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A note on the number of edges in a Hamiltonian graph with no repeated cycle length

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Abstract

Let G be a graph obtained by adding chords to a cycle of length n. Markström asked for the maximum number of edges in G if there are no two cycles in G with the same length. A simple counting argument shows that such a graph can have at most $n + \sqrt{2n} + 1$ edges. Using difference sets in $\mathbb{Z}_n$, we show that for infinitely many n, there is an n-vertex Hamiltonian graph with $n + \sqrt{n - 3/4} - 3/2$ edges and no repeated cycle length.

1 Introduction

Let G be a graph with n vertices. The *cycle spectrum* of G, which we denote by $\mathcal{S}(G)$, is the set of all $k \in \{3, 4, \ldots, n\}$ for which there is a cycle of length k in G. Some of the most basic structural properties of a graph can be stated in terms of the cycle spectrum: G is bipartite if and only if $\mathcal{S}(G)$ contains no odd integer, G is a tree if and only if $\mathcal{S}(G) = \emptyset$, G is Hamiltonian if and only if $n \in \mathcal{S}(G)$, and finally, G is pancyclic if and only if $\mathcal{S}(G) = \{3, 4, \ldots, n\}$. In addition, one may also be interested in the number of cycles in G of a given length. In this case, it is natural to consider the multiset version of $\mathcal{S}(G)$, which we denote by $\mathcal{S}^m(G)$. More precisely, an integer k appears l times in $\mathcal{S}^m(G)$ if G has exactly l cycles of length k. The n-vertex graph G is *uniquely pancyclic* if

$$\mathcal{S}^m(G) = \{3, 4, \ldots, n\}.$$

That is, for every $k \in \{3, 4, \ldots, n\}$, G has exactly one cycle of length k. The existence of uniquely pancyclic graphs has been studied by Shi [9] and Markström [8]. It is an open problem of whether or not there exists infinitely many uniquely pancyclic

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graphs. Currently, only seven uniquely pancyclic graphs are known (see [8] for these graphs). In addition to existence questions, one can also ask extremal type questions such as how large or small $\mathcal{S}(G)$ can be given that G has a fixed number of edges (see [1], Problem 4.3). In the other direction, what is the maximum number of edges in an n-vertex graph with no repeated cycle length? According to Bondy and Murty, this question was asked by Erdős (see [2], Problem 11 on page 247). Let us write $f(n)$ for the maximum number of edges in an n-vertex graph with no repeated cycle length. The best known lower bound on $f(n)$ is due to Lai [6] who proved that

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{7654}{19071}}. \tag{1}$$

This disproved the earlier conjecture of Lai [5] that $\lim_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} = \sqrt{2.4}$. A result of Boros, Caro, Füredi, and Yuster [3] implies that

$$f(n) \leq n + (1.98 + o(1))\sqrt{n}.$$

Determining an asymptotic formula for $f(n)$ is an unsolved problem.

While studying uniquely pancyclic graphs, Markström [8] proved an upper bound on the number of chords in such a graph (since a uniquely pancyclic graph of order n must contain a cycle of length n, it may be constructed by adding chords to a cycle of length n). His upper bound is exceeded by Lai's lower bound on $f(n)$. The reason for this is that Lai's construction that proves (1) has no cycle of length $\Omega(n)$. This motivated Markström to pose the problem of determining the maximum number of edges in an n-vertex Hamiltonian graph with no repeated cycle length (see Problem 2.2 in [8]). This problem was reiterated in the suvey of Lai and Liu [7]. Let $g(n)$ be the maximum number edges in an n-vertex, Hamiltonian graph with no repeated cycle length. Using difference sets in the cyclic group $\mathbb{Z}_n$, we prove the following.

Theorem 1.1 *If q is a power of a prime and $n = q^2 + q + 1$, then*

$$g(n) \geq n + \sqrt{n - 3/4} - 3/2.$$

If G is an n-vertex graph with no repeated cycle length, then we can obtain an upper bound on the number of edges of G as follows. We view G as obtained from C_n by adding some number of chords, say k. Each pair of chords determines at least one cycle and so, since no cycle length is repeated, $\binom{k}{2} < n$. This gives the upper bound $g(n) < n + \sqrt{2n} + 1$. This argument, which can be refined (see [8]) shows that Theorem 1.1 gives the correct order of magnitude of $g(n)$. Like in the case with $f(n)$, we suspect that determining an asymptotic formula for $g(n)$ could be difficult, but at the same time, it is possible that adding the Hamiltonicity constraint makes the problem easier. In the next section, we give the proof of Theorem 1.1.

