BIPARTITE 2-ARC-TRANSITIVE GRAPHS

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Abstract: Let Γ be a finite connected regular bipartite 2-arc transitive graph. It is shown that Γ is a cover of a possibly smaller graph Σ, which is also connected and regular of the same valency as Γ, and there is a subgroup G of Aut Σ such that G is 2-arc transitive on Σ and every nontrivial normal subgroup of G has at most two orbits on vertices. Such graphs Σ for which the subgroup G has an abelian normal subgroup with two orbits are investigated. It is shown that Σ is a 2-arc transitive Cayley graph for either (a) an elementary abelian 2-group, or (b) a group $< N, \tau >$, where N is an elementary abelian group of odd order and τ, an element of order 2, inverts every element of N. The graphs Σ arising in (a) have been classified recently by A.A. Ivanov and the author.

1. Introduction

Let $\Gamma = (V, E)$ be a finite connected graph with finite vertex set V and edge set E. A 2-*arc* of Γ is an ordered triple $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ of vertices such that $\alpha_0 \neq \alpha_2$ and $\{\alpha_0, \alpha_1\}$ and $\{\alpha_1, \alpha_2\}$ are both edges of Γ. If a group G acts on Γ as a group of automorphisms (possibly not faithfully) then G is said to be *vertex-*

transitive or *2-arc transitive* on Γ if G is transitive on V or on the set of 2-arcs of Γ respectively. Also Γ is called vertex-transitive or 2-arc transitive if its automorphism group Aut Γ is vertex-transitive or 2-arc transitive on Γ respectively. This paper begins an investigation of the structure of bipartite 2-arc transitive graphs taking as its starting point the following elementary result, a proof of which can be found in [**6**, Lemma 3.2]. It shows that often a 2-arc transitive graph is a cover of a smaller 2-arc transitive graph of the same valency. To explain what we mean we need the concept of a quotient graph: if P is a partition of the vertex set V of a graph Γ, the *quotient graph* Γ_P corresponding to P is the graph with vertex set P such that distinct parts $B, B' \in P$ are joined by an edge in Γ_P if and only if, for some $\alpha \in B, \alpha' \in B', \{\alpha, \alpha'\}$ is an edge of Γ. Further, Γ is called a *cover* of Γ_P if no part of P contains an edge of Γ and, if $\{B, B'\}$ is an edge of Γ_P, then each point of B is joined to exactly one point of B' and each point of B' is joined to exactly one point of B. If P is the set of orbits in V of a group N of automorphisms of Γ then we write Γ_N for the quotient graph.

Lemma 1.1 Let G be a group of automorphisms of a finite connected graph $\Gamma = (V, E)$ such that G is vertex-transitive and 2-arc transitive on Γ. Suppose that G has a normal subgroup N which has more than two orbits in V. Then

(a) G is vertex-transitive and 2-arc transitive on the quotient graph Γ_N,

(b) Γ is a cover of Γ_N, and

(c) N is semiregular on V.

(A permutation group on V is *semiregular* on V if each element fixes none or all of the points of V; if in addition it is transitive on V then it is said to be *regular* on V.) Lemma 1.1 implies that every finite 2-arc transitive graph Γ is a cover of some, possibly smaller, quotient Γ_N, with the same valency as Γ, and with the property that some subgroup H of Aut Γ_N is 2-arc transitive on Γ_N and every nontrivial normal subgroup of H has at most two orbits on the vertices of Γ_N. It suggests that a solution to the following problem is central to an understanding of finite 2-arc transitive graphs.

Problem 1.2 Classify all finite connected regular graphs $\Gamma = (V, E)$ which admit a group G of automorphisms such that

(*) G is 2-arc transitive on Γ and every nontrivial normal subgroup of G has at most two orbits on V.

A permutation group on a set V with the property that every nontrivial normal subgroup is transitive on V is said to be *quasiprimitive.* The structure of finite quasiprimitive permutation groups was investigated in detail in [7]. That paper also contained an analysis of various types of *quasiprimitive* 2-arc transitive graphs Γ, that is, graphs for which Aut Γ contains a 2-*arc* transitive subgroup which is quasiprimitive on vertices. Thus there is some understanding of graphs arising in Problem 1.2 for which all nontrivial normal subgroups of the group G are vertex-transitive. The remaining class of graphs arising from Problem 1.2 consists of bipartite graphs, since a connected regular graph with 2-arc-transitive group G of automorphisms is bipartite if and only if some normal subgroup of G has two orbits on vertices. Bipartite graphs with a group G of automorphisms with property (*) are the subject of this paper.

If $G \leq \text{Aut}\,\Gamma$ is vertex-transitive on a bipartite graph $\Gamma = (V, E)$ then G has a subgroup G^+ of index 2 with two orbits on V, say $\triangle_1$ and $\triangle_2$, the two parts of the bipartition of V. We shall use this notation (for G^+, $\triangle_1$ and $\triangle_2$) throughout the paper. We shall say that G is *bi-quasiprimitive* on Γ if G^+ is quasiprimitive and faithful on each of $\triangle_1$ and $\triangle_2$. (A group acting on a set $\triangle$ is said to be *faithful* on $\triangle$ if the only element which fixes $\triangle$ pointwise is the identity.) The nature of finite bipartite graphs satisfying property (*) of Problem 1.2 will be discussed in Section 2. Theorem 2.1 of that section shows that either such graphs are bi-quasiprimitive, or they have a complete bipartite quotient graph. Both cases are investigated further.

