslash\Pi$. By Result 2.3, there is a unique generator Σ_1 of $\mathcal{Q}_n$ that contains Q and meets Π in a $(g-1)$-space. Hence Q is on a line of $\mathcal{Q}_n$ with the $2^g - 1$ points of $\Pi \cap \Sigma_1$; and Q is on a 2-secant of $\mathcal{Q}_n$ with the $(2^{g+1} - 1) - (2^g - 1) = 2^g$ points of $\Pi\backslash\Sigma_1$. If R is a point of $\mathcal{C}_{\rm iii}$, then as $Q, R \in \mathcal{C}$, they are adjacent in Γ_s and so QR is a 2-secant of $\mathcal{Q}_n$. Hence the points of $\mathcal{C}_{\rm iii}$ lie in $\Pi\backslash\Sigma_1$, and so $|\mathcal{C}_{\rm iii}| \leq 2^g$.

As $|\mathcal{C}| = 2^{g+1} - 1$, we need equality in all three of these bounds, that is, $|\mathcal{C}_{\rm i}| = 2^s - 1$, $|\mathcal{C}_{\rm ii}| = 2^g - 2^s$, and $|\mathcal{C}_{\rm iii}| = 2^g$. Moreover,

$$\mathcal{C}_{\rm i} = \alpha_s \cap \Pi_1, \quad \mathcal{C}_{\rm ii} = \Sigma\backslash(\alpha_s \cup \Pi_1), \quad \mathcal{C}_{\rm iii} = \Pi\backslash\Sigma_1. \tag{3}$$

To show that $\mathcal{C}$ is a g-clique of Class B, we need to show that $\Pi = \Pi_1$ and $\Sigma = \Sigma_1$. Suppose that $\Pi \neq \Pi_1$, so $\Pi \cap \Pi_1$ has dimension at most $g-1$, that is $|\Pi \cap \Pi_1| \leq 2^g - 1$. As Π contains $\mathcal{C}_{\rm iii}$, and $|\mathcal{C}_{\rm iii}| = 2^g > |\Pi \cap \Pi_1|$, there exists a point $R' \in \mathcal{C}_{\rm iii}$ with $R' \in \Pi\backslash\Pi_1$. By Result 2.3, there exists a unique generator Π_2 of $\mathcal{Q}_n$ which contains R' and meets Σ in a $(g-1)$-space. Further, for each point $X \in \Sigma\backslash\Pi_2$, XR' is a 2-secant of $\mathcal{Q}_n$. Thus $\mathcal{C}_{\rm ii} \subset \Sigma\backslash\Pi_2$. By (3), $\mathcal{C}_{\rm ii} = \Sigma\backslash(\alpha_s \cup \Pi_1)$, moreover we have $|\Sigma\backslash(\alpha_s \cup \Pi_1)| = |\Sigma\backslash(\alpha_s \cup \Pi_2)|$. Hence $\Sigma \cap \Pi_1 = \Sigma \cap \Pi_2$, and so $\Pi_1 \cap \Pi_2$ is a $(g-1)$-space in Σ. Recall that $R \in \Pi_1$, and by assumption $R' \in \Pi_2\backslash\Pi_1$, so $\Pi_1 \neq \Pi_2$. Thus $\langle \Pi_1, \Pi_2\rangle$ is a $(g+1)$-space, and so by Result 2.2, meets $\mathcal{Q}_n$ in exactly the two generators Π_1, Π_2. Now $R, R' \in \mathcal{C}_{\rm iii}$, so $\{R, R'\}$ is an edge of Γ_s, and so RR' is a line of $\mathcal{Q}_n$. As $R' \in \Pi_2\backslash\Pi_1$, and RR' is a line of $\mathcal{Q}_n$ in $\langle \Pi_1, \Pi_2\rangle$, we have $R \in \Pi_2$. So $R \in \Pi_2 \cap \Pi_1 \subset \Sigma$, contradicting the choice of $R \notin \Sigma$. Hence $\Pi = \Pi_1$. Thus Σ meets Π in a $(g-1)$-space, so by the construction of Σ_1, we have $\Sigma = \Sigma_1$. Substituting into (3), we see that $\mathcal{C}$ is a g-clique of Class B. □

Lemma 5.5 *The maximum size of a clique in* Γ_s *is* $2^{g+1} - 1$.

Proof Suppose Γ_s, $s > 0$, contains a clique $\mathcal{K}$ of size 2^{g+1}. Let X be a vertex in $\mathcal{K}$, then $\mathcal{K}\backslash X$ is a g-clique, and so by Lemma 5.4, $\mathcal{K}\backslash X$ has Class A or B. Table 2 gives the number of vertices of each type in the two different g-cliques. As $s > 0$ and $\mathcal{K}\backslash X$ has Class A or B, $\mathcal{K}\backslash X$ contains vertices of both type (i) and (ii). Let P be a vertex of type (i) in $\mathcal{K}$ and Q a vertex of type (ii) in $\mathcal{K}$. If $\mathcal{K}\backslash P$ has Class A, then using Table 2, we see that $\mathcal{K}\backslash Q$ satisfies neither column, and so is not a g-clique of Γ_s, a contradiction. Similarly, if $\mathcal{K}\backslash P$ has Class B, then $\mathcal{K}\backslash Q$ satisfies neither column, and

Table 2: Number of vertices of each type in each g-clique

	g-clique A	g-clique B
vertex type (i)	$2^{s+1}-1$	2^s-1
vertex type (ii)	$2^{g+1}-2^{s+1}$	2^g-2^s
vertex type (iii)	0	2^g

so is not a g-clique of Γ_s. So there are no cliques of size 2^{g+1}, hence the g-cliques are the maximal cliques of Γ_s. A similar argument proves the result when $s=0$. □

In summary, we have classified the maximal cliques of Γ_s as follows.

