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Equivalence classes of eccentric digraphs

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Dedicated to the memory of Mirka Miller

Abstract

The eccentric digraph operator takes a graph G (either directed or undirected) as a basis and transforms it into a digraph $ED(G)$ with the same vertices as G and where there is an arc from a vertex u to a vertex v if and only if v is a farthest vertex from u in G, that is, v is an eccentric vertex of u. The eccentric digraph $ED(G)$ induces a partition of the set of all digraphs of given order. In this paper, we deal with some properties of the partition.

1 Introduction: On Mirka Miller's eccentric digraph research

In the early 2000s Mirka Miller gave a talk about eccentric digraphs in my faculty at Lleida University, Spain. It was the very first time that I met Mirka and she was able to captivate every student mind into graph theory research through her talk. We started a succesful research line in eccentric digraphs (together with Joan Gimbert and Joe Ryan) and Mirka became my PhD external supervisor. Five papers on eccentric digraphs, including one manuscript in a conference proceedings, were coauthored by Mirka ([4, 5, 8, 9, 10]). Since Mirka initiated this line of reseach, a lot of papers related to eccentric digraphs have been published, including a survey [11]. Mirka posted a lot of open problems in this area; some of them were successfully resolved and some others remain as open questions. One of these remaining open questions is related to the equivalence classes of eccentric digraphs (one of Mirka's favourite problems). We give a brief description of this problem and some new results in Section 2.

Buckley [6] defines the *eccentric digraph* of a graph G, denoted by $ED(G)$, as the digraph on the same vertex set as G but with an arc from a vertex u to a vertex v if and only if v is a farthest vertex from u in G, that is, $\mathrm{dist_G}(u,v) = e(u)$. If there is no $u \to v$ path in G then $\mathrm{dist_G}(u,v) = \infty$ and hence v is an eccentric vertex of u. Before Buckley's definition, several digraph operators involving distance between vertices in a digraph had been defined for distinct purposes during the last fifty years. In this context, Singleton defined the *antipodal graph* $A(G)$ of a graph G as a new graph with the same vertex set as G, and where there is an edge joining two vertices if and only if they are antipodal, that is, they are at maximum distance from each other. Antipodal graphs were used to prove that there are no irregular Moore graphs (see [16]). Later, Johns and Sleno [13] extended that definition to directed graphs and they started some theoretical research about this operator, subsequently continued by Acharya and Acharya [1], Rajendran [15], Aravamudhan and Rajendran [2, 3], Johns [12] and Chartrand et al. [7]. Since Boland and Miller considered the eccentric digraph of any digraph in [5], several authors have been working on this particular digraph operator (see [8, 9, 10]). Other related digraph operators involving distance between vertices such as the *super-eccentric graph* and the *radial graph* have been also defined. The reader is referred to [14] for a brief history of these operators and for information about the $\mathcal{P}$-metric operator as a generalization of every digraph operator involving distance between vertices in a digraph.

Given a positive integer k, the k^{th} iterated eccentric digraph of G is written as $ED^k(G) = ED(ED^{k-1}(G))$ where $ED^0(G) = G$. An interesting line of investigation concerning the iterated sequence of eccentric digraphs was initiated in [10]. Thus, since there is a finite number of distinct digraphs with a given vertex set, for every digraph G there exist smallest integer numbers $p > 0$ and $t \geq 0$ such that $ED^t(G) = ED^{p+t}(G)$. We call $p = p(G)$ the *period* of G and $t = t(G)$ the *tail* of G. In the definitions just given, we assume that the vertices of the graphs are labelled. It is also natural to consider the corresponding unlabelled version. For every digraph G there exist smallest integer numbers $p' > 0$ and $t' \geq 0$ such that $ED^{t'}(G) \cong ED^{p'+t'}(G)$, where $\cong$ denotes graph isomorphism. We call p' the *iso-period* of G and t' the *iso-tail* of G. These quantities are denoted by $p'(G)$ and $t'(G)$, respectively. Clearly $p'(G) \mid p(G)$. In [10] it is shown that the period of the odd cycles $p(C_{2m+1})$ is $k+1$ if $m = 2^k$, proving that the period may take on any value. This is also true for the iso-period, as was demonstrated in [9]. The same question remains, as an open problem, for $t(G)$, since the equality $t'(G) = t(G)$ was proved in [9].