One of the types of bi-quasiprimitive 2-arc transitive graphs identified in Theorem 2.3 in Section 2 is the so-called "affine type" where the 2-arc transitive group G has a nontrivial abelian normal subgroup. The main purpose of this paper is to describe the groups G which arise in this case. We show in particular, see Theorem 1.3 below, that all the graphs in this case are Cayley graphs for some (not necessarily abelian) group. (For a subset X of a group M, such that $1_M \notin X$ and $x \in X$ implies $x^{-1} \in X$, the *Cayley graph* for M with respect to X is the graph with vertex set M such that $\{y, z\}$ is an edge if and only if $yz^{-1} \in X$.)

Theorem 1.3 Let Γ be a finite connected regular bipartite graph with a group G of automorphisms which is 2-arc transitive on Γ. Assume that the subgroup G^+ of G fixing setwise the two parts $\triangle_1$ and $\triangle_2$ of the bipartition acts faithfully on each part. Suppose moreover that every nontrivial normal subgroup of G contained in G^+ is transitive on $\triangle_1$ and $\triangle_2$, and that G^+ has a nontrivial abelian normal subgroup.

(a) Then G has a normal subgroup N contained in G^+ which is an elementary abelian group and is regular on $\triangle_1$ and on $\triangle_2$.

(b) For a given pair α, β of adjacent vertices of Γ, the graph Γ has an automorphism τ of order 2 (which is not necessarily in G) which normalises G^+, interchanges α and β, and inverts every element of N. Moreover the group $L = < G, \tau >$ satisfies all the hypothesis on G above, and is such that N is the unique minimal normal subgroup of L^+, where L^+ is the subgroup of L fixing $\triangle_1$ and $\triangle_2$ setwise.

(c) The group $M = < N, \tau >$ is a normal subgroup of L which is regular on vertices. Moreover one of the following holds.

   (i) M is an elementary abelian 2-group. Regarding M as the additive group of a finite vector space V over $GF(2)$ we have $L \leq AGL(V)$, a group of affine transformations of V containing the group M of translations, with L^+ acting irreducibly on a subspace of co-dimension 1 corresponding to N.

   (ii) N is an elementary abelian p-group for some odd prime p, and either $\Gamma = C_{2p}$ is a cycle of length $2p$, or N is self-centralising in L. In the latter case, regarding N as the additive group of a vector space V, we have $L \leq AGL(V)$ with L and L^+ irreducible on V.

The graphs arising in part (c) are all isomorphic to Cayley graphs for the group

M. In case (c)(i), the group M can be identified with a vector space in such a way that all translations are admitted as automorphisms. Such graphs are called *affine*. Thus the graphs arising in case (c)(i) are the finite, affine, bi-quasiprimitive, 2-arc transitive graphs, and this class of graphs has been classified completely by Ivanov and the author in [**2**]. It is hoped that a classification of all graphs arising in case (c)(ii) may also be possible. Theorem 1.3 will be proved in Section 3. In Section 4 we give a general construction of a class of bipartite graphs satisfying the conditions of Theorem 1.3(c)(ii), and we prove that all graphs satisfying those conditions are isomorphic to a graph given by this construction unless the stabilisers in the group L of points lying in $\triangle_1$ and in $\triangle_2$ form two distinct conjugacy classes of complements for N in L^+.

Notation and Preliminaries For a graph Γ and vertex α, the set of vertices β such that $\{\alpha,\beta\}$ is an edge is denoted by $\Gamma_1(\alpha)$. If G is 2-arc transitive on Γ then clearly G_α must be transitive on the set of 2-arcs of the form (β,α,γ), and so G_α is transitive on the set of ordered pairs (β,γ) of distinct points of $\Gamma_1(\alpha)$, that is G_α is *doubly transitive* on $\Gamma_1(\alpha)$. We refer to an ordered pair (α,β) of adjacent vertices of Γ as a *1-arc* of Γ.

If G is a permutation group on a set Ω and $\alpha \in \Omega$ then $\alpha^G = \{\alpha^g \mid g \in G\}$ denotes the orbit of G in Ω containing α. Now let G be transitive on Ω. Then G_α-orbits in Ω are called *suborbits* of G. Further G induces a natural action on $\Omega \times \Omega$ by $(\alpha,\beta)^g = (\alpha^g,\beta^g)$ for $g \in G$, and $\alpha,\beta \in \Omega$, and there is a one-to-one correspondence between G-orbits in $\Omega\times\Omega$ and G_α-orbits in Ω, where a G-orbit $\triangle$ in $\Omega\times\Omega$ corresponds to the G_α-orbit $\triangle(\alpha) = \{\beta \mid (\alpha,\beta) \in \triangle\}$. Also there is a natural pairing of G-orbits in $\Omega \times \Omega$ where the pair of $\triangle$ is $\triangle^* = \{(\beta,\alpha) \mid (\alpha,\beta) \in \triangle\}$; $\triangle$ is said to be *self-paired* if $\triangle = \triangle^*$ and *non-self-paired* otherwise. The G_α-orbit $\triangle(\alpha)$ is said to be self-paired or non-self-paired according as $\triangle$ is self-paired or non-self-paired.