Theorem 5.6 *Let $\mathcal{Q}_n$ be a non-singular quadric of* $\mathrm{PG}(n,2)$ *of projective index $g \geq 1$, and let Γ_s, $0 \leq s < g$, be the graph constructed in Theorem 3.3. If $\mathcal{C}$ is a maximal clique of Γ_s, then $\mathcal{C}$ is a g-clique of Class A or B.*

5.2 Counting maximal cliques

In the previous section, we classified the maximal cliques in the graph Γ_s, we count them here.

Theorem 5.7 *Let $\mathcal{Q}_n$ be a non-singular quadric in* $\mathrm{PG}(n,2)$ *of projective index $g \geq 1$. Let Γ be the point-graph of $\mathcal{Q}_n$ and let Γ_s, $0 \leq s < g$, be the graph constructed in Theorem* 3.3.

1. *Let $\mathcal{Q}_n = \mathcal{E}_{2r+1}$, then*
   - *(a) Γ has $(2^2+1)(2^3+1)\cdots(2^{r+1}+1)$ maximal cliques.*
   - *(b) Γ_s has $(2^2+1)(2^3+1)\cdots(2^{r-s}+1)\times\left(2^{r+2}-2^{r-s+1}+1\right)$ maximal cliques.*
2. *If $\mathcal{Q}_n = \mathcal{H}_{2r+1}$, then*
   - *(a) Γ has $(2^0+1)(2^1+1)\cdots(2^r+1)$ maximal cliques.*
   - *(b) Γ_s has $(2^0+1)(2^1+1))\cdots(2^{r-s-1}+1)\times\left(2^{r+1}-2^{r-s}+1\right)$ maximal cliques.*
3. *If $\mathcal{Q}_n = \mathcal{P}_{2r}$, then*
   - *(a) Γ has $(2^1+1)(2^2+1)\cdots(2^r+1)$ maximal cliques.*
   - *(b) Γ_s has $(2^1+1)(2^2+1)\cdots(2^{r-s-1}+1)\times\left(2^{r+1}-2^{r-s}+1\right)$ maximal cliques.*

Proof For part 1, we work in $\mathrm{PG}(2r+1,2)$ and let $\mathcal{Q}_n = \mathcal{E} = \mathcal{E}_{2r+1}$ have point-graph Γ. The maximal cliques of Γ correspond exactly to the generators of $\mathcal{E}$. By [7, Theorem 22.5.1], the number of generators of $\mathcal{E}$ is

$$(2^2+1)(2^3+1)\cdots(2^{r+1}+1)$$

proving 1(a). For part 1(b), let α_s be a subspace of $\mathcal{E}$, $0 \leq s < g$, and let Γ_s be the graph constructed from Γ as in Theorem 3.3. Let n_{A}, n_{B} be the number of maximal cliques of Γ_s of Class A and B respectively. By Lemma 5.1, $n_{\rm A}$ is equal to the number of generators of $\mathcal{E}$ that contain α_s, and so by [7, Theorem 22.4.7],

$$n_{\rm A} = (2^2+1)(2^3+1)\cdots(2^{r-s}+1). \tag{4}$$

To count the maximal cliques of Class B, by Lemma 5.1 we need to count the number of pairs of generators Σ, Π of $\mathcal{E}$ such that Σ contains α_s, and Π meets Σ in a $(g-1)$-space not containing α_s. The number of choices for Σ is the number of generators of $\mathcal{E}$ that contain α_s, which is given in (4), and is $n_{\rm A}$. Once Σ is chosen, we count the number of choices for Π. The number of $(g-1)$-spaces contained in Σ but not containing α_s equals the number of $(g-1)$-spaces contained in Σ minus the number of $(g-1)$-spaces contained in Σ which contain α_s. This is $(2^{g+1}-1)-(2^{g-s}-1) = 2^{g+1}-2^{g-s}$. By [7, Lemma 22.4.8], the number of generators of $\mathcal{E}$ that meet Σ in a fixed $(g-1)$-space is 4. Hence the number of choices for Π is $(2^{g+1}-2^{g-s}) \times 4 = 2^{g+3}-2^{g-s+2}$. As the projective index of $\mathcal{E}$ is $g = r-1$, we have $n_{\rm B} = n_{\rm A}\big(2^{g+3}-2^{g-s+2}\big) = n_{\rm A}\big(2^{r+2}-2^{r-s+1}\big)$. Hence the total number of maximal cliques of Γ_s is $n_{\rm A}+n_{\rm B} = n_{\rm A}\big(2^{r+2}-2^{r-s+1}+1\big)$ as required. This completes the proof of part 1. The proofs of parts 2 and 3 are similar. □

Theorem 5.8 *Let $\mathcal{Q}_n$ be a non-singular quadric in* $\mathrm{PG}(n,2)$ *of projective index $g \geq 1$. Let Γ_s, $0 \leq s < g$, be the graph constructed in Theorem* 3.3. *Let X be a fixed vertex of Γ_s, then the number of maximal cliques of Γ_s containing X according to the type of X is given in Table 3.*