2 Equivalence classes of eccentric digraphs

The following equivalence relation can be defined on the set of all labelled digraphs of a given order:

$$G_1 \backsim G_2 \text{ if and only if } ED^m(G_1) = ED^k(G_2) \text{ for } m,k \in \mathbb{Z}^+.$$

Of course, one can consider the analogous definition for the corresponding unlabelled version:

$$G_1 \backsim G_2 \text{ if and only if } ED^m(G_1) \cong ED^k(G_2) \text{ for } m,k \in \mathbb{Z}^+.$$

Hence the eccentric digraph operator ED induces a partition of the set of all digraphs of a given order. This partition is shown in [10] for order 3. We will focus on the unlabelled version of the definition. Let $[G]$ denote the equivalence class of (unlabelled) G induced by ED. Some questions regarding the size of $[G]$ are the following:

Problem 2.1 ([10]) Among all digraphs G on n vertices, what is the minimum, maximum and average cardinality of $[G]$?

An equivalence class $[G]$ is said to be a *periodic class* if every digraph in the class is periodic, that is, $t'(H) = 0$ for all $H \in [G]$. A digraph G is a *fixed point* if $ED(G) \cong G$.

Problem 2.2 ([10]) Which digraphs are fixed points? For general n, identify some periodic classes.

One way to study these problems is to define a directed graph as in the next proposition.

Proposition 2.3 *Let $\mathcal{D}_n = (\mathcal{V}, \mathcal{A})$ be the directed graph with vertex set $\mathcal{V}$ consisting of all the non-isomorphic digraphs of order n, and arc set $\mathcal{A}$ defined by having an arc from G to $ED(G)$ for every $G \in \mathcal{V}$. Then the following conclusions hold:*

- $|\mathcal{A}| = |\mathcal{V}|$;
- *there is a directed loop from G to G whenever $G \cong ED(G)$; and*
- *the weakly connected components of $\mathcal{D}_n$ are the equivalence classes induced by the eccentric digraph operator.*

PROOF: Every digraph G has a unique eccentric digraph $ED(G)$ and hence the number of arcs of $\mathcal{D}_n$ is precisely the number of non-isomorphic digraphs of order n. By definition, when $G \cong ED(G)$ there is a directed loop from G to itself. Finally, if G_1 and G_2 are two digraphs belonging to the same equivalence class $[G]$, then there exists a digraph H such that $ED^{m_1}(G_1) \cong ED^{k_1}(H)$ and $ED^{m_2}(G_2) \cong ED^{k_2}(H)$, that is, they belong to the same weakly connected component of $\mathcal{D}_n$. □

We have developed an algorithm in Python (using NetworkX library) to build $\mathcal{D}_n$ for $n \leq 6$. There are 6, 22 and 194 different equivalence classes of $\mathcal{D}_n$, for $n = 3, 4$ and 5, respectively. Some information about their cardinalities is presented next.

2.1 Maximal equivalence classes

A maximal equivalence class contains the largest number of digraphs in a weakly connected component of $\mathcal{D}_n$. There is an equivalence class containing 41 digraphs in $\mathcal{D}_4$