2. Bipartite graphs satisfying property (*).

It will follow from the first result of this section that a bipartite graph with the property (*) of Problem 1.2 either is bi-quasiprimitive or has a complete bipartite quotient.

Theorem 2.1 Let $\Gamma = (V, E)$ be a finite connected regular bipartite graph with a group G of automorphisms which is 2-arc transitive on Γ. Let G^+ be the subgroup of index 2 in G which fixes setwise the two parts $\triangle_1, \triangle_2$ of the bipartition, and suppose that every nontrivial normal subgroup of G contained in G^+ is transitive on $\triangle_1$ and $\triangle_2$. Then one of the following holds.

(a) Γ is a complete bipartite graph.

(b) G^+ is faithful and quasiprimitive on $\triangle_i, i = 1, 2$, that is Γ is bi-quasiprimitive.

(c) G^+ is faithful on $\triangle_i, i = 1, 2$, but G has a normal subgroup of the form $M_1 \times M_2$, where M_1 and M_2 are minimal normal subgroups of G^+ which are interchanged by G, and M_i is intransitive on $\triangle_i, i = 1, 2$.

Proof of Theorem 2.1 Let $\Gamma = (V, E)$ be a finite connected regular bipartite graph with bipartition $V = \triangle_1 \cup \triangle_2$. Let $G \leq \mathrm{Aut}\ \Gamma$ be 2-arc transitive on Γ with the property that every nontrivial normal subgroup of G contained in G^+ is transitive on $\triangle_1$ and $\triangle_2$. Suppose first that the pointwise stabilizer K_1 of $\triangle_1$ in G is nontrivial. Then, for $\alpha \in \triangle_1$, K_1 is a normal subgroup of G_α and so, as G_α is doubly transitive on $\Gamma_1(\alpha)$, either K_1 is transitive on $\Gamma_1(\alpha)$ or K_1 fixes $\Gamma_1(\alpha)$ pointwise. Since K_1 is normal in G^+, all its orbits in $\triangle_2$ have the same length, and consequently K_1 does not fix any points of $\triangle_2$. Thus K_1 is transitive on $\Gamma_1(\alpha)$ (since $\Gamma_1(\alpha) \subseteq \triangle_2$). Let $g \in G \setminus G^+$. Then $K_2 = K_1^g$ is the pointwise stabilizer in G of $\triangle_2$, and K_2 is transitive on $\Gamma_1(\beta) \subseteq \triangle_1$, where $\beta = \alpha^g$. Moreover $K_1 \cap K_2 = 1$ and therefore $< K_1, K_2 > \ \simeq K_1 \times K_2$, a normal subgroup of G contained in G^+. By our assumptions $K_1 \times K_2$ is transitive on $\triangle_1$ and $\triangle_2$ and hence $\triangle_1 = \Gamma_1(\beta), \triangle_2 = \Gamma_1(\alpha)$, and Γ is the complete bipartite graph $K_{v,v}$ where $v = \mid \Gamma_1(\alpha) \mid$. Thus case (a) holds.

So from now on we shall assume that G^+ acts faithfully on $\triangle_1$ and $\triangle_2$. If G^+ is quasiprimitive on one of $\triangle_1, \triangle_2$, then it will be quasiprimitive on both and so G will be bi-quasiprimitive on Γ, and (b) holds. So assume that G^+ is not quasiprimitive on $\triangle_1$ or $\triangle_2$. Then there is a nontrivial normal subgroup M_1 of G^+ which is not transitive on $\triangle_1$. Clearly we may assume that M_1 is a minimal normal subgroup of G^+. For $g \in G \setminus G^+$ the subgroup $M_2 = M_1^g$ is normal in G^+ and not transitive on $\triangle_2$. By our assumptions M_1 is not normal in G so $M_2 \neq M_1$. Also $M_1 \cap M_2$ is normal in G and intransitive on $\triangle_1$ and $\triangle_2$, and therefore $M_1 \cap M_2 = 1$. Hence $< M_1, M_2 > \simeq M_1 \times M_2$ is a normal subgroup of G, contained in G^+, and so $M_1 \times M_2$ is transitive on $\triangle_1$ and $\triangle_2$. Thus case (c) of Theorem 2.1 holds and the proof is complete.

Theorem 2.1 shows that, in order to solve Problem 1.2 for bipartite graphs, it is necessary to determine all bi-quasiprimitive 2-arc-transitive graphs and all graphs arising in case (c). A further analysis of the structure of bi-quasiprimitive 2-arc transitive graphs will be made below, but first we give a discussion of the graphs which arise in case (c). In particular we show that they have a quotient graph which is a complete bipartite graph and admits G acting faithfully and transitively on vertices and on ordered pairs of adjacent vertices.