Table 3: Number of maximal cliques of Γ_s containing X

$\mathcal{Q}_n$	type of X	$0 \leq s < g-1$	$s = g-1$
$\mathcal{E}_{2r+1}$	(i)	$(2^2+1)\cdots(2^{r-s}+1) \times \big(2^{r+1}-2^{r-s+1}+1\big)$	$5(2^{r+1}-7)$
	(ii)	$(2^2+1)\cdots(2^{r-s-1}+1) \times \big(2^{r+1}-2^{r-s}+1\big)$	$2^{r+1}-3$
	(iii)	$(2^2+1)\cdots(2^{r-s}+1)$	5
$\mathcal{H}_{2r+1}$	(i)	$(2^0+1)\cdots(2^{r-s-1}+1) \times \big(2^r-2^{r-s}+1\big)$	$2(2^r-1)$
	(ii)	$(2^0+1)\cdots(2^{r-s-2}+1) \times \big(2^r-2^{r-s-1}+1\big)$	2^r
	(iii)	$(2^0+1)\cdots(2^{r-s-1}+1)$	2
$\mathcal{P}_{2r}$	(i)	$(2^1+1)\cdots(2^{r-s-1}+1) \times \big(2^r-2^{r-s}+1\big)$	$3(2^r-3)$
	(ii)	$(2^1+1)\cdots(2^{r-s-2}+1) \times \big(2^r-2^{r-s-1}+1\big)$	2^r-1
	(iii)	$(2^1+1)\cdots(2^{r-s-1}+1)$	3

Proof First consider the case where $\mathcal{Q}_n = \mathcal{E} = \mathcal{E}_{2r+1}$ in $\mathrm{PG}(n,2) = \mathrm{PG}(2r+1,2)$. Let α_s be a subspace of $\mathcal{E}$, $0 \leq s < g$, and let Γ_s be the graph constructed from the point-graph Γ of $\mathcal{E}$, as in Theorem 3.3. Let P be a vertex of Γ_s of type (i), so in $\mathrm{PG}(2r+1,2)$, $P \in \alpha_s$. All the maximal cliques of Γ_s of Class A contain α_s. So by (4), P lies in $n_{\rm A} = (2^2+1)(2^3+1)\cdots(2^{r-s}+1)$ maximal cliques of Class A. To form a maximal clique of Γ_s of Class B that contains P, we need two generators Σ, Π of $\mathcal{E}$ such that Σ contains α_s, Π meets Σ in a $(g-1)$-space not containing α_s, and $P \in \Pi$. We count the number of pairs Σ, Π satisfying this. First, the number of choices for Σ equals the number of generators of $\mathcal{E}$ containing α_s which is n_{A}. The number of $(g-1)$-spaces of Σ that contain P is $2^g - 1$, and the number of $(g-1)$-spaces of Σ that contain α_s and P is $2^{g-s} - 1$. Hence the number of $(g-1)$-spaces of Σ that contain P, but do not contain α_s is $(2^g - 1) - (2^{g-s} - 1) = 2^g - 2^{g-s}$. By [7, Lemma 22.4.8], the number of generators of $\mathcal{E}$ that meet Σ in a fixed $(g-1)$-space is 4. In total, the number of maximal cliques of Class B containing P is $n_{\mathrm{A}} \times (2^g - 2^{g-s}) \times 4 = n_{\mathrm{A}}\big(2^{r+1} - 2^{r-s+1}\big)$ as $\mathcal{E}$ has projective index $g = r-1$. Hence the total number of maximal cliques of Γ_s containing P is $n_{\mathrm{A}}\big(2^{r+1} - 2^{r-s+1} + 1\big)$ as required.

Now let Q be a vertex of Γ_s of type (ii). The number of maximal cliques of Class A containing Q equals the number of generators of $\mathcal{E}$ containing α_s and Q which by [7, Theorem 22.4.7] is $(2^2+1)(2^3+1)\cdots(2^{r-s-1}+1)$. To count the maximal cliques of Γ_s that contain Q, we need to count pairs of generators Σ, Π of $\mathcal{E}$ such that Σ contains α_s and Q, and Π meets Σ in a $(g-1)$-space not containing α_s or Q. The number of choices for Σ is calculated above to be $(2^2+1)(2^3+1)\cdots(2^{r-s-1}+1)$. Further, the number of $(g-1)$-spaces in Σ is $2^{g+1} - 1$; the number of $(g-1)$-spaces of Σ containing α_s is $2^{g-s} - 1$; the number of $(g-1)$-spaces of Σ containing α_s and Q is $2^{g-s-1} - 1$; and the number of $(g-1)$-spaces of Σ containing Q is $2^g - 1$. Hence the number of $(g-1)$-spaces of Σ that do not contain α_s and do not contain Q is $(2^{g+1}-1) - (2^{g-s}-1) - (2^g - 1) + (2^{g-s-1} - 1) = 2^g - 2^{g-s-1}$. As before, each of these $(g-1)$-spaces lies in 4 suitable choices for the generator Π of $\mathcal{E}$. Hence the number of maximal cliques of Class B containing Q is $(2^2+1)(2^3+1)\cdots(2^{r-s-1}+1) \times (2^g - 2^{g-s-1}) \times 4 = (2^2+1)(2^3+1)\cdots(2^{r-s-1}+1)\big(2^{r+1} - 2^{r-s}\big)$ as $\mathcal{E}$ has projective index $g = r-1$. Hence the total number of maximal cliques containing Q is $(2^2+1)(2^3+1)\cdots(2^{r-s-1}+1)\big(2^{r+1} - 2^{r-s} + 1\big)$ as required.

Let R be a vertex of Γ_s of type (iii), so $\langle R, \alpha_s \rangle$ is not contained in $\mathcal{E}$, hence R is in zero maximal cliques of Class A. To count the maximal cliques of Γ_s of Class B containing R, we need to count pairs of generators Σ, Π of $\mathcal{E}$ such that Σ contains α_s, Π meets Σ in a $(g-1)$-space not containing α_s, and Π contains R. The number of choices for Σ equals the number of generators of $\mathcal{E}$ containing α_s which is n_{A} by (4). As Σ contains α_s, it contains no points of type (iii), so $R \notin \Sigma$. So by Result 2.3, there is a unique generator of $\mathcal{E}$ that contains R and meets Σ in a $(g-1)$-space denoted H. Further, if H contained α_s, then $\langle R, \alpha_s \rangle$ would be contained in $\mathcal{E}$, and so R would be type (ii), a contradiction, so H does not contain α_s. So for each Σ, there is a unique choice for Π that can be used to form a Class B maximal clique containing R. Hence the number of maximal cliques of Γ_s containing R is $n_{\mathrm{A}} = (2^2+1)(2^3+1)\cdots(2^{r-s}+1)$ as required. This completes the proof for the case $\mathcal{Q}_n = \mathcal{E}_{2r+1}$. The cases when $\mathcal{Q}_n$ is $\mathcal{H}_{2r+1}$ and $\mathcal{P}_{2r}$ are similar. □

