BLOCKING SETS IN HANDCUFFED DESIGNS

SALVATORE MILICI

Dipartimento di Matematica - Università di Catania
Viale A.Doria, 6 - 95125 Catania, Italy

Abstract. In this paper we determine the spectrum of possible cardinalities of a blocking set in a $H(v,3,\lambda)$ and in a $H(v,4,1)$. Moreover we construct, for each admissible $v \geq 9$, a $H(v,3,1)$ without blocking sets.

Introduction

A handcuffed design with parameters v,k,λ, or for short an $H(v,k,\lambda)$, consists of a set of ordered k-subsets of a ν-set, called handcuffed blocks. In a block $(a_1,a_2,\ldots,a_k)$ each element is said to be "handcuffed" to its neighbours, so that the block contains k-1 handcuffed pairs $(a_1,a_2),(a_2,a_3),\ldots,(a_{k-1},a_k)$, the pairs being considered unordered. The elements in a block are distinct, so the handcuffed pairs are distinct as well. A collection of b handcuffed blocks forms a handcuffed design if

(i) each element of the ν-set appears in exactly r of the blocks

(ii) each (unordered) pair of distinct elements of the ν-set is handcuffed in exactly λ of the blocks.

It can be shown [9] that every element of V must occur in the interior (that is, not in the first or last position) of exactly u blocks. Further the following equalities can be shown:

$$u=\frac{\lambda(v-1)(k-2)}{2(k-1)}, \qquad r=\frac{\lambda k(v-1)}{2(k-1)}, \qquad b=\frac{\lambda v(v-1)}{2(k-1)}. \qquad (1)$$

It is well known [12] that a $H(v,3,\lambda)$ exists if and only if $v\equiv1 \pmod 4$ for $\lambda\equiv1,3 \pmod 4$, $v\equiv1 \pmod 2$ for $\lambda\equiv2 \pmod 4$, all $v\geq3$ for $\lambda\equiv0 \pmod 4$, and a $H(v,4,1)$ exists if and only if $v\equiv1 \pmod 3$.

Let (V,B) be a handicuffed design $H(v,k,\lambda)$. A subset S of V is called a *blocking set* if, for every $b\in B$, $b\cap S\neq\emptyset$ and $b\cap(V\setminus S)\neq\emptyset$.

Blocking sets have been investigated in projective spaces [1,2,13], in t-designs [3,4,6,7,8,10,11] and in G-designs [5].

Let

$BS(v,k,\lambda)=\{h$: $\exists$ $H(v,k,\lambda)$ *having a blocking set* S *with* $|S|=h\}$.

In this paper, we completely determine $BS(v,3,\lambda)$ and $BS(v,4,1)$ for all admissible v. In particular we prove that

$$BS(v,3,\lambda)=\begin{cases} \frac{v}{2} & \text{for } v\equiv0 \pmod 2 \\ \left\{\frac{v-1}{2}, \frac{v+1}{2}\right\} & \text{for } v\equiv1 \pmod 2 \end{cases}$$

and that

$$BS(v,4,1)=\left\{\frac{v-1}{3}, \ldots, \frac{2v+1}{3}\right\} \text{ for } v\equiv1 \pmod 3 .$$

Moreover, in section 4, we exhibit, for every $v\equiv1 \pmod 4$, $v\geq9$, a $H(v,3,1)$ without blocking sets.

2. Blocking sets in $H(v,3,\lambda)$.

In this section we determine $BS(v,3,\lambda)$ for all admissible v.

LEMMA 2.1.