Discussion of case (c)

Suppose case (c) of Theorem 2.1 holds. Let P be the partition of V consisting of the M_1-orbits in $\triangle_1$ and the M_2-orbits in $\triangle_2$. Clearly P is a G-invariant partition so G acts as a group of automorphisms of the quotient graph Γ_P. Moreover since M_1 fixes setwise all parts of P in $\triangle_1$ and is transitive on the parts of P in $\triangle_2$ it follows that Γ_P is a complete bipartite graph. However Γ is nothing like a cover of this quotient graph Γ_P. The case where M_1 is abelian will be investigated in Theorem 1.3. Here we make some general observations. Clearly we may assume (if necessary by interchanging M_1 and M_2) that the M_1-orbits in $\triangle_1$ are no bigger than the M_1- orbits in $\triangle_2$.

Proposition 2.2 Let the M_1-orbits in $\triangle_i$ have length d_i, for $i = 1, 2$, with $d_1 \leq d_2$. Then one of the following holds.

(a) $d_1 = d_2$ and M_1 and M_2 are both semi-regular on V.

(b) $d_2 = | \Gamma_1(\alpha) | d_1$, for $\alpha \in \triangle_1$ and $\beta \in \Gamma_1(\alpha)$ and setting $M = M_1$, we have $M_\beta < M_\alpha < M$, and β^M is the disjoint union of $\Gamma_1(\gamma)$ for $\gamma \in \alpha^M$.

Proof Let $\alpha \in \triangle_1$ and $\beta \in \Gamma_1(\alpha)$ and set $M = M_1$. We claim that $M_\beta \leq M_\alpha$. Suppose to the contrary that M_β does not fix α. Then as M_β is a normal subgroup of G_β and G_β is doubly transitive on $\Gamma_1(\beta)$ it follows that M_β is transitive on $\Gamma_1(\beta)$. Moreover M_α does not fix β (for if it did we would have $M_\alpha \leq M_\beta$ whence $d_1 = d_2$ and $M_\alpha = M_\beta$ which we are assuming is not the case). Then as above M_α is transitive on $\Gamma_1(\alpha)$. So for each vertex γ we have M_γ transitive on $\Gamma_1(\gamma)$. Since Γ is connected this implies that M is transitive on $\triangle_1$ and $\triangle_2$ which is a contradiction. Thus $M_\beta \leq M_\alpha$.

Next suppose that M is not semiregular on V. Then $M_\alpha \neq 1$. If M_α fixed $\Gamma_1(\alpha)$ pointwise then we would have $M_\alpha = M_\beta$. Since this would be true for all pairs of adjacent vertices α, β it would follow from connectivity that $M_\alpha = M_\beta = 1$, that is M is semiregular. As this is not the case, M_α acts nontrivially and hence transitively on $\Gamma_1(\alpha)$. Thus $d_2 = | M : M_\beta | = | M : M_\alpha || M_\alpha : M_\beta | = d_1 | \Gamma_1(\alpha) |$. We have $\Gamma_1(\alpha)$ contained in the M-orbit β^M containing β. Moreover, since $M_\beta < M_\alpha < M$ it follows that $\Gamma_1(\alpha)$ is a block of imprimitivity for M in β^M. In fact β^M is the disjoint union of the sets $\Gamma_1(\gamma)$ as γ ranges over α^M.

Next we analyse the possible structure of bi-quasiprimitive 2-arc transitive groups of automorphisms.

Theorem 2.3 Let $\Gamma = (V, E)$ be a finite connected regular bipartite graph with a group G of automorphisms which is 2-arc transitive and bi-quasiprimitive. Then the subgroup G^+ of index 2 in G fixing the parts $\triangle_1$ and $\triangle_2$ of the bipartition setwise has a unique minimal normal subgroup $N \cong T^k$, where $k \geq 1$, and T is a simple group, and $(G^+)^{\triangle_1} \cong (G^+)^{\triangle_2}$ is of type I (affine), II (almost simple), III(b)(i) (product action), or III(c) (twisted wreath), as described in [7].

Proof of Theorem 2.3 By assumption G^+ is faithful and quasiprimitive on each of $\triangle_1$ and $\triangle_2$. By [7] the socle N of G^+ is isomorphic to T^k for some simple group T and some integer $k \geq 1$. If N is elementary abelian, then $(G^+)^{\triangle_1} \cong (G^+)^{\triangle_2}$ is of type I of [7] , so we may assume that T is a nonabelian simple group. If $k = 1$ then $(G^+)^{\triangle_1} \cong (G^+)^{\triangle_2}$ is of type II of [7] , so we assume from now on that $k > 1$. It remains to show that G^+ is not of type III(a) or III(b)(ii) of [7] . Let $\alpha \in \triangle_1$ and $\beta \in \Gamma_1(\alpha)$.