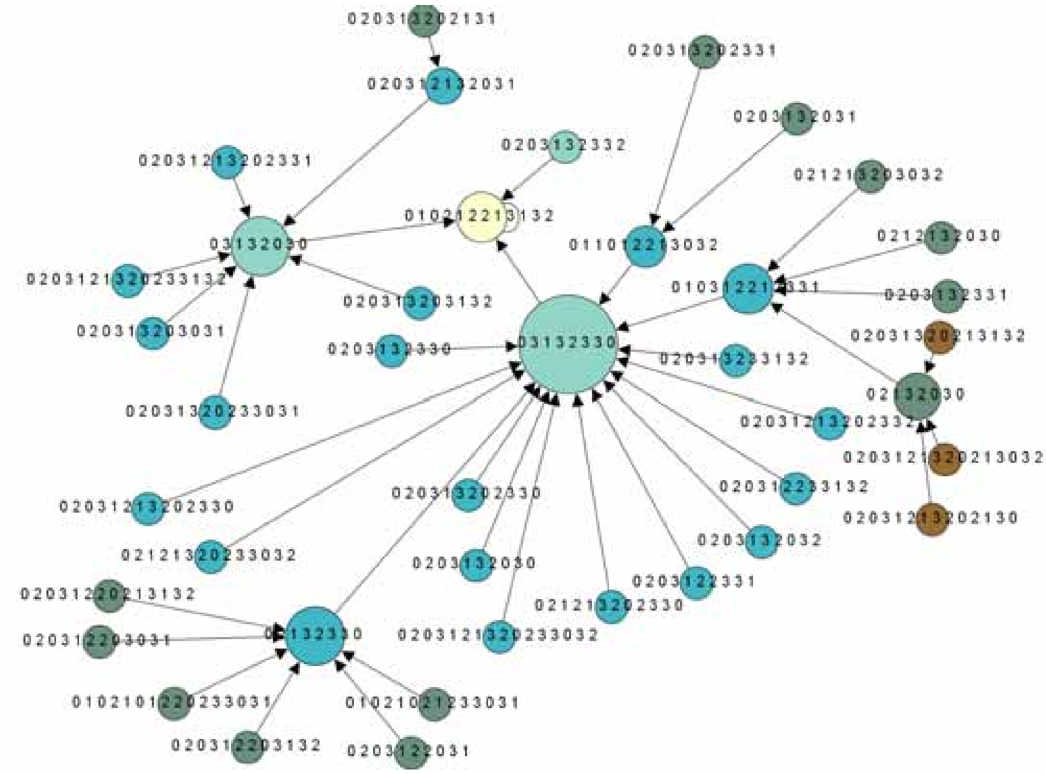

Figure 1: The largest equivalence class of $\mathcal{D}_4$. The label of the vertices denotes the arcs of the corresponding digraph on 4 vertices, where the vertices of each digraph are labelled from 0 to 3.

(see Fig. 1). Meanwhile the largest equivalence class of $\mathcal{D}_5$ has 3958 digraphs (about 41% of the total number of digraphs). In general, it seems difficult to determine the cardinality of the largest equivalence class of $\mathcal{D}_n$. Nevertheless, our computer approach has been useful to observe that there is a digraph H_n acting as the strongest attractor for many digraphs into the largest equivalence class for $n = 4, 5$.

Proposition 2.4 *Let H_n be the digraph with vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and arc set $A = \{(v_i, v_{n-1}) \,|\, 0 \leq i \leq n-2\} \cup \{(v_{n-1}, v_0)\}$. Let H' be any digraph, with vertex set V, satisfying the following conditions:*

- *H' contains the arcs (v_{n-1}, v_i), for any $1 \leq i \leq n-2$, and does not contain their corresponding reverse arcs.*
- *H' has neither the arc (v_0, v_{n-1}) nor (v_{n-1}, v_0).*
- *The subgraph of H' induced by $V \setminus \{v_{n-1}\}$ is strongly connected.*

Then $ED(H') = H_n$.

PROOF: Just observe that vertices v_0 and v_{n-1} are mutually eccentric in H' and every vertex $v \in V \setminus \{v_{n-1}\}$ has v_{n-1} as its unique eccentric vertex. □

H_4 is the vertex with maximum indegree in Fig. 1, where 15 digraphs have H_4 as their eccentric digraph (and this is the largest value for a digraph with four vertices). Moreover, 563 and 63838 digraphs have H_5 and H_6 as their eccentric digraph, respectively. There is a large number of digraphs H' satisfying the conditions in the proposition given above. This could explain the large number of digraphs having H_n as their eccentric digraph. Nevertheless, one can find other digraphs, different to H', having H_n as their eccentric digraph.