If S *is a blocking set in* $H(v,3,\lambda)$ (V,B), *then*

$|S|\in\left\{ \frac{v-1}{2}, \frac{v+1}{2} \right\}$ *for* v *odd and* $|S|=\frac{v}{2}$ *for* v *even.*

Proof. Let $|S|=w$ and let $d_i(x)$ be the number of blocks containing an element $x\in V$ in the interior. Since any block containing a handcuffed pair of S contains necessarily an element of S in the interior and $d_i(x)=\frac{\lambda(v-1)}{4}$, we obtain

$$\sum_{x\in S} d_i(x)= \frac{\lambda w(v-1)}{4} \geq\lambda \frac{w(w-1)}{2} \qquad (1)$$

This implies

$$2w\leq v+1 . \qquad (2)$$

Since V-S is also a blocking set, by (2) we obtain

$$2w\geq v-1 . \qquad (3)$$

The proof follows from (2) and (3). ■

LEMMA 2.2.

There is a $H(v,3,\lambda)$ (V,B) *having a blocking set* S *such that*

i) $|S|\in\left\{ \frac{v-1}{2}, \frac{v+1}{2} \right\}$ *if* $v\equiv1 \pmod 4$ *and* $\lambda\equiv1,3 \pmod 4$

ii) $|S|\in\left\{ \frac{v-1}{2}, \frac{v+1}{2} \right\}$ *if* $v\equiv1 \pmod 2$ *and* $\lambda\equiv0,2 \pmod 4$

iii) $|S|=\frac{v}{2}$ *if* $v\equiv0 \pmod 2$ *and* $\lambda\equiv0 \pmod 4$

Proof. Let $V=\{1,2,\dots,v\}$

i) Let $B_1=\bigcup_{j=1}^{\frac{v-1}{4}}\{(i,i+2j,i+1),i\in Z_v\}$ and $B_2=\bigcup_{j=1}^{\frac{v-1}{4}}\left\{\left(i,\frac{v-1}{2}+2j+i,i+1\right),i\in Z_v\right\}$.

Repeating each block of B_1 and B_2 respectively $\frac{\lambda+1}{2}$ and $\frac{\lambda-1}{2}$ times we obtain a set of blocks B such that (V,B) is a $H(v,3,\lambda)$ with blocking sets $\{2,4,\dots,v-1\}$ and $\{1,3,\dots,v\}$.

ii) Let $B_1=\bigcup_{j=1}^{\frac{v-1}{2}}\{(i,i+2j,i+1),i\in Z_v\}$.

Repeating each block of B_1 $\frac{\lambda}{2}$ times we obtain a set of blocks B such that (V,B) is a $H(v,3,\lambda)$ with blocking sets $\{2,4,\dots,v-1\}$ and $\{1,3,\dots,v\}$.

iii) Let $B_1=\bigcup_{j=1}^{\frac{v-2}{2}}\{(i,i+2j,i+1),i\in Z_v\}$ and $T_0=\{(i,i+1,i+2),i\in Z_v\}$.

Repeating each block of B_1 $\frac{\lambda}{2}$ times we obtain a set of blocks $\overline{B}$ such that $(V,\overline{B}\cup T_0)$ is a $H(v,3,\lambda)$ with blocking sets $\{2,4,\dots,v-1\}$ and $\{1,3,\dots,v\}$. ■

From Lemmas 2.1 and 2.2 we obtain the following theorem.

THEOREM 2.1.

$BS(v,3,\lambda)=\frac{v}{2}$ *for* $v\equiv0 \pmod 2$ *and* $BS(v,3,\lambda)=\left\{\frac{v-1}{2},\frac{v+1}{2}\right\}$ *for* $v\equiv1 \pmod 2$.

3. Blocking sets in H(v,4,1).

Let (V,B) be a $H(v,4,1)$ and $I(v)=\left\{\frac{v-1}{3},\dots,\frac{2v+1}{3}\right\}$. In this section we determine $BS(v,4,1)$ for all $v\equiv1 \pmod 3$.

LEMMA 3.1.