Suppose that $(G^+)^{\triangle_1}$ is of type III(a) of [7]. Then $(G^+)^{\triangle_2}$ is also of type III(a), and we may assume that $G^+ \leq W := \{(a_1, \ldots, a_k) \cdot \pi \mid a_i \in \text{Aut } T,\ \pi \in S_k,\ a_i \equiv a_j \bmod \text{Inn } T \text{ for all } i,j\}$, $G_\alpha \leq \{(a, \ldots, a) \cdot \pi \mid a \in \text{Aut } T,\ \pi \in S_k\}$, and $G_\beta \leq \{(a, a^{\phi_2}, \ldots, a^{\phi_k}) \cdot \pi \mid a \in \text{Aut } T,\ \pi \in S_k\}$, for some $\phi_2, \ldots, \phi_k \in \text{Aut } T$. Then $G_{\alpha\beta} = G \cap \{(a, \ldots, a) \cdot \pi \mid a \in \cap_{2 \leq i \leq k} C_{\text{Aut } T}(\phi_i),\ \pi \in S_k\}$. Since $G_\alpha \neq G_\beta$, at least one of the ϕ_i is nontrivial, and hence $N_\alpha \cong T$ acts nontrivially on $\Gamma_1(\alpha)$. Therefore T is the socle of a doubly transitive permutation group such that the stabiliser in T of a point in this representation is the intersection of the centralisers of $k-1$ automorphisms of T, at least one of which is nontrivial. Checking through the list of such groups in [**1**] we see that this is

impossible. Hence G^+ is not of type III(a) of [7].

Suppose that $(G^+)^{\triangle_1}$ is of type III(b)(ii) of [7] . Then, by [7], also $(G^+)^{\triangle_2}$ is of type III(b)(ii). We have $G^+ \leq W := H \text{wr} S_l$ where $l > 1$, l is a proper divisor of k, H has socle $T^{k/l}$, and $N_\alpha \cong T^l$. By connectivity, N_α acts nontrivially on $\Gamma_1(\alpha)$, and it follows that $G_\alpha^{\Gamma_1(\alpha)}$ has socle T. By connectivity also, the subgroup of G_α fixing $\Gamma_1(\alpha)$ pointwise does not have T as a composition factor, and this implies that $l = 1$ which is a contradiction. Thus Theorem 2.3 is proved.

3. Proof of Theorem 1.3.

In this section we prove Theorem 1.3. The crucial part of the proof is in establishing that the automorphism τ exists. We prove the existence of τ under weaker conditions in the following theorem which is essentially Lemma 3.2 of [**3**].

Theorem 3.1 Let Γ be a finite connected regular bipartite graph with a group G of automorphisms which acts transitively and faithfully on both parts $\triangle_1$ and

$\triangle_2$ of the bipartition of the vertex set of Γ. Suppose that G has an abelian normal subgroup N which is regular on both $\triangle_1$ and $\triangle_2$. Then Γ is vertex-transitive. Moreover, for a pair α, β of adjacent vertices of Γ, there is an automorphism τ of Γ of order 2 which interchanges α and β, normalizes G, and inverts every element of N (acting by conjugation).

Proof of Theorem 3.1 Let $\alpha \in \triangle_1$ and $\beta \in \Gamma_1(\alpha)$, and let $H = G_\alpha$, and $K = G_\beta$. Then $G = HN = KN$ (whence $H \simeq G/N \simeq K$). Moreover we may identify $\triangle_1$ with the set $[G : H]$ of right cosets of H in G, and $\triangle_2$ with $[G : K]$ in such a way that elements of G act by right multiplication. Then we have $\alpha = H, \beta = K, \triangle_1 = \{Hn \mid n \in N\}$ and $\triangle_2 = \{Kn \mid n \in N\}$, and there is a subset X of N such that $\Gamma_1(\alpha) = \{Kx \mid x \in X\}$. The edges of Γ are therefore precisely those pairs of vertices of the form $\{Hn, Kxn\}$ for $n \in N, x \in X$. Let τ be the permutation of the vertex set given by $(Hn)\tau = Kn^{-1}$ and $(Kn)\tau = Hn^{-1}$, for $n \in N$. The image of the edge $\{Hn, Kxn\}$ under τ is $\{Kn^{-1}, H(xn)^{-1}\} = \{Kx(xn)^{-1}, H(xn)^{-1}\}$, since N is abelian, which is again an edge. Thus τ is an automorphism of Γ, and τ interchanges $\triangle_1$ and $\triangle_2$, interchanges α and β, and clearly has order 2. So Γ is vertex transitive.

Finally we must show that τ normalizes G and inverts every element of N. It follows from the definition of τ that, for $n \in N$, the automorphism $\tau n \tau$ of Γ maps Hm to Hmn^{-1} and Km to Kmn^{-1} (using the fact that N is abelian), and hence $\tau n \tau$ acts by right multiplication by $n^{-1} \in N$, hence $\tau n \tau = n^{-1}$. Now each $h \in H = KN$ can be written uniquely as $h = k(h)n(h)$ where $k(h) \in K$ and $n(h) \in N$, and each $k \in K$ arises as $k = k(h)$ for a unique $h \in H$. The action of $h \in H$ is as follows: h maps Hn to $Hnh = Hn^h = Hn^{k(h)}$ (since N is abelian) and h maps Kn to $Knh = Kn^{k(h)}n(h)$ for each $n \in N$. Also $\tau k(h) \tau = \tau h n(h)^{-1} \tau$ maps Hn to $Hn^{k(h)}$ and Kn to $Kn^{k(h)}n(h)$ for each $n \in N$. Thus $\tau k(h) \tau = h$ for each $h \in H$, and hence for each $k(h) \in K$. Therefore τ normalizes G. Thus Theorem 3.1 is proved.