6 The graphs Γ_s are all non-isomorphic

Theorem 6.1 *Let $\mathcal{Q}_n$ be a non-singular quadric in* PG$(n, 2)$ *of projective index $g \geq 1$. Let Γ be the point-graph of $\mathcal{Q}_n$ and let Γ_s, $0 \leq s < g$, be the graph constructed in Theorem 3.3. Then Γ_s is isomorphic to Γ if and only if $s = 0$.*

Proof We first show that $\Gamma_0 \cong \Gamma$. To construct Γ_0 from Γ, we let α_0 be a subspace of $\mathcal{Q}_n$ of dimension 0, so α_0 is a point which we denote P. We classify the points of $\mathcal{Q}_n$, and so the vertices of Γ, into type (i), (ii), (iii) with respect to $\alpha_0 = P$. The point P is the only point of $\mathcal{Q}_n$ of type (i). Note that lines in PG$(n, 2)$ contain exactly three points. Consider the involution ϕ acting on the vertices of Γ where: ϕ fixes vertices of type (i) and (iii); and ϕ maps a vertex Q of type (ii) to the vertex of type (ii) that corresponds to the third point of $\mathcal{Q}_n$ on the line PQ. The involution ϕ maps Γ to a graph Γ'. Incidence in Γ' is inherited from Γ, that is, points X and Y are adjacent in Γ (so XY is a line of $\mathcal{Q}_n$) if and only if vertices $\phi(X)$ and $\phi(Y)$ are adjacent in Γ'. The map ϕ is an isomorphism, so $\Gamma \cong \Gamma'$. We now show that $\Gamma' = \Gamma_0$.

By Corollary 4.5, we need to show that the edges of Γ' satisfy Table 1. First note that as there is only one point of type (i) in $\mathcal{Q}_n$, the first row of Table 1 is not relevant. Let Q_1, Q_2 be points of $\mathcal{Q}_n$ of type (ii), and let R, R' be points of $\mathcal{Q}_n$ of type (iii). The incidences in rows 4 and 5 of Table 1 hold in Γ, so as ϕ fixes points of type (i) and (iii), they also hold in Γ'.

To simplify notation, let $Q_1' = \phi^{-1}(Q_1)$ and $\phi^{-1}(Q_2) = Q_2'$. Consider row 2 of Table 1: $\{P, Q_1\}$ is an edge of Γ' if and only if $\{P, Q_1'\}$ is an edge of Γ if and only if $\{P, Q_1, Q_1'\}$ is a line of $\mathcal{Q}_n$. Hence it follows from the definition of ϕ that $\{P, Q_1\}$ is always an edge of Γ' as required.

Consider row 6 of Table 1: $\{Q_1, R\}$ is an edge of Γ' if $\{Q_1', R\}$ is an edge of Γ, that is, if $Q_1'R$ is a line of $\mathcal{Q}_n$. As R is type (iii), the plane $\langle P, Q_1', R\rangle$ is not contained in $\mathcal{Q}_n$, and so by Result 2.2 meets $\mathcal{Q}_n$ in exactly the lines PQ_1', $Q_1'R$. As Q_1 is the third point on the line PQ_1', the line Q_1R is a 2-secant of $\mathcal{Q}_n$ as required.

Consider row 3 of Table 1. Suppose $\{Q_1, Q_2\}$ is an edge of Γ', so $\{Q_1', Q_2'\}$ is an edge of Γ. If the line Q_1Q_2 contains P, then $Q_1' = Q_2$ and $Q_2' = Q_1$, so $\{Q_1, Q_2\}$ is an edge of Γ and so Q_1Q_2 is a line of $\mathcal{Q}_n$ as required. Now suppose Q_1Q_2 does not contain P. Then $\{Q_1', Q_2'\}$ an edge of Γ implies $Q_1'Q_2'$ is a line of $\mathcal{Q}_n$. Hence the plane $\langle P, Q_1', Q_2'\rangle$ contains at least three lines, namely PQ_1', PQ_2' and $Q_1'Q_2'$, and so by Result 2.2, is contained in $\mathcal{Q}_n$. Further, it contains Q_1 and Q_2, so Q_1Q_2 is a line of $\mathcal{Q}_n$ as required. Hence the edges of Γ' satisfy Table 1. So by Corollary 4.5, $\Gamma' = \Gamma_0$.

We now show that Γ_s with $1 \leq s < g$ is not isomorphic to the graph $\Gamma \cong \Gamma_0$ by considering the maximal cliques. We prove the case when $\mathcal{Q}_n = \mathcal{E} = \mathcal{E}_{2r+1}$, the cases where $\mathcal{Q}_n$ is $\mathcal{H}_{2r+1}$ or $\mathcal{P}_{2r}$ are similar. The number of maximal cliques in Γ and Γ_s are given in (1a) and (1b) of Theorem 5.7. These numbers are equal if and only if $2^{r+1} - 2^{r-s+1} + 1 = (2^{r-s+1} + 1) \cdots (2^r + 1)$. If $s \geq 1$, then the right hand side is $\geq 2^{2r+1}$, which is larger than the left hand side. So we have equality if and only if $s = 0$. Hence Γ_s with $1 \leq s < g$ is not isomorphic to Γ. □