2 Proof of Theorem 1.1

For $n \geq 3$, we write C_n for the cycle of length n. We will always assume that the vertices of C_n are $\{1, 2, \ldots, n\}$, and the edges are $\{i, i+1\}$ for $1 \leq i \leq n-1$ together with $\{n, 1\}$. If C is a cycle whose edges are

$$\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{k-1}, x_k\}, \{x_k, x_1\},$$

then we write $x_1, x_2, x_3, \ldots, x_k, x_1$ for C.

Definition 2.1 *Given a positive integer $n \geq 4$ and a set $S \subseteq \{3, 4, \ldots, n-1\}$, let $G_n(S)$ be the graph obtained by adding the edges $\{1, a\}$ to the cycle C_n for each $a \in S$.*

The graph that we construct will be obtained by choosing S and n carefully. The next definition lists the conditions that we need S to satisfy in order for $G_n(S)$ to have no repeated cycle length.

Definition 2.2 *Let $n \geq 4$ and $S \subseteq \{3, 4, \ldots, n-1\}$. We say that S is a* distinct cycle set *if the following two conditions hold:*

1. *the differences $b - a$ with $b, a \in S$ and $b > a$ are all distinct,*
2. *the sets S, $S_n^{\star} := \{n + 2 - a : a \in S\}$, and $S^{-} := \{b - a + 2 : a, b \in S, b > a\}$, are pairwise disjoint.*

Lemma 2.3 *If $n \geq 4$ and $S \subseteq \{3, 4, \ldots, n-1\}$ is a distinct cycle set, then the graph $G_n(S)$ has $n + |S|$ edges and no two cycles in $G_n(S)$ have the same length.*

Proof. It is clear from the definitions that $G_n(S)$ has $n + |S|$ edges. We must show that no two cycles have the same length. Observe that all chords in $G_n(S)$ are incident to the vertex 1. Thus, any cycle in $G_n(S)$ must contain the vertex 1, and furthermore, must pass through exactly zero, one, or two chords. There is only one cycle that contains no chords, namely $1, 2, 3, \ldots, n, 1$. There are two types of cycles that pass through exactly one chord. Given $a \in S$, the sequence $1, 2, \ldots, a-1, a, 1$ forms a cycle of length a and we call this a cycle of Type 1. Also, the sequence $1, a, a+1, \ldots, n-1, n, 1$ forms a cycle of length $n + 2 - a$ and we call a cycle of this form Type 2. The cycles in $G_n(S)$ that pass through exactly two chords, which we call Type 3, are of the form $1, a, a+1, \ldots, b-1, b, 1$ where a, b are elements of S with $b > a$. In short,

- Type 1 are cycles of the form $1, 2, 3, \ldots a, 1$ where $a \in S$ and have length a,
- Type 2 are cycles of the form $1, a, a+1, \ldots, n-1, n, 1$ where $a \in S$ and have length $n + 2 - a$, and
- Type 3 are cycles of the form $1, a, a+1, \ldots, b-1, b, 1$ where $b > a$ are in S and have length $b - a + 2$.

No two distinct cycles of Type 1 will have the same length. If a Type 1 has the same length as a Type 2, then there are elements $a, b \in S$ with $a = n + 2 - b$. This implies $S \cap S_n^\star \neq \emptyset$ which cannot occur since S is a distinct cycle set. Similarly, if a Type 1 has the same length as a Type 3, then $S \cap S^- \neq \emptyset$ which cannot occur. No two distinct cycles of Type 2 will have the same length. If a Type 2 has the same length as a Type 3, then $S_n^\star \cap S^- \neq \emptyset$. Lastly, if two distinct cycles of Type 3 have the same length, then there are elements $a, b, c, d \in S$ with $b - a + 2 = d - c + 2$ and $b > a$, $d > c$. This cannot occur since the differences $b - a$ with $b > a$ and $a, b \in S$ are all distinct. ∎

Before our next lemma, we recall the definition of a perfect difference set. A set $A \subset \mathbb{Z}_n$ is a *perfect difference set* if every nonzero element of $\mathbb{Z}_n$ can be written uniquely as a difference of two elements of A.