Now we use this result to complete the proof of Theorem 1.3.

Proof of Theorem 1.3 Now $G \leq \text{Aut}\,\Gamma$ is vertex-transitive and 2-arc transitive on Γ, G^+ is faithful on $\triangle_1$ and $\triangle_2$, and every nontrivial normal subgroup of G contained in G^+ is transitive on $\triangle_1$ and $\triangle_2$. Also G^+ has a nontrivial abelian normal subgroup. Let P be an abelian minimal normal subgroup of G^+, so P is an elementary abelian p-group for some prime p. If P is normal in G then P must be transitive and hence regular on $\triangle_1$ and $\triangle_2$, and we have part (a) with $N = P$. So suppose that P is not normal in G. Then for $g \in G \setminus G^+, Q = P^g \neq P$ and $N \cong P \times Q$ is normal in G and hence transitive on $\triangle_1$ and $\triangle_2$. Since N is (elementary) abelian it is regular on $\triangle_1$ and $\triangle_2$.

Let α, β be adjacent vertices of Γ. Then by Theorem 3.1 there is an automorphism τ of Γ of order 2 which interchanges α and β, normalizes G^+, and inverts every element of N. In the case where $N = P \times Q$, with P normal in G^+ but not normal in G, the element τ normalizes P and hence $\tau \notin G$. In this case $L = < G, \tau >$ contains G and L^+ (the subgroup of L fixing $\triangle_1$ and $\triangle_2$ setwise) as subgroups of index 2. For $g \in G \setminus G^+, g\tau \in L^+$ and so N is a minimal normal subgroup of L^+. Thus in all cases N is a minimal normal subgroup of L^+. Since N is self-centralizing in L^+, N is the unique minimal normal subgroup of L^+. Clearly L satisfies all the conditions about normal subgroups imposed on G.

If N is a 2-group then $M = < N, \tau >$ is an elementary abelian normal subgroup of L which is regular on vertices and (c)(i) holds. So assume that N is an elementary abelian p-group for some odd prime p. Suppose that N is not self-centralizing in L. Then there is an element y of order 2 in $L \setminus L^+$ which centralizes N, and we have another regular normal subgroup of L, namely $N_2 =< N, y >$. In this case we may identify the vertices of Γ with N_2 so that $\alpha = 1, \triangle_1 = N, \triangle_2 = Ny, N_2$ acts by right multiplication and L_α acts by conjugation. With the new identification, $\Gamma_1(\alpha) = X_2 y$ for some $X_2 \subseteq N$, and Γ is a Cayley graph for N_2. Since Γ is undirected, $X_2 y$, and hence X_2, must be closed under forming inverses (that is $x \in X_2$ implies $x^{-1} \in X_2$). This means that the two-element subset $\{xy, x^{-1}y\}$ of $\Gamma_1(\alpha)$ is a block of imprimitivity for the doubly transitive action of L_α on $\Gamma_1(\alpha)$ and therefore $\mid \Gamma_1(\alpha) \mid = 2$, that is Γ is a cycle. Since N is a minimal normal subgroup of L^+ it follows that $|N| = p$, and $< N, \tau >$ is a regular normal subgroup of L, so $\Gamma = C_{2p}$. Finally if N is self-centralising in L then $M = < N, \tau >$ is normal in L and regular, and $L \leq AGL(N)$, with L and L^+ irreducible on N.

4. Construction and discussion of bi-quasiprimitive graphs satisfying Theorem 1.3(c)(ii).

We give a construction of a graph satisfying the conditions of Theorem 1.3(c)(ii) from a finite primitive permutation group G with an abelian regular normal subgroup N and a non-self-paired doubly transitive suborbit. The graph constructed is a Cayley graph for the nonabelian group $N.<\tau> = N.2$ where τ inverts each element of N. All these properties of the graph constructed are proved in Theorem 4.2 below. The construction is significant since we shall prove, in Theorem 4.3, that all graphs satisfying the conditions of Theorem 1.3(c)(ii) arise in this way unless the group L^+ contains more than one conjugacy class of complements for N.

Construction 4.1 Let H be a finite primitive permutation group on Ω with an elementary abelian regular normal subgroup N. Suppose also that $H_\alpha, \alpha \in \Omega$, is doubly transitive on a non-self-paired orbit $\Gamma(\alpha)$. This means that N is a p-group for some odd prime p, that Ω can be identified with N in such a way that N acts by right multiplication and H_α (where α is identified with 1_N) acts by conjugation, and $\Gamma(\alpha)$ is a subset X of N which is disjoint from its inverse $X^{-1} = \{x^{-1} \mid x \in X\}$.