Theorem 6.2 *Let $\mathcal{Q}_n$ be a non-singular quadric in* $\mathrm{PG}(n,2)$ *of projective index $g \geq 1$. Let Γ be the point-graph of $\mathcal{Q}_n$ and let Γ_s, $0 \leq s < g$, be the graph constructed in Theorem 3.3. Then the graphs $\Gamma_0, \Gamma_1, \ldots, \Gamma_{g-1}$ are distinct up to isomorphism.*

Proof We prove the case when $\mathcal{Q}_n = \mathcal{E} = \mathcal{E}_{2r+1}$, the cases where $\mathcal{Q}_n$ is $\mathcal{H}_{2r+1}$ or $\mathcal{P}_{2r}$ are similar. Let s_1, s_2 be two integers with $0 \leq s_1 < s_2 < g$. The number of maximal cliques in Γ_{s_1} and Γ_{s_2} are given in Theorem 5.7(1b). These two numbers are equal if and only if

$$2^{r+2} - 2^{r-s_2+1} + 1 \quad = \quad (2^{r-s_2+1}+1)\cdots(2^{r-s_1}+1)\big(2^{r+2} - 2^{r-s_1+1} + 1\big). \qquad (5)$$

As $s_1 < s_2$, the right hand side is greater than 2^{2r+2-s_1}, which is greater than 2^{r+1} as $s_1 < s_2 < g = r-1$. Hence the right hand side is greater than the left, so they cannot be equal. Thus Γ_{s_1} and Γ_{s_2} are not isomorphic if s_1 and s_2 are distinct. □

6.1 Kantor's graphs

In [8], Kantor constructs a strongly regular graph Γ_K from a non-singular quadric $\mathcal{Q}_n$ in $\mathrm{PG}(n,q)$ with the same parameters as the point-graph Γ of $\mathcal{Q}_n$. Kantor conjects that the graph Γ_K is not the same as Γ except in the case when $\mathcal{Q}_n = \mathcal{H}_7$. It is not known in general whether Γ_K is isomorphic to $\Gamma \cong \Gamma_0$. We show that Γ_K is not isomorphic to the graphs Γ_s when $s > 0$. Kantor's construction works when the quadric $\mathcal{Q}_n$ contains a spread, however, we do not need to describe the details of Kantor's graphs to prove non-isomorphism.

Theorem 6.3 *Let $\mathcal{Q}_n$ be a non-singular quadric in* $\mathrm{PG}(n,2)$ *of projective index $g \geq 1$. Let Γ_s, $0 < s < g$ be the graph constructed in Theorem 3.3. Let Γ_K be the graph constructed from $\mathcal{Q}_n$ in [8]. Then Γ_K is not isomorphic to Γ_s, $0 < s < g$.*

Proof We use [8, Lemma 3.3] which shows that the vertices of Γ_K can be partitioned into maximal cliques. We show that the vertices of Γ_s, $0 < s < g$, cannot be partitioned into maximal cliques. Let $\mathcal{C}, \mathcal{C}'$ be two maximal cliques of Γ_s. We consider three cases. If $\mathcal{C}, \mathcal{C}'$ are both of Class A, then they both contain α_s, and so are not disjoint. If $\mathcal{C}$ is Class A and $\mathcal{C}'$ is Class B, then $\mathcal{C}$ contains α_s, and $\mathcal{C}'$ meets α_s in a $(s-1)$-space. Hence as $s > 0$, $\mathcal{C}'$ contains at least one point of α_s, so $\mathcal{C}, \mathcal{C}'$ are not disjoint in this case.

Now consider the case where $\mathcal{C}, \mathcal{C}'$ are maximal cliques of Γ_s of Class B. Both $\mathcal{C}, \mathcal{C}'$ meet α_s in a subspace of dimension $s-1$. If $s \geq 2$, then two subspaces of dimension $s-1$ contained in an s-space meet in at least a point, and so $\mathcal{C}, \mathcal{C}'$ share at least a point. Thus if $s \geq 2$, any two maximal cliques of Γ_s share at least one vertex, and so the vertices of Γ_s cannot be partition into maximal cliques, and hence Γ_s, $2 \leq s < g$ is not isomorphic to Γ_K.

Now suppose $s = 1$, so α_1 is a line. A partition of the vertices of Γ_1 into maximal cliques partitions the points of α_1. As every maximal clique of Γ_1 contains a point of α_1, we are looking for a partition of Γ_1 into three maximal cliques of Class B, one through each point of α_1. We show there is no such partition. First, a maximal clique has $2^{g+1}-1$ points, so three pairwise disjoint maximal cliques contain $x = 3(2^{g+1}-1)$ points, with either $g = r-1$ or r. As $0 < s < g$, it follows that $g \geq 2$. Thus for the elliptic and parabolic case we have $r \geq 3$ and for the hyperbolic case we have $r \geq 2$. However, as $q = 2$, $\mathcal{E}_{2r+1}$ contains $2^{2r+1} - 2^r - 1$ points, $\mathcal{H}_{2r+1}$ contains $2^{2r+1} + 2^r - 1$ points and $\mathcal{P}_{2r}$ contains $2^{2r} - 1$ points. None of these numbers is equal to x when $r \geq 2$. Hence we cannot partition the vertices of Γ_s, $s > 0$ into maximal cliques. Thus by [8, Lemma 3.3], Γ_s is not isomorphic to Γ_K. □

7 The automorphism group of Γ_s

Let $\mathcal{Q}_n$ be a non-singular quadric in $\mathrm{PG}(n, 2)$ of projective index $g \geq 1$. Let Γ be the point-graph of $\mathcal{Q}_n$. Let α_s be an s-space contained in $\mathcal{Q}_n$, $0 \leq s < g$, construct the partition of the points of $\mathcal{Q}_n$ given in Definition 3.2, and let Γ_s be the graph constructed in Theorem 3.3. If $s = 0$, then by Theorem 6.1, $\Gamma_0 = \Gamma$ so $\mathrm{Aut}(\Gamma_0) = \mathrm{Aut}\,\Gamma$. In this section we determine the automorphism group of the graph Γ_s, $0 < s < g$.