Conjecture 2.5 Let $\mathcal{D}_n$ be as defined in Proposition 2.3, $n > 2$. Then H_n is the vertex of $\mathcal{D}_n$ with maximum indegree. Moreover, H_n belongs to the largest weakly connected component of $\mathcal{D}_n$, that is, $[H_n]$ is the maximal equivalence class.

2.2 Minimal equivalence classes and fixed points

The minimal equivalence classes, that is, those containing the minimum number of digraphs, are isolated vertices in $\mathcal{D}_n$, for $2 < n \leq 6$. There are only two of them for $n = 3$ and just one for $n = 4$ (see Fig. 2). Besides, there are 15 isolated vertices of $\mathcal{D}_5$. Of course, any isolated vertex G is a fixed point ($ED(G) \cong G$), although

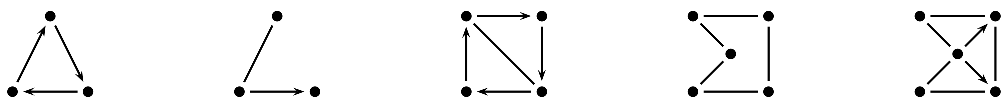

Figure 2: Isolated vertices of $\mathcal{D}_3$ and $\mathcal{D}_4$ and two isolated vertices of $\mathcal{D}_5$ (out of 15).

the converse is not true. The complete graph K_n is indeed a fixed point, but $[K_n]$ contains also the null graph $\overline{K_n}$, so K_n is not an isolated vertex of $\mathcal{D}_n$. The next two propositions show the relation between both isolated vertices and fixed points. First, we recall some definitions and results that appear in [8]. Given a digraph G of order n, a *reduction* of G, denoted by G^-, is derived from G by removing all its arcs incident from vertices with outdegree $n-1$. We refer to the digraph $\overline{G^-}$ as the complement of the reduction of G. We say that a digraph G is *eccentric* if there exists a digraph H such that $ED(H) \cong G$. It is known (see [8]) that a digraph G is eccentric if and only if $ED(\overline{G^-}) = G$.

Proposition 2.6 *If G is an isolated vertex of $\mathcal{D}_n$, then $G \cong \overline{G^-}$.*

PROOF: The ecentric digraph of an isolated vertex G must coincide with itself, that is, $ED(G) \cong G$. This means that G is an eccentric digraph. Hence, by the characterization of eccentric digraphs, we have that $ED(\overline{G^-}) \cong G$. But since G is an isolated vertex, every digraph H such that $ED(H) \cong G$ must satisfy $H \cong G$. Hence, $\overline{G^-} \cong G$. □

Proposition 2.7 *If G is an eccentric digraph such that $G \cong \overline{G^-}$, then G is a fixed point.*

Proof: From $G \cong \overline{G^-}$ we have that $ED(G) \cong ED(\overline{G^-})$. Now, since G is eccentric, $ED(\overline{G^-}) \cong G$, and as a consequence $ED(G) \cong G$. □

Both propositions may help us to find either isolated vertices and fixed points in $\mathcal{D}_n$. For instance, one of these isolated vertices of $\mathcal{D}_5$ (out of 15) is the circulant digraph on 5 vertices with set of generators $\{1, 2\}$. In general, due to Proposition 2.7, the circulant digraph on $2m+1$ vertices and set of generators $\{1, 2, \ldots, m\}$, denoted by $C(2m + 1, \{1, 2, \ldots, m\})$, is a fixed point. Moreover, it seems that $C(2m + 1, \{1, 2, \ldots, m\})$ is an isolated vertex of $\mathcal{D}_n$, where $n = 2m + 1$, but we do not have a proof.

Conjecture 2.8 $\mathcal{D}_n$ contains isolated vertices for any $n > 2$ and $C(2m+1, \{1, 2, \ldots, m\})$ is an isolated vertex in $\mathcal{D}_{2m+1}$.

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