Define a graph Γ to have vertex set $V = N \times Z_2 = \{(n, i) \mid n \in N, i \in Z_2\}$ such that $(n, i), (m, j)$ are adjacent if and only if $i \neq j$ and $nm \in X$.

Theorem 4.2 The graph Γ in Construction 4.1 is bipartite with bipartition $\Delta_i = N \times \{i\}, i \in Z_2$, and admits H as a group of automorphisms fixing each set Δ_i setwise, where, for $h \in H_\alpha$, and $x \in N$,

$$(n, i)^h = (n^h, i), \quad (n, 0)^x = (nx, 0) \quad \text{and} \quad (n, 1)^x = (nx^{-1}, 1)$$

for all $(n, i) \in N \times Z_2$. Also Γ admits as an automorphism the map τ defined by

$$(n, i)^\tau = (n, i+1) \quad \text{for all} \quad (n, i) \in N \times Z_2.$$

Further, τ inverts each element of N, and the subgroup $< H, \tau >$ of Aut Γ is vertex-transitive and 2-arc transitive on Γ, and Γ satisfies all the conditions of Theorem 1.3(c)(ii).

Proof of Theorem 4.2 We have $H = NH_\alpha$, where $\alpha \in \Omega$, and Ω may be identified with N in such a way that N acts on Ω by right multiplication, and H_α (where $\alpha = 1_N$) acts by conjugation. Then $\Gamma(\alpha)$ is identified with a subset X of N and, since $\Gamma(\alpha)$ is not self-paired, X is disjoint from $X^{-1} = \{x^{-1} \mid x \in X\}$. This means in particular that N is a p-group for some odd prime p.

Now consider the graph $\Gamma = (V, E)$. Since N is abelian, $nm \in X$ if and only if $mn \in X$, and so the adjacency relation is symmetric (and Γ really is an *undirected* graph). Also Γ is bipartite with bipartition $\triangle_i = N \times \{i\}, i = 0, 1$. Since H_α acts on $\Omega = N$ by conjugation, the set X is H_α-invariant and so elements h of H_α acting by conjugation on V, that is $(n,i)^h = (n^h, i)$, are automorphisms of Γ. Next for $x \in N$ the map $(n,0)^x = (nx, 0)$, and $(n,1)^x = (nx^{-1}, 1)$ can be seen to be an automorphism as follows. If $e = \{(n,i),(m,j)\}$ is an edge, then $i \neq j$, say $i = 0, j = 1$, and $nm \in X$. The image of e under x is $\{(nx,0),(mx^{-1},1)\}$ and, as $nxmx^{-1} = nm$ (N is abelian), this is also an edge. Finally the map τ, where $(n,i)^\tau = (n, i+1)$, is an automorphism since $e^\tau = \{(n,0),(m,1)\}^\tau = \{(n,1),(m,0)\}$ is an edge for each edge e of Γ. For $x \in N$, we have $(n,i)^{\tau x \tau} = (n,i)^{x^{-1}}$ for all $(n,i) \in V$, and hence $\tau x \tau = x^{-1}$, that is τ inverts every element of N. Similarly we see that τ centralizes H_α.

Consider the subgroup $G = < H, \tau >$ of Aut Γ. Since τ interchanges $\triangle_0 = N \times \{0\}$ and $\triangle_1 = N \times \{1\}$, and since N is transitive on both $\triangle_1$ and $\triangle_2, G$ is vertex-transitive on Γ. Moreover, since H_α is doubly transitive on X it follows that H_α is transitive on the set of 2-arcs with $(1_N, 0)$ as middle vertex, these being all of the form $((x,1),(1_N,0),(y,1))$ where $x, y \in X, x \neq y$. Hence G is 2-arc transitive on Γ. To complete the proof that all the conditions of Theorem 1.3 (c)(ii) hold for G, we must verify that N is self-centralizing in G (for we know that H and hence G is irreducible on N). Suppose that some element $g \in G \setminus N$ centralizes N. Since N is self-centralizing in $H, g \in G \setminus H$, so $g = h\tau$ for some $h \in H$. Since

$H = NH_\alpha$ we may assume that $h \in H_\alpha$. (Then, as noted above $g = h\tau = \tau h$.) Now $(n,0)^{x\tau h} = (n^h x^h, 1)$ and $(n,0)^{\tau h x} = (n^h x^{-1}, 1)$ for $n, x \in N$, and so h inverts x. However $X^h = X$ and $X \cap X^{-1} = \phi$, so this is a contradiction. Thus N is self-centralizing in G, and $G \leq AGL(N)$.

Now we prove that, for a graph satisfying the conditions of Theorem 1.3(c)(ii), either it arises as in Construction 4.1, or the stabilisers in the group L of points of $\triangle_1$ and of $\triangle_2$ form two distinct conjugacy classes of subgroups of L^+. Thus an important step in the classification of the graphs satisfying the conditions of Theorem 1.3(c)(ii) is the classification of all finite primitive permutation groups with an abelian regular normal subgroup and a non-self-paired doubly transitive suborbit. It is hoped that the latter classification will be completed by Ivanov and the author. Finite primitive permutation groups with a doubly transitive suborbit were first studied by W.A. Manning in 1927, see [4] and [**8**, 17.7]. Moreover it was shown in [5] that such a group is either almost simple, or has a unique minimal normal subgroup which is regular. Thus a classification of the affine examples is important for a solution to this problem about primitive permutation groups as well as our problem about 2-arc transitive graphs.