First note that the group of collineations of $\mathrm{PG}(n, 2)$ fixing $\mathcal{Q}_n$ is $\mathrm{PGO}(n + 1, 2)$, see [7]. Moreover, if $n \geq 4$, then the group of automorphisms of Γ is $\mathrm{Aut}\,\Gamma \cong \mathrm{PGO}(n + 1, 2)$, see [10, Chapter 8].

The partition of the points of $\mathcal{Q}_n$ given in Definition 3.2 also partitions the vertices of Γ and Γ_s, $0 \leq s < g$. Vertices of type (i) in Γ correspond in $\mathrm{PG}(n, 2)$ to the points of α_s. Let $(\mathrm{Aut}\,\Gamma)_{\alpha_s}$ denote the subgroup of automorphisms of Γ that fix the set of vertices of type (i). As the graphs Γ, Γ_s have the same set of vertices, if ϕ is a map acting on the vertices of Γ, then ϕ is also a map acting on the vertices of Γ_s. We will prove the following relationship between their automorphism groups.

Theorem 7.1 *Let $\mathcal{Q}_n$ be a non-singular quadric in $\mathrm{PG}(n, 2)$ of projective index $g \geq 1$ with point-graph Γ. Let α_s be an s-space of $\mathcal{Q}_n$, $0 < s < g$, and let Γ_s be the graph constructed in Theorem 3.3. Then $\mathrm{Aut}(\Gamma_s) = (\mathrm{Aut}\,\Gamma)_{\alpha_s}$.*

In order to prove this theorem, we need a series of preliminary lemmas, the first relies on an application of Witt's Theorem, so we begin with a discussion on applying Witt's Theorem to non-singular quadrics of $\mathrm{PG}(n, 2)$, see [9, Chapter 7] for more details. Let V be a vector space of dimension $n + 1$ over $\mathrm{GF}(2)$, and let $f(x_0, \ldots, x_n)$ be a quadratic form on V with associated bilinear form $b(x, y) = b(x + y) - b(x) - b(y)$. The radical of f in V is the subspace $\mathrm{rad}\, f = \{u \in V : b(u, v) = 0 \text{ for all } v \in V\}$. Let U be a subspace of V and suppose there exists a linear isometry $\varphi\colon U \to V$ with respect to f (that is, φ is an invertible linear map and $f(u) = f(\varphi(u))$ for all $u \in U$. Then Witt's theorem says that there exists a linear isometry $\zeta\colon V \to V$ such that $\zeta(u) = \varphi(u)$ for all $u \in U$ if and only if $\varphi(U \cap \mathrm{rad}\, f) = \varphi(U) \cap \mathrm{rad}\, f$. We interpret this in the projective space $\mathrm{PG}(n, 2)$ associated with V. Let $\mathcal{Q}_n$ be a non-singular quadric in $\mathrm{PG}(n,2)$ with homogeneous equation $f(x_0,\ldots,x_n)=0$. If n is odd, then $\operatorname{rad} f=\emptyset$. If n is even, then $\mathcal{Q}_n=\mathcal{P}_n$ and $\operatorname{rad} f$ is the nucleus point N of $\mathcal{P}_n$. As an example, let Π_1,Π_2 be subspaces of $\mathcal{Q}_n$ of the same dimension. If $\mathcal{Q}_n$ has a nucleus N, then $N\notin\mathcal{Q}_n$, so neither Π_1 nor Π_2 contain N. As there exists a collineation of $\mathrm{PG}(n,2)$ that maps Π_1 to Π_2, it follows from Witt's theorem that there exists a collineation of $\mathrm{PG}(n,2)$ that fixes $\mathcal{Q}_n$ and maps Π_1 to Π_2. We use Witt's Theorem to prove the following lemma.

Lemma 7.2 *Let $\mathcal{Q}_n$ be a non-singular quadric in* $\mathrm{PG}(n,2)$ *of projective index $g\geq 1$. Let s be an integer, $0\leq s<g$, let α_s be an s-space of $\mathcal{Q}_n$, and partition the points of $\mathcal{Q}_n$ into types* (i), (ii), (iii) *as in Definition 3.2. Then the subgroup of* $\mathrm{PGO}(n+1,2)$ *fixing α_s is transitive on the points of each type.*

Proof Let P,P' be two points of $\mathcal{Q}_n$ of type (i), so $P,P'\in\alpha_s$. There is a collineation of $\mathrm{PG}(n,2)$ that fixes α_s, and maps P to P'. Hence by Witt's theorem, there is a collineation of $\mathrm{PG}(n,2)$ fixing α_s and $\mathcal{Q}_n$, and mapping P to P'. Hence $\mathrm{PGO}(n+1,2)_{\alpha_s}$ is transitive on the points of $\mathcal{Q}_n$ of type (i).

Let Q,Q' be points of $\mathcal{Q}_n$ of type (ii), so $\Pi=\langle Q,\alpha_s\rangle$ and $\Pi'=\langle Q',\alpha_s\rangle$ are $(s+1)$-spaces contained in $\mathcal{Q}_n$. There is a collineation of $\mathrm{PG}(n,2)$ that maps Π to Π', fixes α_s, and maps Q to Q'. Hence by Witt's Theorem, there is a collineation of $\mathrm{PG}(n,2)$ that fixes α_s and $\mathcal{Q}_n$, and maps Q to Q'. Hence $\mathrm{PGO}(n+1,2)_{\alpha_s}$ is transitive on the points of $\mathcal{Q}_n$ of type (ii).