If S *is a blocking set in* H(v,4,1) (V,B), *then*

$$|S| \in \left\{ \frac{v-1}{3}, \dots, \frac{2v+1}{3} \right\}.$$

Proof. Let $|S|=w$ and $d_i(x)$ be as in Lemma 2.1. Since $d_i(x)=\frac{v-1}{3}$ and any block containing a handcuffed pair of S contains necessarily at least an element of S in the interior, we obtain

$$\sum_{x\in S} d_i(x)=\frac{(v-1)}{3}w \geq \frac{w(w-1)}{2} \qquad (1)$$

hence

$$w \leq \frac{2v+1}{3}. \qquad (2)$$

Since V-S is also a blocking set, by (2) we have

$$w \geq \frac{v-1}{3}. \qquad (3)$$

The proof follows from (2) and (3). ■

LEMMA 3.2.

Let $v\equiv 1 \pmod 3$. *If* $t\in BS(v,4,1)$ *then* $t+i\in BS(v+3,4,1)$ *for* $i=1,2$.

Proof. Let (V,B) be a H(v,4,1) having blocking set S such that $|S|=t\in BS(v,4,1)$.

Let $V=\{1\}\cup X_1\cup X_2\cup X_3$ with

$$X_j=\left\{x_i^{(j)},\ i=1,2,\dots,\frac{v-1}{3}\right\},\ j=1,2,3.$$

Without loss of generality we may suppose that $S\subseteq\{1\}\cup X_1\cup X_2$ for $|S|=\frac{2v+1}{3}$ and $S\subseteq X_1\cup X_2$ for $\frac{v-1}{3}\leq|S|<\frac{2v+1}{3}$.

Let $A=\{a,b,c\}$ with $A\cap V=\emptyset$. Put $V^*=V\cup A$ and $B^*=B\cup T\cup F$ where $T=\{(1,a,b,c),(b,1,c,a)\}$ and $F=\left\{(x_i^{(1)},a,x_i^{(2)},b),(x_i^{(3)},c,x_i^{(1)},a),\right.$ $\left.(x_i^{(2)},b,x_i^{(3)},c);\ i=1,2,\dots,\frac{v-1}{3}\right\}$.

Then (V^*,B^*) is a $H(v+3,4,1)$ with blocking sets $S\cup\{c\}$ and $S\cup\{a,c\}$. This proves that $t+1,t+2\in BS(v+3,4,1)$. ■

From Lemma 3.2. we obtain easily the following lemma

LEMMA 3.3.

Let $v\equiv1 \pmod 3$, *then* $BS(v,4,1)=I(v)$ *implies* $BS(v+3,4,1)=I(v+3)$.

Since it is easy to see that $BS(4,4,1)=I(4)=\{1,2,3\}$, from Lemmas 3.2 and 3.3 we obtain

THEOREM 3.1.

$BS(v,4,1)=I(v)$ for every $v\equiv1 \pmod 3$.

4. H(v,3,1) without blocking sets.

LEMMA 4.1.

There exists a $H(9,3,1)$ *without blocking sets.*

Proof. Let $v=\{1,\dots,9\}$ and $B=\{(i,i+2,i+1),\ (i,4+i,i+7),\ i\in Z_9\}$. It is easy to see that (V,B) is a $H(9,3,1)$. We now prove that (V,B) has no blocking sets. In fact suppose that S is a blocking set of (V,B). Since $|S|\in\{4,5\}$, there necessarily exists at least one $x\in V$ such that either $x,x+1\in S$ or $x,x+2\in S$. If $x,x+1\in S$, it follows that $x-1,x+2\notin S$ and $x+4\in S$. Hence $x+6\notin H$, $x+7\in S$ and $x+3\notin S$. This is impossible because $(x-1,x+3,x+6)\in B$. If $x,x+2\in S$, then $(x-1,x+1,x+3,x+6)\cap S=\emptyset$. This is impossible because $(x-1,x+3,x+6)\in B$. ■

THEOREM 4.1.

For every $v\equiv 1 \pmod 4)\geq 9$ *there exists a* $H(v,3,1)$ *without blocking sets.*

Proof. The statement follows by applying the $v\longrightarrow v+4$ construction [12] to the $H(9,3,1)$ of the Lemma 4.1 and noting that a $H(v,3,1)$ with blocking sets cannot contain a sub-$H(v,3,1)$ without blocking sets.

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