Theorem 4.3 Let Γ be a graph satisfying all the conditions of Theorem 1.3 (c)(ii) with respect to some group L, but not a cycle. Then L^+ is a finite primitive group on $\triangle_1$ with an abelian regular normal subgroup N and, for some subset X of N, with X disjoint from X^{-1}, Γ is isomorphic to the graph with vertex set $N \times Z_2$ such that $(n,i),(m,j)$ are adjacent if and only if $i \neq j$ and $nm \in X$. Moreover, if the stabilizers of vertices of Γ (which are all complements for N in L^+) form a single L^+-conjugacy class, then Γ is isomorphic to a graph obtained from Construction 4.1.

Proof of Theorem 4.3 Now let $\Gamma = (V, E)$ be a graph satisfying the conditions of Theorem 1.3(c)(ii) with respect to some $L \leq \mathrm{Aut}\,\Gamma$. Then $L \leq AGL(N)$ for some elementary abelian normal p-subgroup N, p an odd prime, and L^+ is irreducible on N. Further Γ is bipartite and N is regular on both parts $\triangle_1$ and $\triangle_2$ of the bipartition, and $M = < N, \tau >$ is normal in L and regular on vertices, where τ

inverts each element of N, τ interchanges adjacent vertices $\alpha \in \triangle_1$ and $\beta \in \triangle_2$, and $\tau^2 = 1$. Thus L^+ acts faithfully on $\Omega = \triangle_1$ with an abelian regular normal subgroup N. Moreover L^+ is primitive on $\triangle_1$, for if $L_\alpha \le R \le L^+$ then $N \cap R$ is normal in $NL_\alpha = L^+$, and, since N is a minimal normal subgroup of L^+, either $R = L_\alpha$ or $R = L^+$.

As in the proof of Theorem 3.1, since L^+ is transitive on $\triangle_1$ and $\triangle_2$, we may identify $\triangle_1$ and $\triangle_2$ with the set of cosets in L^+ of $H = L_\alpha$ and $K = L_\beta$ in such a way that L^+ acts by right multiplication. To see how τ acts in terms of this identification, note that $L^+ = HN = KN$ so that N is a transversal for both H and K in L^+. Then for $n \in N$, the vertex Hn is mapped by τ to $(Hn)^\tau = (H)^{n\tau} = H^{\tau n^\tau} = H^{\tau n^{-1}} = (K)^{n^{-1}} = Kn^{-1}$. Similarly τ maps Kn to Hn^{-1} for each $n \in N$. Further let $X = \{x \in N | Kx^{-1} \in \Gamma_1(\alpha)\}$. Then the edges of Γ are precisely the pairs $\{Hn, Kx^{-1}n\}$ for $n \in N, x \in X$. Let us relabel V by the map $\varphi : V \to N \times Z_2$ defined by $(Hn)\varphi = (n, 0), (Kn)\varphi = (n^{-1}, 1)$ for $n \in N$. Then $(n, 0)$ is adjacent to $(m, 1)$ if and only if $mn \in X$.

Note that there are many ways to make this identification of V with $N \times Z_2$. For example, for each $y \in N$, the element τy of $M = < N, \tau >$ is also an involution which inverts each element of N and interchanges α and β^y. (Of course α and β^y may not be adjacent.) If we replace K by its conjugate $K^y = L_{\beta^y}$, and replace τ by τy in defining the map above then we obtain a map $\phi_y : V \to N \times Z_2$ and a consequent identification of Γ with a graph with $(n, 0)$ adjacent to $(m, 1)$ if and only if $mn \in yX$. In particular, if all vertex stabilizers in L are conjugate in L^+, then $H = L_\alpha$ is the stabilizer of a vertex $\beta^y \in \triangle_2$ for some $y \in N$. So we have $H = K^y$, and $\Gamma_1(\alpha) = \{(yx, 1) = (K^y y^{-1} x^{-1})\varphi_y = (Hy^{-1}x^{-1})\varphi_y | x \in X\}$. (Recall that N is abelian.) An element $h \in H$ maps $Hy^{-1}x^{-1}$ to $Hy^{-1}x^{-1}h = H(y^{-1}x^{-1})^h = K^y(y^{-1}x^{-1})^h$ and, as $(y^{-1}x^{-1})^h \in N$, it follows that $(yx, 1)^h = (K^y(y^{-1}x^{-1})^h)\varphi_y = ((yx)^h, 1)$. (In particular $(yx)^h \in yX$.) Hence the action of H on $\Gamma_1(\alpha)$ is equivalent to its action by conjugation on the subset yX of N. Thus Γ is isomorphic to a graph obtained by Construction 4.1.

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