Let R,R' be points of $\mathcal{Q}_n$ of type (iii), so $\Pi=\langle R,\alpha_s\rangle$ and $\Pi'=\langle R',\alpha_s\rangle$ are $(s+1)$-spaces which are not contained in $\mathcal{Q}_n$. Now Π is an $(s+1)$-space, and $\Pi\cap\mathcal{Q}_n$ contains α_s and the point $R\notin\alpha_s$, hence by Result 2.2, $\Pi\cap\mathcal{Q}_n$ is exactly two s-spaces. Similarly, $\Pi'\cap\mathcal{Q}_n$ is two s-spaces, one being α_s. So there is an automorphism of $\mathrm{PG}(n,2)$ that maps Π to Π', fixes α_s, and maps R to R'. As Π,Π' are not contained in $\mathcal{Q}_n$, in order to apply Witt's Theorem, we need to consider the nucleus N of $\mathcal{Q}_n$ when n is even. Suppose n is even, so $\mathcal{Q}_n=\mathcal{P}_n$, and $\mathcal{P}_n$ has nucleus a point $N\notin\mathcal{P}_n$. We show that neither Π nor Π' contain N. Let $P\in\alpha_s\subset\mathcal{P}_n$ and let Σ_P be the tangent hyperplane to $\mathcal{P}_n$ at P. So Σ_P contains N and all the lines of $\mathcal{P}_n$ through P. Let $\Sigma=\cap_{P\in\alpha_s}\Sigma_P$, then Σ contains N and points of type (i) and (ii), but no points of type (iii). As $\langle\alpha_s,N\rangle$ is an $(s+1)$-space contained in Σ, it contains no points of type (iii). As the $(s+1)$-space Π contains points of type (iii), Π meets $\langle\alpha_s,N\rangle$ in exactly the s-space α_s. Thus $N\notin\Pi$, and similarly $N\notin\Pi'$. Hence by Witt's Theorem there is a collineation of $\mathrm{PG}(n,2)$ that fixes α_s and $\mathcal{Q}_n$, and maps R to R'. Thus $\mathrm{PGO}(n+1,2)_{\alpha_s}$ is transitive on the points of $\mathcal{Q}_n$ of type (iii). □

We now show that if $s>0$, then $\mathrm{Aut}(\Gamma_s)$ has at least three orbits on the vertices of Γ_s, namely the vertices of each type.

Lemma 7.3 *For $0<s<g$, the vertices of Γ_s of different types lie in a different number of maximal cliques.*

Proof We prove the result for the case $\mathcal{Q}_n=\mathcal{E}_{2r+1}$, the cases when $\mathcal{Q}_n$ is $\mathcal{H}_{2r+1}$ and $\mathcal{P}_{2r}$ are similar. Comparing the number of cliques through vertices of type (i), (ii) and (iii) in Γ_s from Theorem 5.8, it is sufficient to show that $k_!, k_2, k_3$ are distinct where

$$k_1 = (2^{r-s}+1)(2^{r+1}-2^{r-s+1}+1), \quad k_2 = 2^{r+1}-2^{r-s}+1, \quad k_3 = 2^{r-s}+1.$$

If $0 < s < g-1$, then $k_1-k_2 = 2^{r-s}(2^{r+1}-2^{r-s+1}) > 0$ and $k_2-k_3 = 2^{r+1}-2^{r-s+1} > 0$. Hence $k_1 > k_2 > k_3$, that is vertices of different types lie in a different number of maximal cliques. If $0 < s = g-1$, then $r \geq 3$ and so k_1, k_2, k_3 are distinct. □

Lemma 7.4 *If* $0 < s < g$, *then* $(\mathrm{Aut}\,\Gamma)_{\alpha_s} \subseteq \mathrm{Aut}(\Gamma_s)$. *Further,* $\mathrm{Aut}(\Gamma_s)$ *has exactly three orbits on the vertices of* Γ_s, *namely the vertices of each type.*

Proof Recall that Γ is the point-graph of a non-singular quadric $\mathcal{Q}_n$ in $\mathrm{PG}(n,2)$ with projective index $g \geq 1$; α_s is an s-space contained in $\mathcal{Q}_n$; and the vertices of Γ are partitioned into types (i), (ii) and (iii) as given in Definition 3.2. Note that as $0 < s < g$, we need $g > 1$, and so $n \geq 5$.

Let $\phi \in (\mathrm{Aut}\,\Gamma)_{\alpha_s}$, so ϕ is an automorphism of Γ that fixes the set of vertices of Γ of type (i). As Γ, Γ_s have the same set of vertices, ϕ acts on the vertices of Γ_s, and fixes the set of vertices of Γ_s of type (i). Further, ϕ induces a bijection denoted $\bar{\phi}$ acting on the points of $\mathcal{Q}_n$ and fixing α_s. As $n \geq 5$, we have $\mathrm{Aut}\,\Gamma \cong \mathrm{PGO}(n+1,2)$ (see [10, Chapter 8]) so $\bar{\phi} \in \mathrm{PGO}(n+1,2)_{\alpha_s}$. By Lemma 7.2, $\bar{\phi}$ preserves the type of a point in $\mathcal{Q}_n$, hence ϕ preserves the type of a vertex in Γ_s. By Corollary 4.5, the edges of Γ_s are described in Table 1. As the collineation $\bar{\phi}$ maps lines (respectively 2-secants) of $\mathcal{Q}_n$ to lines (2-secants) of $\mathcal{Q}_n$, the map ϕ preserves adjacencies and non-adjacencies of vertex pairs of Γ_s. That is $\phi \in \mathrm{Aut}(\Gamma_s)$, and so $(\mathrm{Aut}\,\Gamma)_{\alpha_s} \subseteq \mathrm{Aut}(\Gamma_s)$.