Lemma 2.4 *If $n \geq 4$ and $A \subset \mathbb{Z}_n$ is a perfect difference set, then there is a distinct cycle set $S \subseteq \{3, 4, \ldots, n-1\}$ with $|S| \geq |A| - 2$.*

Proof. Let $A \subseteq \mathbb{Z}_n$ be a perfect difference set. There is a unique ordered pair $(a_0, b_0) \in A \times A$ such that $a_0 - b_0 \equiv 2 (\text{mod } n)$. Let

$$B = \{a - b_0 (\text{mod } n) : a \in A\}.$$

Since B is a translate of A, B is a perfect difference set in $\mathbb{Z}_n$. Observe that B contains

$$a_0 - b_0 \equiv 2 (\text{mod } n) \text{ and } b_0 - b_0 \equiv n (\text{mod } n).$$

We may view B as a subset of $\{1, 2, \ldots, n\}$ and we let $S = B \backslash \{2, n\}$. The set S has at least $|A| - 2$ elements, and has the property that all of the differences $b - a$ with $b, a \in S$ and $b > a$ are distinct. To complete the proof, we must show that the sets S, S^-, and $S_n^\star$ are pairwise disjoint (see Definition 2.2 for the definitions of S^- and $S_n^\star$). In each of the three cases, we will argue by contradiction.

If $S \cap S^- \neq \emptyset$, then there are elements $a, b, c \in S$ with $c = b - a + 2$ where $b > a$. The equation $c = b - a + 2$ implies

$$c - 2 \equiv b - a (\text{mod } n).$$

Each of a, b, c, and 2 belong to B and since B is a subset of a perfect difference set, we must have $c = 2$ or $b = 2$. However, both c and b belong to S and $2 \notin S$. We conclude that $S \cap S^- = \emptyset$.

Suppose $S \cap S_n^\star \neq \emptyset$. There are elements $a, b \in S$ with $b = n + 2 - a$. This implies $b \equiv n + 2 - a (\text{mod } n)$ so that $b - 2 \equiv n - a (\text{mod } n)$. Now $b, 2, n$, and a are all elements of B so that $b = 2$ or $a = 2$. Again, this is a contradiction since $a, b \in S$ but $2 \notin S$.

Lastly, suppose that $S^- \cap S_n^\star \neq \emptyset$. There are elements $a, b, c \in S$ with $b - a + 2 = n + 2 - c$. We can cancel 2 and then take the resulting equation modulo n to get

$$b - a \equiv n - c (\text{mod } n).$$

The elements b, a, n, and c all belong to B. We must have $b = n$ or $n = c$, but $n \notin S$ so this cannot occur.

The preceding three paragraphs show that the sets S, S^-, and $S_n^\star$ are all pairwise disjoint. Therefore, S is a distinct cycle set. ■

Proof of Theorem 1.1. Whenever q is a power of a prime, there is a perfect difference set $A_q \subseteq \mathbb{Z}_{q^2+q+1}$ with $q+1$ elements. This classical result is due to Singer [10]. By Lemma 2.4, there is a distinct cycle set $S_q \subseteq \{3, 4, \dots, q^2+q\}$ with $|S_q| = q-1$. By Lemma 2.3, the graph $G_{q^2+q+1}(S_q)$ has q^2+2q edges and no repeated cycle length. Therefore,

$$g(q^2+q+1) \geq q^2+2q$$

whenever q is a power of a prime. ■

3 Concluding Remarks

Theorem 1.1 implies that

$$\limsup_{n\to\infty} \frac{g(n)-n}{\sqrt{n}} \geq 1. \tag{2}$$

Our approach, that of taking C_n and adding chords incident to a single vertex, will not lead to improvements upon (2). This is because for any $S \subseteq \{3, 4, \dots, n\}$, if $G_n(S)$ has no repeated cycle lengths, then S forms a Sidon set in $\{1, 2 \dots, n\}$. This is a set with the property that all sums of pairs of elements are distinct. A famous result of Erdős and Turán [4] says that a Sidon set in $\{1, 2, \dots, n\}$ has at most $(1+o(1))\sqrt{n}$ elements and thus, $|S| \leq (1+o(1))\sqrt{n}$.

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