Further, by Lemma 7.2, $\mathrm{PGO}(n,2)_{\alpha_s}$ is transitive on the points of $\mathcal{Q}_n$ of each type, so $(\mathrm{Aut}\,\Gamma)_{\alpha_s}$ is transitive on the vertices of Γ of each type. Hence $\mathrm{Aut}(\Gamma_s)$ has at most three orbits on the vertices of Γ_s. By Lemma 7.3, $\mathrm{Aut}(\Gamma_s)$ has at least three orbits on the vertices of Γ_s. Hence $\mathrm{Aut}(\Gamma_s)$ has exactly three orbits on the vertices of Γ_s, namely the vertices of each type. □

Lemma 7.5 *For* $0 \leq s < g$, $\mathrm{Aut}(\Gamma_s) \subseteq \mathrm{Aut}\,\Gamma$.

Proof First note that if $s = 0$, then by Theorem 6.1, $\Gamma_0 = \Gamma$ so $\mathrm{Aut}(\Gamma_0) = \mathrm{Aut}\,\Gamma$. Suppose $s > 0$, and let $\phi \in \mathrm{Aut}(\Gamma_s)$. As Γ and Γ_s have the same set of vertices, ϕ is a bijection acting on the vertices of Γ. We show that ϕ preserves adjacencies and non-adjacencies of vertices in Γ.

By Corollary 4.5 and Table 1, the only difference in adjacencies between vertices in Γ and Γ_s are between a vertex of type (ii) and a vertex of type (iii). Let X, Y be two vertices of Γ, there are two cases to consider. Firstly, if the pair X, Y consists of one vertex of type (ii) and one vertex of type (iii), then X, Y are adjacent in Γ if and only if X, Y are non-adjacent in Γ_s. Secondly, if the pair X, Y does not consist of one vertex of type (ii) and one vertex of type (iii), then X, Y are adjacent in Γ if and only if X, Y are adjacent in Γ_s. In either case, as ϕ preserves adjacency and non-adjacency in Γ_s, ϕ preserves the adjacency or non-adjacency of the vertex pair X, Y in Γ. Hence $\phi \in \operatorname{Aut}\Gamma$ as required. □

Proof of Theorem 7.1 By Lemma 7.5, $\operatorname{Aut}(\Gamma_s) \subseteq \operatorname{Aut}\Gamma$, and so $(\operatorname{Aut}(\Gamma_s))_{\alpha_s} \subseteq (\operatorname{Aut}\Gamma)_{\alpha_s}$. As $s > 0$, by Lemma 7.4, $\operatorname{Aut}(\Gamma_s)$ fixes the set of vertices of type (i), that is $(\operatorname{Aut}(\Gamma_s))_{\alpha_s} = \operatorname{Aut}(\Gamma_s)$, hence $\operatorname{Aut}(\Gamma_s) \subseteq (\operatorname{Aut}\Gamma)_{\alpha_s}$. By Lemma 7.4, $(\operatorname{Aut}\Gamma)_{\alpha_s} \subseteq \operatorname{Aut}(\Gamma_s)$, hence $(\operatorname{Aut}\Gamma)_{\alpha_s} = \operatorname{Aut}(\Gamma_s)$ as required. □

Finally we show that given a graph Γ_s, we can reconstruct the graph Γ and the quadric $\mathcal{Q}_n$. If $s = 0$ then $\Gamma = \Gamma_0$ by Theorem 6.1. So suppose $0 < s < g$, and define a graph Γ whose vertices are the vertices of Γ_s. The proof of Lemma 7.3 shows that the number of maximal cliques through a vertex of Γ_s of type (i) is greater than the number of maximal cliques through a vertex of type (ii), which is greater than the number of maximal cliques through a vertex of type (iii). Hence we can partition the vertices of Γ_s into their types by using the number of maximal cliques through them. Define the edges of Γ to be the same as the edges of Γ_s, except swapping the adjacencies between vertices of type (ii) and (iii). Then by Corollary 4.5, Γ is the point-graph of the quadric $\mathcal{Q}_n$ used to construct Γ_s. We can now reconstruct the quadric $\mathcal{Q}_n$ from Γ as follows. The maximal cliques of Γ are exactly the generators of $\mathcal{Q}_n$ in $\mathrm{PG}(n, 2)$. By intersecting the generators of $\mathcal{Q}_n$, we can recover firstly the $(g-1)$-spaces of $\mathcal{Q}_n$, and so on, constructing the lattice of subspaces of the generators. Hence we can construct the points of $\mathcal{Q}_n$, all the lines contained in $\mathcal{Q}_n$, the planes contained in $\mathcal{Q}_n$, ..., the g-spaces contained in $\mathcal{Q}_n$.

8 Conclusion

In summary, Table 4 lists the parameters of the strongly regular graphs arising from the point-graph of each type of non-singular quadric. Further, we list the number of new non-isomorphic graphs with these parameters arising from our construction (that is, not including $\Gamma_0 = \Gamma$).

Table 4: Parameters of the strongly regular graphs Γ_s, $0 \le s < g$

quadric	$\mathcal{E}_{2r+1}$, $r \ge 2$	$\mathcal{H}_{2r+1}$, $r \ge 1$	$\mathcal{P}_{2r}$, $r \ge 2$
v	$2^{2r+1} - 2^r - 1$	$2^{2r+1} + 2^r - 1$	$2^{2r} - 1$
k	$2^{2r} - 2^r - 2$	$2^{2r} + 2^r - 2$	$2^{2r-1} - 2$
λ	$2^{2r-1} - 2^r - 3$	$2^{2r-1} + 2^r - 3$	$2^{2r-2} - 3$
μ	$2^{2r-1} - 2^{r-1} - 1$	$2^{2r-1} + 2^{r-1} - 1$	$2^{2r-2} - 1$
number of new non-isomorphic graphs	$r-2$	$r-1$	$r-2$

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